ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર-ક્રમાંક
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MATHEMATICS

Standard 9

(Semester I)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

Gujarat State Board of School Textbooks
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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is pleasure for the Gujarat State Board of School Textbooks, to place before the students this textbook of **Mathematics for Standard 9 (Semester I)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions, from people interested in education, to improve the quality of the textbook.

Dr. Bharat Pandit
Director
Date : 03-03-2015

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

CONTENTS

About This Textbook...

The Gujarat Secondary and Higher Secondary Education Board has prepared a new syllabus for school curriculum with the help of Gujarat State Textbook Board, learned school teachers, college teachers and university teachers in order to make a student of Gujarat State secure his leading position at national level in present scenario. This syllabus is equivalent to NCERT syllabus. NCF 2005 was kept in mind while preparing this syllabus. Then a panel of subject experts was formed to prepare textbook based on this syllabus.

In a very short period the first draft of the textbook was prepared in English and then translated into Gujarati. Considering global thinking, it was decided to prepare the textbook first in English.

A panel of experts from school, colleges and universities held a workshop of three days and thoroughly discussed the content and made some suggestions. Amendments were carried out in the English as well as Gujarati draft accordingly. A professor of English also was helpful in suggesting some changes in English language. After that there was again a workshop of four days for the Gujarati draft of the textbook. Again suggestions were obtained from experts and amendments were carried out.

The final draft was thus prepared. The experts and members of syllabus committee along with authors reviewed the textbook in the office of Gujarat State Secondary and Higher Secondary Board.

This textbook is prepared according to new syllabus. The NCERT textbook based on NCERT syllabus is having classical approach to geometry. But Gujarat State is using modern approach using set theory for last four decades. Thus the first chapter on set theory is additional compared to the NCERT textbook. The whole curriculum is divided into two semesters. Therefore this book is also divided into two parts. Since standard 8 is included in primary education, content of geometry as specified in the NCERT textbook for standard 8 at present has to be covered in this book.

In the first semester set theory, number system, polynomials, coordinate geometry, graphs of linear equations along with four chapters of geometry are included. All chapters are explained in a lucid language and yet logical consistency is preserved, plenty of illustrations are used to explain the concepts. We have kept in mind that student at the remote end of the state also can study with his own skill. Concept, illustrations and exercises are formed accordingly. Printing in two colours and attractive title in four colours are an added assets to the textbook.

Four chapters of geometry, calculations of area and volume and statistics are included in the textbook of second semester. Also keeping in view the need of a science student, chapter on logarithm is given at the end.

Some teachers of Central Board schools also reviewed the draft of the textbook. They also found the draft of the textbook very useful and they praised the explanation and the illustrations.

In the golden jubilee year of the birth of Gujarat State, we have tried to see that the students of the state get a textbook equivalent to curriculum specified by NCERT. So that students may maintain leading position at national level using the textbook.

In both the semester books detailed explanation is given using figures, diagrams and graphs.

Complete syllabus of NCERT is covered by the textbook. Moreover examples and exercises are introduced to be equally useful to the teachers and students. The main aim of this textbook is to make student of Gujarat face challenges at national level and still study with interest and without burden.

- Author

Chapter 1

SET OPERATIONS

- *The essence of mathematics lies in its freedom.*
- *I see it but I don't believe it !*

– Georg Cantor

1.1 Introduction

The concept of a set is a base for all branches of mathematics. The theory of sets was developed by mathematician **Georg Cantor** (1845-1918 A.D). We have learnt some important and primary facts about sets in std. VIII.

In day-to-day life, we often talk about a group of same kind of objects; e.g., a herd of cows, a pack of cards, a team of players etc. This type of a **well-defined collection of objects is considered as a set.**

The father of set theory

The main inventor of set theory was the mathematician Georg Cantor. He was born on 3rd March, 1845 in St. Petesburg, Russia. He took his school education in St. Petesburg. In 1856, he moved to Germany. He was president of **Berlin Mathematical Society** (1864-1865). He achieved doctorate degree in 1867. He taught at a girls school in Berlin. In 1872, he was promoted as an extraordinary professor in Halle. He was a friend of Dedekind. He got some very surprising results in Mathematics. In 1873, he proved that rational numbers are countable. The birth of set theory dates to 1873, when Georg Cantor proved the uncountability of real line, actually December 7, 1873. Hilbert described Cantor's work as the finest product of mathematical genius and one of the supreme achievement of purely intellectual human activity. Some powerful people who disagreed with him severely criticized him for this. But today while those who troubled him are forgotten, Georg Cantor is remembered and widely respected. He died on 6th January, 1918 in Halle, Germany.

1.2 Important Points for Revision

- A set is a well-defined collection of objects.
- A set without any member (element) is called a null set or an empty set.
- A set having only one member is called a singleton.
- ∈ (belongs to) is an undefined symbol.
- If x is a member of the set A, we write $x \in A$
- If x is not a member of the set A, we write $x \notin A$.
- A set total number of members of which is a positive integer is called a finite set and a set which is not finite is called an infinite set. Null set is considered to be a finite set.
- If all the elements of a set A are present in the set B, then the set A is called a subset of the set B. This fact is denoted by $A \subset B$.

Important points about subsets :

(1) Empty set is a subset of every set. Thus, for any set A, $\emptyset \subset A$.

(2) Every set is a subset of itself. Thus, for any set A, $A \subset A$.

(3) If a set A has n elements, then number of its subsets is 2^n.

(4) $N \subset Z \subset Q \subset R$.

- Generally, while dealing with a problem, we consider some definite set and its subsets. Such a definite set is called the **universal set** with reference to that problem. The **universal set** is denoted as U.

  A set which is a universal set for one problem need not be a universal set for another problem. For example, In Geometry, space or plane is a universal set. For interrelations of integers, set of integers Z is a universal set. For the solution of linear equations, the set of real numbers is a universal set.

- The set of all the elements which are in U but not in the given set A is called the **Complement of the set A**. It is denoted by A'.

  Thus, $A' = \{x \mid x \in U, x \notin A\}$

  so from the above definition, we get the following results.

  (1) $A \cup A' = U$ and (2) $A \cap A' = \emptyset$

- If two sets have same elements, they are said to be **equal sets. If every member of set A is a member of set B and every member of set B is a member of set A, then set A and set B are called equal sets. If A and B are equal sets we write A = B.** For equal sets A and B, $A \subset B$ and $B \subset A$.

i.e. if $A \subset B$ and $B \subset A$, then $A = B$.

For example let $A = \{x \mid x \in N, x < 5\}$ and $B = \{1, 2, 3, 4\}$ be two sets.

Then both the sets A and B have the same members $\{1, 2, 3, 4\}$.

So, we say that $A = B$

- If every member of set A corresponds to one and only one member of set B and every member of set B corresponds to one and only one member of set A then the sets A and B are said to be in one-one correspondence with each other and the sets A and B are called **equivalent sets**. If set A is equivalent to set B, we write $A \sim B$.
- Thus, if two finite sets are in one-one correspondence with each other, then they should have the same number of elements.
- **Equal sets are always equivalent sets but equivalent sets need not be equal sets.**

For example, if $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ then $A \sim B$ but $A \neq B$.

But for infinite sets, situation is different.

If, $E = \{2, 4, 6, 8, ...\}$, then $N \sim E$. Because for every element of N, A unique number n is related to the number $2n$ belonging E and for every element of E, a unique number m is related to the number $\frac{m}{2} \in N$. But $E \subset N$.

EXERCISE 1.1

1. Classify the following sets in (a) as empty set or singleton set and in (b) as equal sets or equivalent sets :

(a) (1) $A = \{x \mid x \in Z, x + 1 = 0\}$

(2) $B = \{x \mid x \in N, x^2 - 1 = 0\}$

(3) $C = \{x \mid x \in N, x$ is a prime number between 13 and 17$\}$

(b) (1) $A = \{x \mid x \in N, x \leq 7\}$,

$B = \{x \mid x \in Z, -3 \leq x \leq 3\}$

(2) $A = \{x \mid x \in N, x$ is a multiple of 2, $x < 10\}$,

$B = \{x \mid x \in N, x$ is an even natural number with a single digit$\}$

2. Find the number of subsets of the set $A = \{1, 2, 3\}$. Also write all the subsets of the set A.

3. If $A = \{x \mid x \in Z, x^2 - x = 0\}$, $B = \{x \mid x \in N, 1 \leq x \leq 4\}$, then can we say that $A \subset B$? Why ?

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 8\}$, then find A' and also verify that $A \cup A' = U$.

5. If A = {1, 2, 3}, B = {3, 4, 6}, then find all possible non-empty sets X which satisfy the following conditions :

(1) $X \subset A, X \not\subset B$ (2) $X \subset B, X \not\subset A$ (3) $X \subset A, X \subset B$

6. Examine whether the following statements are true or false :

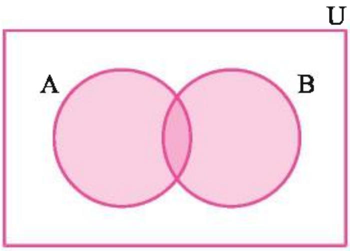
(1) $\{1, 2, 3\} \subset \{1, 2, 3\}$ (2) $\{a, b\} \not\subset \{b, c, a\}$

(3) $\emptyset \notin \{\emptyset\}$ (4) $\{3\} \subset \{1, 2, \{3\}, 4\}$

*

1.3 Properties of the Union Operation

Union set : For any two sets A and B, the set consisting of all the elements which are either in A or in B (or in both) is called the union set of the sets A and B and it is denoted by A ∪ B. The process of finding the union of two set is called the union operation.



Venn diagram of A ∪ B
Figure 1.1

Thus, in symbol, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. Venn-diagram is useful in understanding various relations between sets. In Venn-diagram 1.1 the coloured region describes A ∪ B.

Example 1 : Let A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8} be two sets. Find A ∪ B.

Solution : A ∪ B = {1, 3, 5, 7, 9} ∪ {2, 4, 6, 8}
= {1, 2, 3, 4, 5, 6, 7, 8, 9}

Example 2 : If α = The letters of the word AHMEDABAD
and β = The letters of the word BARODA are two sets, then find $\alpha \cup \beta$.

Solution : Here α = {A, B, D, E, H, M} and
β = {A, B, D, O, R}
$\therefore \alpha \cup \beta$ = {A, B, D, E, H, M} ∪ {A, B, D, O, R}
= {A, B, D, E, H, M, O, R}

Properties : Following are some rules followed by union operation. We will verify them with the help of illustrations.

(1) Union is a Binary Operation : For any two sets A and B, if $A \subset U$ and $B \subset U$, then $(A \cup B) \subset U$.

Suppose $U = \{x \mid x \in N, 1 \le x \le 6\}$, A = {1, 2, 3}, B = {2, 3, 4, 5}
Here, U = {1, 2, 3, 4, 5, 6}
So, $A \subset U$ and $B \subset U$

Now, $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

Clearly, all members of $A \cup B$ are in U.

So, $(A \cup B) \subset U$. This result says union is a binary operation.

(2) Commutative Law : For any two sets A and B, $A \cup B = B \cup A$.

Let $A = \{c, d, e, f\}$, $B = \{p, q, r, s, t\}$ be any two sets.

Then,

$A \cup B = \{c, d, e, f\} \cup \{p, q, r, s, t\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(i)**

Now, $B \cup A = \{p, q, r, s, t\} \cup \{c, d, e, f\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(ii)**

Thus from (i) and (ii), $A \cup B$ and $B \cup A$ have the same elements.

Therefore, $A \cup B = B \cup A$

This law is known as commutative law for union, i.e. union is a commutative operation.

(3) Associative Law :

For any three sets A, B and C, $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $A = \{x \mid x \in N, 1 \leq x \leq 5\}$, $B = \{x \mid x \in N, x$ is an even number, $x < 10\}$,

$C = \{x \mid x \in N, x$ is a multiple of 3, $x < 10\}$

Now, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 6, 9\}$ are given sets.

So, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{2, 4, 6, 8\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

$\therefore$ $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\} \cup \{3, 6, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(i)**

Now, $B \cup C = \{2, 4, 6, 8\} \cup \{3, 6, 9\}$

$= \{2, 3, 4, 6, 8, 9\}$

$\therefore$ $A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{2, 3, 4, 6, 8, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(ii)**

$\therefore$ By results (i) and (ii) we verify that union is associative.

From Venn-diagram 1.2 it can be verified that,

$(A \cup B) \cup C = A \cup (B \cup C)$

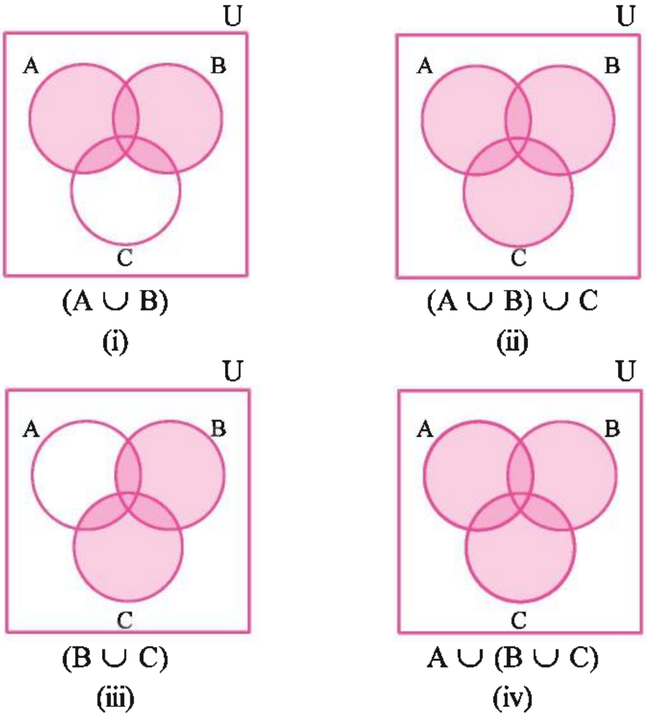


Figure 1.2

In Venn diagram 1.2 coloured region describes the set mentioned below the Venn-diagram.

This law is known as the associative law for union.

i.e. **union is an associative operation.**

(4) For any two sets A and B, $A \subset (A \cup B)$ and $B \subset (A \cup B)$

Let $A = \{x \mid x \in Z, x^2 - 4 = 0\}$, $B = \{x \mid x \in N, x \leq 5\}$ be two sets.

$\therefore$ Here $A = \{-2, 2\}$, $B = \{1, 2, 3, 4, 5\}$

Now $A \cup B = \{-2, 2\} \cup \{1, 2, 3, 4, 5\}$

$= \{-2, 1, 2, 3, 4, 5\}$

Clearly $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

Look at the Venn-diagram 1.3. Here A and B are two sets. The set A consists of the regions R_1 and R_2; the set B consists of the regions R_2 and R_3. So, $A \cup B$ consists of the regions R_1, R_2 and R_3. Thus, the regions R_1 and R_2 are included in the regions R_1, R_2 and R_3 together. i.e. the set A is contained in the set $A \cup B$, so $A \subset (A \cup B)$.

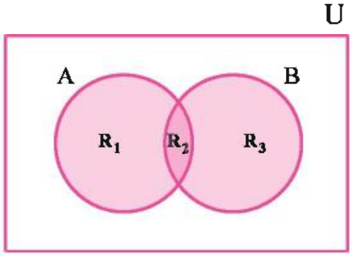


Figure 1.3

Similarly the regions R_2 and R_3 are included in the regions R_1, R_2 and R_3 together. That means the set B is contained in the set A ∪ B. i.e. B ⊂ (A ∪ B)

In general, **for any two sets A and B, A ⊂ (A ∪ B) and B ⊂ (A ∪ B).**

(5) If A ⊂ B, then A ∪ B = B

Let us try to understand the above property by following example.

Example 3 : If α = The set of the letters of the word GATE and β = The set of the letters of the word LOCGATE are two sets, then verify that α ⊂ β and α ∪ β = β

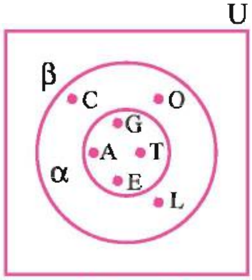


Figure 1.4

Solution : Here α = {G, A, T, E},

β = {L, O, C, G, A, T, E}

∴ Here all the elements of the set α are present in β.

∴ α ⊂ β

Now, α ∪ β = {G, A, T, E} ∪ {L, O, C, G, A, T, E}

= {L, O, C, G, A, T, E}

∴ α ∪ β = β

(6) A ∪ U = U and A ∪ ∅ = A

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} be the universal set and

A = {x | x ∈ N, x is a prime number less than 10}

be a given set

∴ A = {2, 3, 5, 7}

Thus, A ∪ U = {2, 3, 5, 7} ∪ {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} = U

A ∪ ∅ = {2, 3, 5, 7} ∪ ∅

= {2, 3, 5, 7}

= A

Thus, we say that A ∪ U = U and A ∪ ∅ = A.

1.4 Properties of the Intersection Operation

Now we will study some rules about operation of intersection and verify them with the help of illustrations.

Intersection set : For any two sets A and B, the set consisting of all the elements which belong to both the sets A and B is called the intersection set of two sets A and B and it is denoted by A ∩ B.

Thus, in symbols A ∩ B = {x | x ∈ A and x ∈ B}

In Venn-diagram 1.5 the coloured region describes A ∩ B.

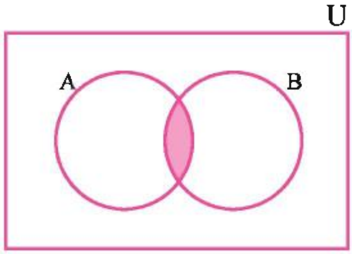


Figure 1.5

Example 4 : Let A = {x | x ∈ N, x is a multiple of 3, $x \leq 15$}

B = {x | x ∈ Z, $0 < x < 10$} be two sets.

Find A ∩ B.

Solution :

Here A = {x | x ∈ N, x is a multiple of 3, $x \leq 15$}

= {3, 6, 9, 12, 15}

B = {x ∈ Z, $0 < x < 10$} = {1, 2, 3, 4, 5, 6, 7, 8, 9}

∴ A ∩ B = {3, 6, 9, 12, 15} ∩ {1, 2, 3, 4, 5, 6, 7, 8, 9} = {3, 6, 9}

Let us verify some properties of intersection by examples.

Properties :

(1) Intersection is a Binary Operation : For two sets A and B, if A ⊂ U, B ⊂ U, then (A ∩ B) ⊂ U.

Suppose U = {x | x ∈ N, $1 \leq x \leq 25$}, A = {1, 4, 9, 16, 25}

B = {4, 8, 12, 16, 20}

Since U = {1, 2, 3,..., 25}, A ⊂ U, B ⊂ U

Now A ∩ B = {1, 4, 9, 16, 25} ∩ {4, 8, 12, 16, 20} = {4, 16}

Clearly, each member of A ∩ B is in U.

∴ (A ∩ B) ⊂ U

So, intersection is a binary operation.

(2) Commutative Law : For any two sets A and B, A ∩ B = B ∩ A

Let A = {1, 2, 3, 4, 5}, B = {3, 4, 6, 8} be any two sets.

Then A ∩ B = {1, 2, 3, 4, 5} ∩ {3, 4, 6, 8} = {3, 4} **(i)**

and B ∩ A = {3, 4, 6, 8} ∩ {1, 2, 3, 4, 5}

= {3, 4} **(ii)**

Thus, A ∩ B = B ∩ A **(from (i) and (ii))**

This law is known as commutative law for intersection i.e. intersection is a commutative operation.

(3) Associative Law : For any three sets A, B and C,

(A ∩ B) ∩ C = A ∩ (B ∩ C)

Let A = {1, 2, 3, 4, 5}, B = {3, 4, 5, 6}, C = {1, 4, 7}

∴ A ∩ B = {3, 4, 5}

∴ (A ∩ B) ∩ C = {4} (i)

B ∩ C = {4}

∴ A ∩ (B ∩ C) = {4} (ii)

Thus, $(A \cap B) \cap C = A \cap (B \cap C)$ ((i) and (ii))

Now, Let us verify the law with the help of Venn-diagram.

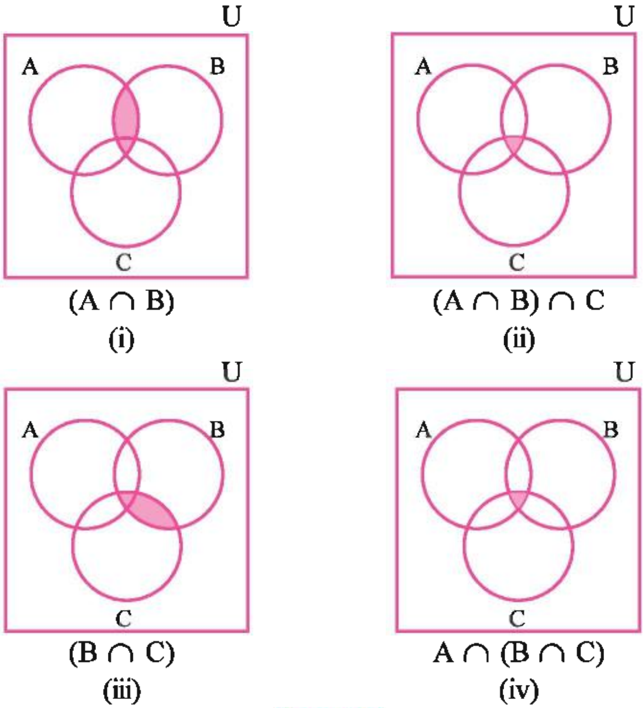


Figure 1.6

In Venn-diagram 1.6, coloured region describes the set mentioned below the Venn-diagram.

It can be seen from the Venn-diagram 1.6 that, $(A \cap B) \cap C = A \cap (B \cap C)$

In general, **for any three sets A, B, and C, $(A \cap B) \cap C = A \cap (B \cap C)$**

This rule is known as the associative law for the operation of intersection.

(4) $(A \cap B) \subset A$ and $(A \cap B) \subset B$.

All the elements of $A \cap B$ belong to the sets A and B.

Hence $(A \cap B) \subset A$ and $(A \cap B) \subset B$

Look at the Venn-diagram 1.7. The set A consists of the regions R_1 and R_2. The set B consists of the regions R_2 and R_3. The region R_2 is common to both A and B. Thus, $A \cap B$ consists of the region R_2. The region R_2 is contained in A as well as in B. Thus, $(A \cap B) \subset A$ and $(A \cap B) \subset B$.

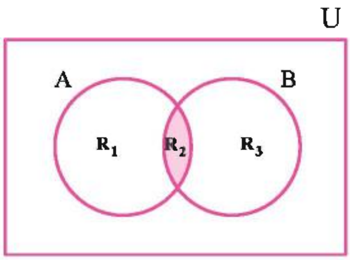


Figure 1.7

(5) If $A \subset B$, then $A \cap B = A$

Let us verify this property by an example.

Example 5 : Let A = $\{x \mid x \in N, x^2 - 9 = 0\}$ and B = $\{x \mid x \in N, x < 5\}$ be two given sets. Verify that A ⊂ B and A ∩ B = A

Solution : Here $x^2 - 9 = 0$

∴ $x^2 - 3^2 = 0$

∴ $(x + 3)(x - 3) = 0$

∴ $x = -3$ or $x = 3$

as $x \in N$, $x = -3$ is not possible.

∴ $x = 3$

Hence A = {3} **(i)**

Now, B = $\{x \mid x \in N, x < 5\}$

= {1, 2, 3, 4} **(ii)**

Hence by the results (i) and (ii), we can say that A ⊂ B.

Now, A ∩ B = {3} ∩ {1, 2, 3, 4} = {3} = A

∴ If A ⊂ B, then A ∩ B = A

Similarly If B ⊂ A, then A ∩ B = B. (Verify it by yourself)

(6) A ∩ ∅ = ∅ and A ∩ U = A

Let U = {1, 2, 3, 4, 5}, A = {2, 3}

Then obviously A ∩ U = {2, 3} = A and A ∩ ∅ = ∅

Disjoint sets : For any two non-empty sets A and B, if A ∩ B = ∅ then the sets A and B are said to be disjoint

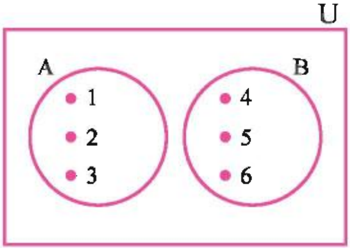
Example 6 : If A = {1, 2, 3}, B = $\{x \mid x \in N, 3 < x < 7\}$ are two sets, are they disjoint ?

Solution : Here B = {4, 5, 6}

Hence A ∩ B = {1, 2, 3} ∩ {4, 5, 6} = ∅

So there is no element common to both A and B. Hence we say that A and B are disjoint sets.

By Venn-diagram 1.8 we can also understand the above definition very easily.



A ∩ B = ∅

Figure 1.8

1.5 Distributive Laws

We are familiar with the distributive law of multiplication over addition for real numbers.

For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$

For example if $a = 3$, $b = 4$, $c = 5$, then

L.H.S. = $a \times (b + c) = 3 \times (4 + 5) = 3 \times 9 = 27$

R.H.S. = $(a \times b) + (a \times c) = (3 \times 4) + (3 \times 5) = 12 + 15 = 27$

∴ L.H.S. = R.H.S.

But the converse does not hold. Thus, the distributive law of addition over multiplication is not satisfied.

i.e. $a + (b \times c) \neq (a + b) \times (a + c)$. For set operations, the situation is different. Union is distributive over intersection and intersection is distributive over union.

(1) Distributivity of Union over Intersection : For any three sets A, B and C, we have A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Suppose A = {p, q, r, s}, B = {q, r}, C = {r, s, t} are three sets.

Taking B ∩ C = {q, r} ∩ {r, s, t} = {r}

∴ L.H.S. = A ∪ (B ∩ C) = {p, q, r, s} ∪ {r} = {p, q, r, s} **(i)**

Now A ∪ B = {p, q, r, s} ∪ {q, r} = {p, q, r, s}

A ∪ C = {p, q, r, s} ∪ {r, s, t} = {p, q, r, s, t}

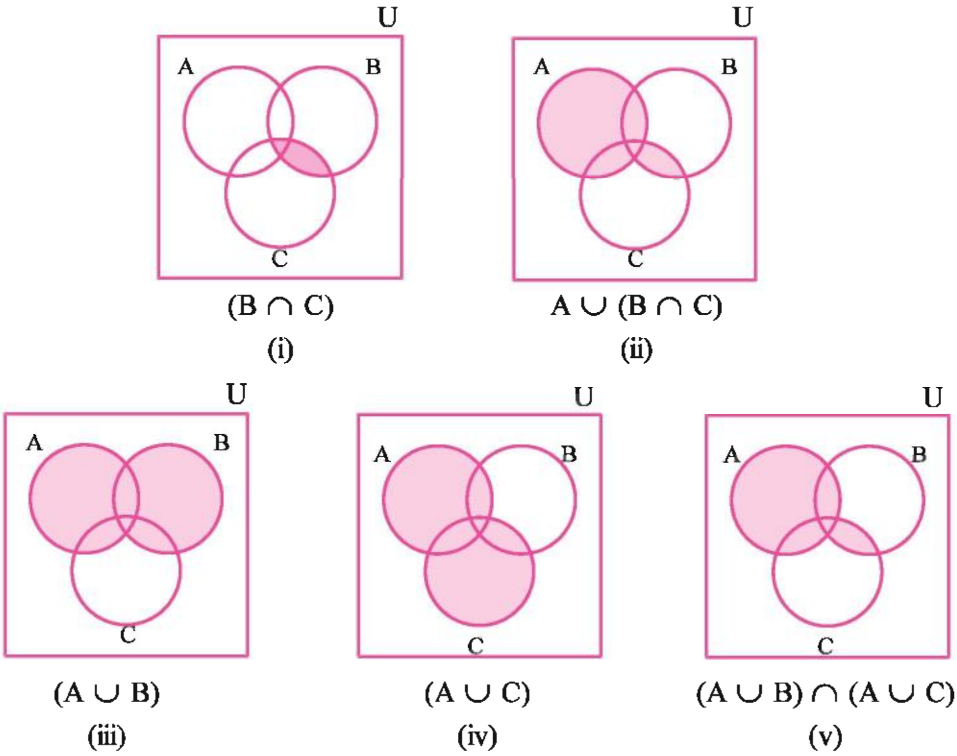
∴ R.H.S. = (A∪B) ∩ (A ∪ C) = {p, q, r, s} ∩ {p, q, r, s, t}

= {p, q, r, s} **(ii)**

Thus, from (i) and (ii), it is clear that

A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

This law can be verified using Venn-diagram as follows :



(B ∩ C)
(i)

A ∪ (B ∩ C)
(ii)

(A ∪ B)
(iii)

(A ∪ C)
(iv)

(A ∪ B) ∩ (A ∪ C)
(v)

Figure 1.9

In Venn-diagram 1.9, coloured region describes the set mentioned below the Venn-diagram.

$\therefore \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

In words; **Union is distributive over intersection**

(2) Distributivity of Intersection over Union :

For any three sets A, B and C, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5, 6\}$ be three sets. Then $B \cup C = \{2, 3, 4\} \cup \{3, 4, 5, 6\} = \{2, 3, 4, 5, 6\}$

Now $A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5, 6\} = \{2, 3, 4\}$ **(i)**

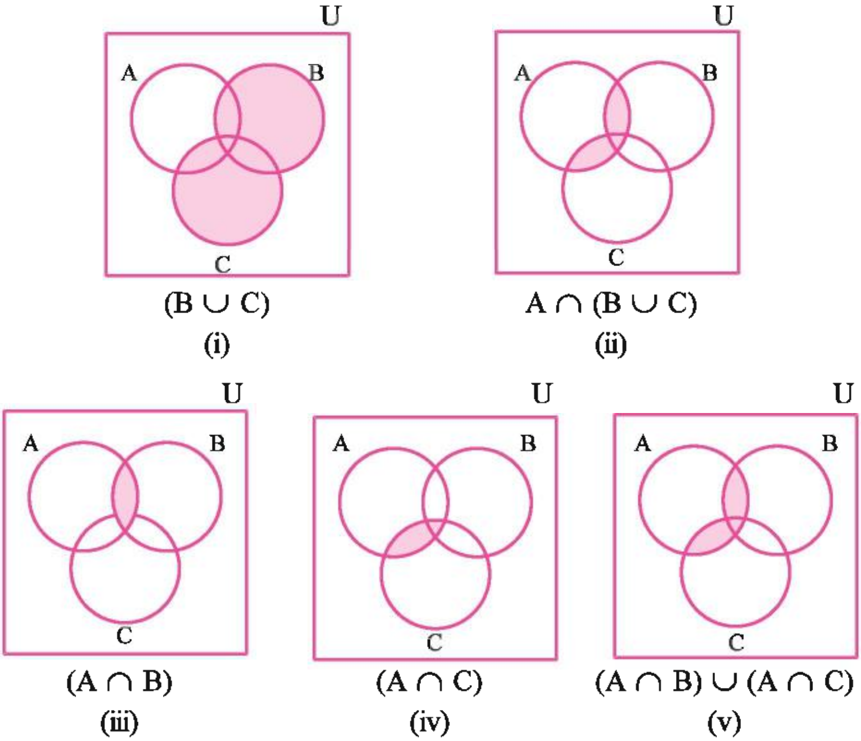
$A \cap B = \{1, 2, 3, 4\} \cap \{2, 3, 4\} = \{2, 3, 4\}$ and

$A \cap C = \{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}$, we get

$(A \cap B) \cup (A \cap C) = \{2, 3, 4\} \cup \{3, 4\} = \{2, 3, 4\}$ **(ii)**

Thus, by (i) and (ii), **the distributivity of intersection over union is verified.**

We can also verify this by following Venn-diagrams :



$(B \cup C)$ (i) | $A \cap (B \cup C)$ (ii) | $(A \cap B)$ (iii) | $(A \cap C)$ (iv) | $(A \cap B) \cup (A \cap C)$ (v)

Figure 1.10

In Venn-diagram 1.10, coloured region describes the set mentioned below the Venn-diagram.

$\therefore$ A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

In general, for any three sets A, B and C, A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).

In words, **intersection is distributive over union.**

EXERCISE 1.2

1. Verify that A ∪ B = B ∪ A for the sets A = {1, 2, 3, 4} and B = {3, 4, 5, 6}.
2. If A = {x | x ∈ N, x is a factor of 12} and B = {x | x ∈ N, $2 < x < 7$}, then find A ∩ B.
3. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 3, 4, 5}, B = {4, 5, 6} and C = {1, 3, 5, 7}, then verify the distributivity of union over intersection.
4. If A = {x | x ∈ N, x is a prime factor of 12} and B = {x | x ∈ N, x is a prime factor of 20} are given sets, then find A ∩ B.
5. Let A = {x | x ∈ N, $x < 10$}, B = {x | x ∈ N, x is a multiple of 3; x less then 15}, C = {x | x ∈ Z, $-4 < x < 4$} be three sets, then verify A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).
6. If A = {1, 2, 3, 4}, B = {x | x ∈ N, $4 \le x \le 6$} are given sets, then find A ∩ B. Are they disjoint sets ?

*

1.6 Properties of Complement of a Set

Complement of a set : The set consisting of all the elements of U which are not in the given set A is called the complement of a set A. It is denoted by A'. The process of finding the complement of a set is called complementation.

Thus, in symbols, A' = {x | x ∈ U, x ∉ A}

From the definition, it is clear that,

(1) A member of U which is not in the set A is in set A', and a member of U which is not in A' is in the set A. Thus each member of U is either in A or in A'. So, A ∪ A' = U.

(2) A member of U which is in A, is not in A', and a member of U which is in A', is not in A. Thus A and A' have no common members. This means that A ∩ A' = ∅.

If an element of U is in A, it cannot be in A' and hence it must be in (A')' and vice-versa. Thus, (A')' = A.

Let us understand the above results by the following example.

Example 7 : Let U = {−1, 0, 1, 2, 3, 4, 5} be the universal set and A = {−1, 0, 1} is a given set. Then verify

(1) A ∪ A' = U   (2) A ∩ A' = ∅   (3) (A')' = A

Solution : Here A = {–1, 0, 1}. U = {–1, 0, 1, 2, 3, 4, 5}

∴ A' = {2, 3, 4, 5}

(1) Now, A ∪ A' = {–1, 0, 1} ∪ {2, 3, 4, 5}

= {–1, 0, 1, 2, 3, 4, 5} = U

(2) A ∩ A' = {–1, 0, 1} ∩ {2, 3, 4, 5} = ∅

because there is no common element. So they are disjoint.

(3) A' = {2, 3, 4, 5}, (A')' = {–1, 0, 1} = A

According to the definition, the members of U which are not in A are in A'. But there is no member of U which is not in U. So there is no member in U'. So U' is the null set. i.e. U' = ∅.

Similarly, **∅' consists of all members of U, which are not in ∅. But there is no member in ∅. Therefore ∅' = U.**

1.7 De Morgan's Laws

For A ⊂ U and B ⊂ U we have the following results which are known as De Morgan's laws

(i) (A ∩ B)' = A' ∪ B' (ii) (A ∪ B)' = A' ∩ B'.

By Venn-diagram, we can verify the above laws.

Verification of (A ∪ B)' = A' ∩ B'

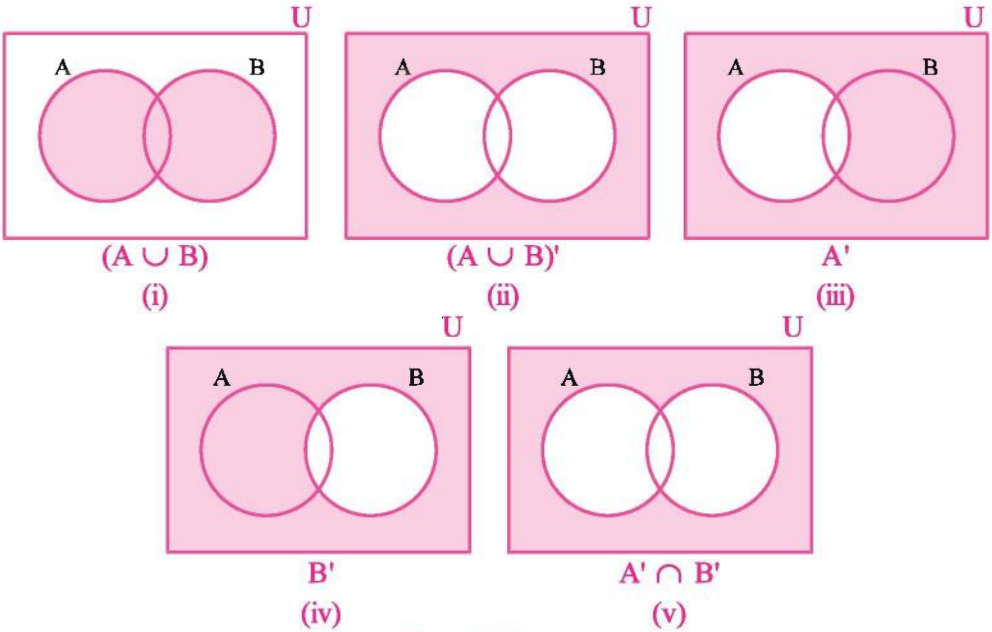


Figure 1.11

In Venn-diagram 1.11, coloured region describes the set mentioned below the Venn-diagram.

From the Venn-diagram 1.11, it is clear that $(A \cup B)' = A' \cap B'$

Similarly, we can verify the other law $(A \cap B)' = A' \cup B'$ by Venn-diagram (do it by yourself).

EXERCISE 1.3

1. If $U = \{x \mid x \in N, x < 10\}$, $A = \{2x \mid x \in N, x < 5\}$ and $B = \{1, 3, 4, 5\}$, then find $(A \cup B)'$ and $(A \cap B)'$.
2. If $U = \{a, b, c, d, e, f, g, h\}$, $P = \{b, c, d, e, f\}$, $Q = \{a, c, d, e, g\}$, then verify De Morgan's laws.
3. Let U = N. If $B \subset A$, then find $A' \cap B'$ and A'. What do you conclude ?
4. If $A = \{x \mid x \in Z, x^3 = x\}$, $B = \{x \mid x \in Z, x^2 = x\}$, $C = \{x \mid x \in N, x^2 = x\}$, then considering $U = \{-1, 0, 1, 2\}$, verify the following results

   (1) $(B \cup C)' = B' \cap C'$ (2) $(C')' = C$ (3) $(B \cap C)' = B' \cup C'$
5. If $U = \{x \mid x \in N, (x + 1)^2 < 40\}$, $A = \{x \mid x \in N, x < 4\}$ and $B = \{2x \mid x \in N, x < 3\}$, then find A', B' and verify De Morgan's laws.

EXERCISE 1

1. Examine from the following which are the subsets of the other set ?

   $A = \{-1, 5\}$, $B = \{-2, -1, 0, 1, 2\}$, $C = \{2, 4, 6, 8, 10, 12\}$

   $D = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$
2. If $A = \{x \mid x \in Z, 4 \le (x + 1)^2 < 25\}$, $B = \{-2, -1, 0, 1, 2\}$, then find $A \cup B$ and $A \cap B$.
3. If $A \subset B$, then can A and B be disjoint, when $A \neq \emptyset$, $B \neq \emptyset$? Why ?
4. If $A = \{x \mid x \in N, x \text{ is a factor } 18\}$, $B = \{x \mid x \in N, x \text{ is a multiple of } 3, x < 20\}$ and $U = \{x \mid x \in N, x \le 20\}$, then verify De Morgan's Laws.
5. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{x \mid x \in N, x < 10\}$, $C = \{3x \mid x \in N, x < 20\}$ and $U = \{1, 2, 3, \ldots\ldots 20\}$, then verify the distributive laws.
6. If $U = \{1, 3, 4\}$, $A = \{1\}$, then find the complement of the set A.
7. If U = Z, then find the complements of

   (1) $A = \{2x \mid x \in Z\}$ and (2) $B = \{2x - 1 \mid x \in Z\}$
8. Write all the subsets of the set $A = \{-1, 0, 1, 2\}$
9. If $A = \{6, 8, 10, 12, 14\}$, $B = \{8, 9, 10, 11, 12, 13\}$, $C = \{7, 8, 9, 10, 12, 14\}$, then prove that $(A \cap C) \cup B = (A \cup B) \cap (B \cup C)$.

10. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If U = {x | $x \in$ N, $x < 5$}, A = {x | $x \in$ N, $x \leq 2$} then A′ = ... ☐

(a) {1, 2} (b) {1, 2, 3, 4, 5} (c) {3, 4} (d) {3, 4, 5}

(2) $\emptyset$ ...... {$\emptyset$} ☐

(a) $\subset$ (b) $\notin$ (c) = (d) $\not\subset$

(3) If A = {1, 2, 3}, B = {3, 4, 5}, then A $\cup$ B = ... ☐

(a) {1, 2, 3, 4, 5} (b) {3} (c) {1, 2} (d) $\emptyset$

(4) If A = {x | $x \in$ N, $x \leq 7$} and B = {2, 4, 6}, then B ...... A. ☐

(a) = (b) $\subset$ (c) $\not\subset$ (d) ~

(5) If A = {1, 2, 3}, B = {2, 3, 4}, C = {3, 4, 5}, then (A $\cap$ B) $\cap$ C′ = ... where U = {1, 2, 3, 4, 5} ☐

(a) {1} (b) {2} (c) {1,2} (d) {2,3}

(6) If A = {x | $x \in$ N, $x \leq 3$}, B = {1, 2, 3}, U = N, then A and B are ...... sets. ☐

(a) equal (b) singleton (c) null (d) complements of each other

(7) If A = {1, 2, 3, 4} .... is a correct statement. ☐

(a) 3 $\notin$ A (b) {1} $\in$ A (c) {2} $\in$ A (d) {3, 4} $\subset$ A

(8) If A = {1, 2, 3, 4}, then number of subsets of A are = ... ☐

(a) 2 (b) 4 (c) 8 (d) 16

(9) ...... is a singleton. ☐

(a) A = {$x \in$ R : $x^2 - x = 0$}

(b) B = {x | $x \in$ N, $2x = 3$}

(c) C = {x | $x \in$ R : $x^2 = -4$}

(d) D = {x | $x \in$ Z, x is neither positive nor negative}

(10) If A = {0, 1, 2, 4}, B = {1, 3, 5, 7, 9}, C = {0, 1, 4, 3, 9}, then (A $\cap$ B) $\cup$ C = ... ☐

(a) A (b) B (c) C (d) A $\cup$ B

(11) If A $\cup$ B = $\emptyset$, then ...... ☐

(a) A $\neq \emptyset$ and B $\neq \emptyset$ (b) A = $\emptyset$ and B $\neq \emptyset$

(c) A $\neq \emptyset$ and B = $\emptyset$ (d) A = $\emptyset$ and B = $\emptyset$

(12) If A = {x | $x \in$ N, $x \leq 4$}, B = {−1, 0, 1, 2, 3}, C = {0, 1, 2}, then (A $\cup$ B) $\cap$ (A $\cup$ C) = ... ☐

(a) {1, 2, 3, 4} (b) {0, 1, 2}

(c) {0, 1, 2, 3, 4} (d) {−1, 0, 1, 2, 3, 4}

(13) If A = {1, 2, 3, 4}, B = {2, 4, 5, 6}, U = {1, 2, 3, 4, 5, 6, 7}, then A' ∩ B' = ...

(a) ∅ (b) {1, 2, 3, 4, 5, 6} (c) {7} (d) {3, 4, 5, 6}

(14) ∅ ∩ U' = ...

(a) ∅ (b) U (c) {U} (d) { ∅ }

(15) (A ∩ B')' = ...

(a) A ∪ B' (b) A' ∪ B (c) A ∪ B (d) A ∩ B

*

Summary

In this chapter,

1. Revision of set theory learnt in standard 8 is given.
2. Union operation, Intersection operation, Complement operation are defined and their properties are studied.
3. Distributive laws and De-Morgan's laws are explained.

•

Set theory is the branch of mathematics that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics.

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

The language of set theory could be used in the definitions of nearly all mathematical objects, such as functions, and concepts of set theory are integrated throughout the mathematics curriculum. Elementary facts about sets and set membership can be introduced in primary school, along with Venn and Euler diagrams, to study collections of commonplace physical objects. Elementary operations such as set union and intersection can be studied in this context.

CHAPTER 2

NUMBER SYSTEM

Give me a place to stand and I will move the earth. - **Archimedes**

Numbers are the free creation of the human mind. - **R. Dedekind**

2.1 Introduction

In our earlier classes we have learnt about the number line. Let us review it.

We take a line and select one point on it and call it O and associate 0 with it. Now if we move towards the right side of O we get numbers like 1, 2, 3,... . Picking all such numbers and collecting in one bag, can you think about the total number of numbers we collected ? Of course, infinitely many. This collection is denoted by N. Thus, N = {1, 2, 3,...}.

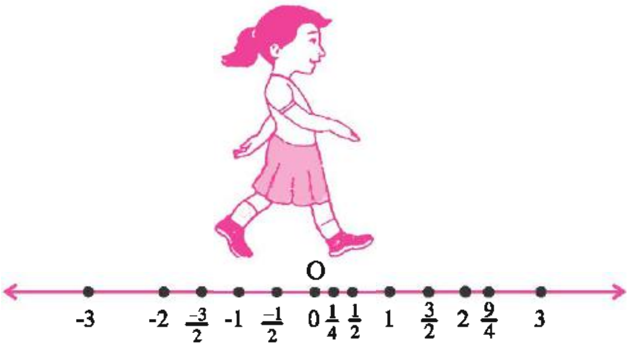


Figure 2.1

Now again, we return to O and if we pick up 0 and put it also in the same bag, this new collection is denoted by W. Thus W is the set of whole numbers.

i.e., W = {0, 1, 2, 3,...}.

We observed that, $N \subset W$.

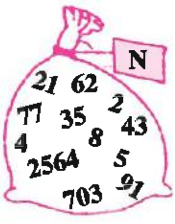


Figure 2.2

Figure 2.3

Now let us enrich our collection by moving towards the left side of O on the line. What kind of numbers we get ? of course, negative integers $-1, -2, -3, \ldots$. We add them to the same bag. This collection is denoted by Z. Thus Z is the set of integers.

$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. $W \subset Z$

Figure 2.4

$\therefore \quad N \subset W \subset Z$

Still there are numbers left on the number line like $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, -\frac{1}{2}$, etc. If we put all such numbers in the same bag, then we have a collection of numbers called **rational numbers** and it is denoted by Q. Rational numbers can be expressed in the form of $\frac{p}{q}$, where $p \in Z$, $q \in N$ and p and q have no common factors other than 1. $Q = \left\{\frac{p}{q} \mid p \in Z, q \in N\right\}$. For any $a \in Z$ we can write 'a' as $a = \frac{a}{1}$ so $a \in Q$. For example, -31 can be written as $\frac{-31}{1}$. Thus all integers are also rational, so $Z \subset Q$. Thus, $N \subset W \subset Z \subset Q$.

Here, 'Rational' comes from the word 'ratio' and Q comes from the word 'quotient'.

We also know that rational numbers do not have a unique representation in the form $\frac{p}{q}$. For example, $\frac{1}{3} = \frac{2}{6} = \frac{5}{15} = \frac{10}{30} = \frac{35}{105}$, and so on. These are equivalent

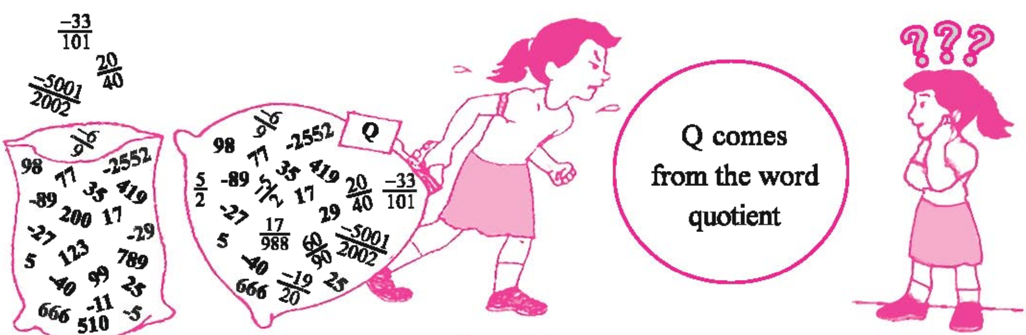


Figure 2.5

rational numbers. If we want to represent a rational number $\frac{p}{q}$ on the number line, then we assume that p and q have no common factors other than 1, i.e. p and q are co-prime. In short $\frac{p}{q}$ should be in its simplest form.

Example 1 : Are the following statements true or false ? Give the reason.

(1) Every whole number is an integer.

(2) Every rational number is a whole number.

(3) Every integer is a rational number.

Solution :

(1) True, because $W \subset Z$

(2) False, because $\frac{3}{4} \in Q$ but $\frac{3}{4} \notin W$.

(3) True, because $Z \subset Q$

Now let us see how to find a rational number or rational numbers between two given rational numbers.

Example 2 : Find four rational numbers between 2 and 3.

Solution : To find a rational number between given numbers 2 and 3, we find their average $\frac{2+3}{2} = \frac{5}{2}$ which is a rational number between 2 and 3. Similarly other rational numbers can be obtained by successive averaging as follows.

$$\frac{\frac{5}{2}+2}{2} = \frac{9}{4}, \quad \frac{\frac{9}{4}+2}{2} = \frac{17}{8}, \quad \frac{\frac{5}{2}+3}{2} = \frac{11}{4}$$

$\frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \frac{11}{4}$ are four rational numbers between 2 and 3. A general method is described below.

Method 1 : Let $a, b \in Q$. We want to find a rational number between a and b. As such there are infinitely many numbers between a and b, but we think of getting one number between a and b in a convenient way. As $a < \frac{a+b}{2} < b$, $\frac{a+b}{2}$ is a rational number between a and b. The same method can be applied for getting more numbers between a and b.

Method 2 : To get n rational numbers between a and b ($a < b$), let $d = \frac{b-a}{n+1}$, then $a + d$, $a + 2d$,..., $a + (n - 1)d$, $a + nd$ are n rational numbers between a and b.

Between two rational numbers there are infinitely many rational numbers. This is an important property of Q. This property is called "The rational numbers exhibit gap".

EXERCISE 2.1

1. Are 3, −5 and −3.5 rational numbers ? If yes, then express them in the form of $\frac{p}{q}$, where $p \in$ Z, $q \in$ N.
2. Find six rational numbers between −2 and 5.
3. Find three rational numbers between 4 and 6.
4. Find two rational numbers between $\frac{2}{7}$ and $\frac{4}{9}$.
5. Find four rational numbers between $\frac{2}{5}$ and $\frac{3}{7}$.

*

2.2 Irrational Numbers

Now, our bag is very heavy, so it seems that no numbers are left out. But no ! there are still many numbers left on the number line. Which are they ? They are called **Irrational Numbers**. So let us pick them up and put in our bag. This collection is denoted by R, the set of real numbers. R includes rational and irrational numbers. Thus $Q \subset R$, i.e. $N \subset W \subset Z \subset Q \subset R$.

Thus **real numbers which are not rational are called irrational numbers.** The set of irrational numbers is denoted by I. $I \cap Q = \emptyset$ and $I \cup Q = R$.

Let us study something more about irrational numbers.

Now we will see how to represent some irrational numbers on number line.

Example 3 : Represent $\sqrt{2}$ on the number line.

Solution : Consider a square OABC (Figure 2.6), with each side having length 1 unit. Here by the Pythagoras theorem OB = $\sqrt{1^2+1^2}$ = $\sqrt{2}$; Place O on the number line in such a way that vertex O coincides with zero and A with 1. (figure 2.7)

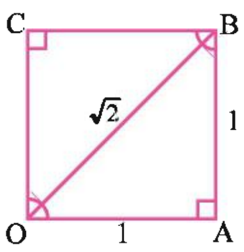


Figure 2.6

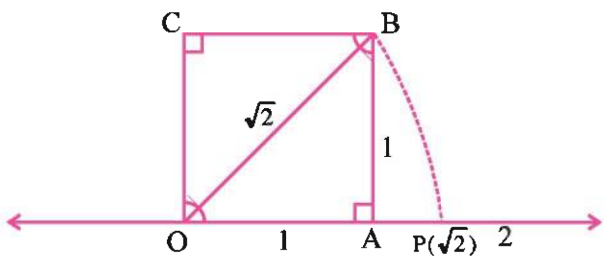
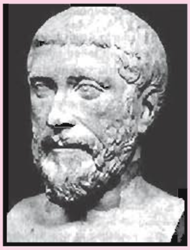


Figure 2.7

We know that OB = $\sqrt{2}$. Draw an arc with center O and radius OB, intersecting the number line at the point P. Then point P corresponds to $\sqrt{2}$ on the number line.

Pythagoras studied properties of numbers which would be familiar to mathematician today. Such as even and odd numbers, triangular numbers and perfect numbers. The theorem known as Pythagoras's theorem was known to the Babylonians one thousand years earlier. He may have been the first to prove it.



Pythagoras
569 BC - 475 BC

Some theorems attributed to Pythagorians are

(1) The sum of the angles of a triangle is equal to two right angles. (2) Constructing the figures of a given area and geometrical algebra e.g. they solve the equation such as $a \cdot (a - x) = x^2$ by geometrical means. (3) 5-regular solids. Pythagoras himself knew how to construct first three. (4) In Astronomy Pythagoras taught that the earth was a sphere at the centre of the universe. (5) He also recognised that the orbit of the moon was inclined to the equator of the earth.

Example 4 : Represent $\sqrt{3}$ on the number line.

There are two methods to locate $\sqrt{3}$ on the number line.

Solution 1 : Let us return to figure 2.7

Construct $\overline{PD}$ perpendicular to $\overline{OP}$ having unit length (figure. 2.8(i))

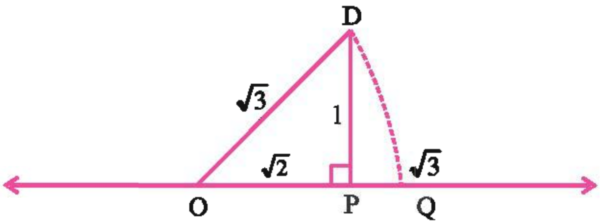


Figure 2.8 (i)

By Pythagoras theorem, OD = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$. Draw an arc with centre O and radius OD, intersecting the number line at the point Q. Then point Q corresponds to $\sqrt{3}$ on the number line.

Solution 2 : Let us return to figure 2.7

Construct $\overline{BD}$ having unit length perpendicular to $\overline{OD}$. (figure 2.8(ii))

By Pythagoras theorem, OB = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$, Draw an arc with centre O and radius OB, intersecting the number line at the point Q. Then point Q corresponds to $\sqrt{3}$ on the number line.

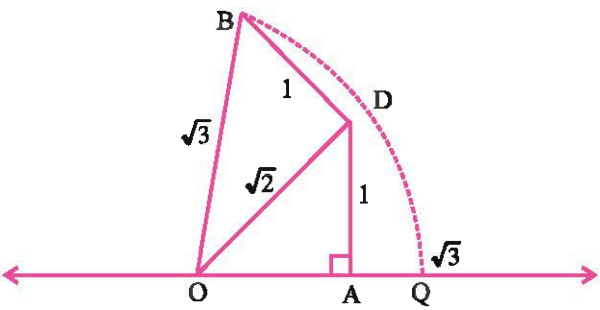
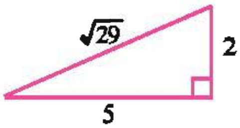


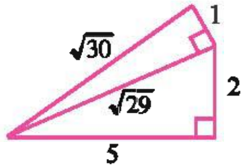
Figure 2.8 (ii)

In the same manner, we can represent $\sqrt{n+1}$ for any positive integer n after $\sqrt{n}$ has been represented or put $n + 1$ as sum of the squares of two or three or four numbers and using Pythagoras theorem we can get $\sqrt{n+1}$.

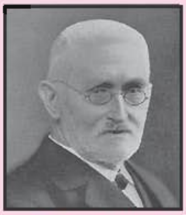
For example : $\sqrt{29} = \sqrt{5^2+2^2}$

$\sqrt{30} = \sqrt{5^2+2^2+1^2}$





Now we can say that, every point on the number line represents a unique real number and every real number represents a unique point on the number line. i.e., there is one-one correspondence between set of real numbers and the set of points on the number line.



R. Dedekind
(1831-1916)

Julius Wilhelm Richard Dedekind

Born : 6th October, 1831 in Braunschweig, duchy of Braunschweig (now Germany).

Died : 12th February, 1916 in Braunschweig, duchy of Braunschweig (now Germany).

- Dedekind did his doctorat work under Gauss' supervision.
- He attended the courses by Dirichlet on the theory of numbers.
- His remarkable piece of work was redefinition of irrational numbers in terms of Dedekind cuts.

EXERCISE 2.2

1. State whether the following statements are true or false. Give reasons for your answer :
   (1) Every rational number is a real number.
   (2) Every integer is an irrational number.
   (3) $\sqrt{4}$ is an irrational number.
   (4) There is a real number whose square is -3.
2. Represent $\sqrt{5}$ on the number line.
3. Represent $\sqrt{17}$ on the number line.

*

Classroom Activity : (Construction of the "Square root Spiral") :

We construct square root spiral in the following way.

Take point O and draw $\overline{OA}$ of unit length. Draw $\overline{AB}$ of unit length perpendicular to $\overline{OA}$ (figure 2.9). Now draw $\overline{BC}$ of unit length perpendicular to $\overline{OB}$. Then draw $\overline{CD}$ of unit length perpendicular to $\overline{OC}$. Continuing in this manner, we can get a spiral known as square root spiral. Here OB = $\sqrt{2}$, OC = $\sqrt{3}$, OD = $\sqrt{4}$,...

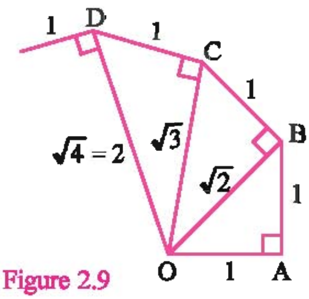


Figure 2.9

2.3 Real Number and Their Decimal Expression

In this section we will obtain the decimal expression of real numbers and using them we will distinguish between rational and irrational numbers.

Let us take some examples of rational numbers; $\frac{1}{3}, \frac{3}{8}, \frac{8}{7}$.

$$\begin{array}{r|l} & 0.333 \\ \hline 3 & 1.0 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 1 \end{array} \qquad \begin{array}{r|l} & 0.375 \\ \hline 8 & 3.0 \\ & \underline{24} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{40} \\ & 00 \end{array} \qquad \begin{array}{r|l} & 1.142857 \\ \hline 7 & 8 \\ & \underline{7} \\ & 10 \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 1 \end{array}$$

Remainders : 1, 1, 1
Divisor : 3

Remainders : 6, 4, 0
Divisor : 8

Remainders : 1, 3, 2, 6, 4, 5, 1
Divisor : 7

From the above examples, we observe the following :

(1) The remainder becomes 0 or the remainders start recurring themselves.

(2) Remainders are less than the divisor and they form a recurring string.

(In case of $\frac{1}{3}$, remainder is recurring and it is less than the divisor 3. In case of $\frac{8}{7}$ there are six digits 1, 3, 2, 6, 4, 5 recurring in order and they are less than the divisor 7.)

(3) If the remainder is recurring, then some digit or a group of digits of quotient are also recurring (In case of $\frac{1}{3}$, 3 is recurring and in case of $\frac{8}{7}$ group 142857 is recurring in the quotient).

These observations are true for all the rational number in the form $\frac{p}{q}$. If we divide p by q, then remainder becomes zero or never becomes zero and the digits of remainder are recurring after some stage.

Now we will observe them individually.

(1) The remainder becomes zero.

In the example $\frac{3}{8}$, we have observed that remainder becomes zero after some stage, its decimal expression is $\frac{3}{8} = 0.375$. Few other examples are $\frac{3}{4} = 0.75$, $\frac{73}{25} = 2.92$.

Here the decimal expression terminates or ends after a certain number of steps. These types of decimal expressions are called terminating.

(2) The remainder never becomes zero.

In the example $\frac{1}{3}$ and $\frac{8}{7}$, remainder is recurring after a certain stage and the decimal expression will go on and on. The digits of quotient of such expression are recurring after certain stage. For example in case of $\frac{1}{3} = 0.333...$, digit 3 repeats and in case of $\frac{8}{7} = 1.142857142857142857...$, digits 1,4,2,8,5,7 repeat in the same order. These types of expressions are non terminating recurring in the same order. In the decimal expression of $\frac{1}{3}$, digit 3 is repeated in its quotient, So we write $\frac{1}{3}$ as $0.\overline{3}$ or $0.\dot{3}$. Similarly for $\frac{8}{7}$, the digits 1,4,2,8,5,7 are repeating in the same order. So we write it as $\frac{8}{7} = 1.\overline{142857}$ or $1.\dot{1}4285\dot{7}$. Similarly if we have an expression 2.7323232..., then we write it as $2.7\overline{32}$. All these expressions are non-terminating and recurring.

Thus **the decimal expression of rational numbers are either terminating or non-terminating recurring. Conversely, if the decimal expression of a number is either terminating or non-terminating recurring, then the number is a rational number.**

In fact if in $\frac{p}{q}$, $q = 2^m \cdot 5^n$, $m, n \in N \cup \{0\}$, then $\frac{p}{q}$ has terminating expression and not otherwise. (Why ? Can you explain ?)

Let us consider a terminating decimal number.

Example 5 : Show that 2.1321 is a rational number.

Solution 1 : We can write $2.1321 = \frac{21321}{10000}$ and it is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$ and hence 2.1321 is a rational number.

Now let us consider a non terminating recurring decimal expression.

Example 6 : Show that $0.666... = 0.\overline{6}$ can be expressed in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

Solution 1 : Let $x = 0.\overline{6}$

$\therefore$ $x = 0.666...$

$\therefore$ $10x = 6.666...$

$\therefore$ $10x = 6 + 0.666...$

$\therefore$ $10x = 6 + x$

$\therefore$ $9x = 6$

$\therefore$ $x = \frac{6}{9}$

$\therefore$ $x = \frac{2}{3}$ is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

$\therefore$ $0.\overline{6}$ represents a rational number.

Note : In fact here 0.666....= 0.6 + 0.06 + 0.006... and this is called an 'infinite geometric series'. If a number is 0.*ppp*... then it is equal to $\frac{p}{9}$.

Similarly, $0.pqpq... = \frac{pq}{99}$; $0.pqrpqr... = \frac{pqr}{999}$ etc... Can you explain $0.\overline{9} = 1$?

Example 7 : Find the $\frac{p}{q}$ form of $2.\overline{237}$.

Solution : Let $x = 2.\overline{237}$

$\therefore \quad x = 2.237237237...$

$\therefore \quad 1000x = 2237.237237...$

$\therefore \quad 1000x = 2235 + 2.237237...$

$\therefore \quad 1000x = 2235 + x$

$\therefore \quad 999x = 2235$

$\therefore \quad x = \frac{2235}{999}$ is in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$.

[**Note :** Straight away $2.\overline{237} = 2 + 0.\overline{237} = 2 + \frac{237}{999} = \frac{2235}{999}$]

Example 8 : Prove that $3.1\overline{23}$ is a rational number and obtain its $\frac{p}{q}$ form.

Solution : Given number $3.1\overline{23}$ is non-terminating recurring, so it is a rational number.

Let $x = 3.1\overline{23}$

$\therefore \quad x = 3.1232323...$

$\therefore \quad 100x = 312.32323...$

$\therefore \quad 100x = 309.2 + 3.12323...$

$\therefore \quad 100x = 309.2 + x$

$\therefore \quad 99x = \frac{3092}{10}$

$\therefore \quad x = \frac{3092}{990} = \frac{1546}{495}$, is in the form $\frac{p}{q}$ where $p \in Z$, $q \in N$.

$\therefore \quad 3.1\overline{23}$ is a rational number.

[**Note :** $x = 3.1 + 0.0232323... = 3.1 + (0.1)(0.232323...)$

$= 3.1 + \left(\frac{1}{10}\right)\left(\frac{23}{99}\right) = \frac{31}{10} + \frac{23}{990}$ etc.]

From above examples we can observe that in example 6, the number is $0.\overline{6}$, here one digit is repeated, so we have multiplied the number by 10. Similarly in example 7, the number is $2.\overline{237}$. Here three digits are repeated. So we have multiplied it by 10^3. In general, if number of digits repeated in the given number is *n*, then we multiply the number by 10^n, $n \in N$.

Express $0.\overline{9}$ in the form $\frac{p}{q}$. What is your answer ? Why is it so ?

Thus, a number is rational if and only if its decimal expression is terminating or non-terminating recurring. So obviously, a number is an irrational number if and only if its decimal expression is non-terminating and non-recurring.

A decimal expression like 0.303303330... is non-terminating and non-recurring and so it is an irrational number. We can write infinitely many such irrational numbers.

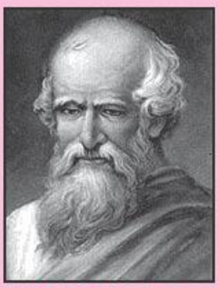
The decimal expression of $\sqrt{2}$, $\sqrt{3}$ can be obtained by the division method. Such square roots are non-terminating and non-recurring.

Let us look at the decimal expression of irrational numbers $\sqrt{2}$ and π.

$\sqrt{2} = 1.414213562373...$

$\pi = 3.1415926535897932384626433\ 832\ 7950...$

- His famous work is measurement of the circle.
- He got value of *pi* between two fractions $3\frac{10}{71}$ and $3\frac{1}{7} = \frac{22}{7}$.
- He got his information by inscribing and circumscribing a circle with a 96-sided regular polygon.
- He proved that volume of an inscribed sphere is $\frac{2}{3}$rd the volume of a circumscribed cylinder.
- He requested that this formula be inscribed on his tomb.
- He discovered density and specific gravity.
- He invented the machine called Archimedes Screw, which is a mechanical water pump.



Archimedes
(287 BC - 212 BC)

Note : In calculation of area, volume, etc, we take π as $\frac{22}{7}$ but π is irrational while $\frac{22}{7}$ is rational, so $\pi \neq \frac{22}{7}$. But $\frac{22}{7}$ or 3.14 are approximate values of π. (Which of 3.14 or $\frac{22}{7}$ is nearer to π ?)

Now we will learn how to obtain an irrational number between two rational numbers.

Example 9 : Find an irrational number between $\frac{2}{7}$ and $\frac{3}{7}$.

Solution : First of all we will find the decimal expression of $\frac{1}{7}$.

$$\begin{array}{r|l} & 0.142857 \\ \hline 7 & 1.0 \\ & \underline{7\ \ } \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 1 \end{array}$$

$\therefore$ $\frac{1}{7} = 0.\overline{142857}$. To find $\frac{2}{7}$ we multiply $0.\overline{142857}$ by 2.

So we will have $\frac{2}{7} = 0.\overline{285714}$. Similarly $\frac{3}{7} = 0.\overline{428571}$

Now to find irrational number between $\frac{2}{7}$ and $\frac{3}{7}$, we will write one number which is non - terminating and non - recurring between their expressions. 0.350350035000... is one such required number.

Example 10 : Find three different irrational numbers between $\frac{4}{7}$ and $\frac{8}{11}$.

Solution : First of all we will find the decimal expression of $\frac{1}{7}$ and $\frac{8}{11}$.

$\frac{1}{7} = 0.\overline{142857}$ (from example 9) and

$$\begin{array}{r|l} & 0.09 \\ \hline 11 & 1.00 \\ & \underline{99} \\ & \ \ 1 \end{array}$$

$\therefore$ $\frac{1}{11} = 0.\overline{09}$

$\therefore$ $\frac{4}{7} = 0.\overline{571428}$ and $\frac{8}{11} = 0.\overline{72}$

$\therefore$ The different irrational numbers between $\frac{4}{7}$ and $\frac{8}{11}$ are,

0.590590059000..., 0.606606660...

(Write one more irrational number by yourself.)

Note : we can obtain infinitely many such irrational numbers between any two given rational numbers.

EXERCISE 2.3

1. Classify the following numbers as rational or irrational.

(1) $\sqrt{25}$ (2) $\sqrt{331}$ (3) 0.41757575... (4) 7.808808880... (5) $\frac{\pi}{7}$ (6) $0.\overline{98}$

2. Convert following rational numbers in decimal form and state the kind of its decimal expression.

(1) $\frac{43}{1000}$ (2) $\frac{33}{5}$ (3) $\frac{5}{6}$ (4) $1\frac{2}{7}$ (5) $\frac{157}{300}$ (6) $\frac{14}{11}$

3. Using $\frac{16}{99} = 0.\overline{16}$, obtain decimal form of $\frac{32}{99}$ and $\frac{80}{99}$.

4. Using $\frac{1}{7} = 0.\overline{142857}$, obtain decimal form of $\frac{3}{7}$ and $\frac{5}{7}$.

5. Express the following in the form of $\frac{p}{q}$, where $p \in Z$, $q \in N$.

(1) $0.\overline{23}$ (2) $0.\overline{1437}$ (3) $3.\overline{47}$

6. Find three different irrational numbers between the rational numbers $\frac{3}{5}$ and $\frac{5}{8}$.

*

2.4 Representing Real Numbers on the Number Line

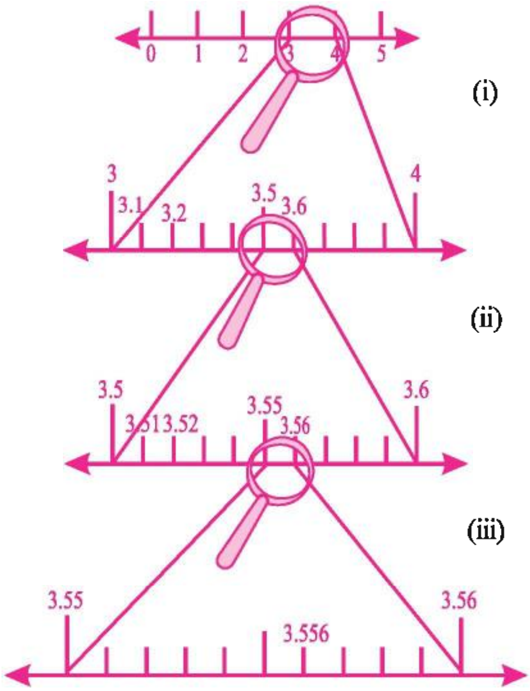


Figure 2.10

In section 2.3 we have seen that we can express any real number in a decimal expression form. After writing the real number into decimal expression form, it is easy to represent it on the number line.

Suppose we want to represent 3.556 on the number line.

We know that 3.556 lies between 3 and 4, so we look for portion between 3 and 4 (Fig. 2.10(i)) on the number line. We divide this portion in 10 equal parts and we mark them. The first mark on the right of 3 represents 3.1, the second 3.2,... as shown in figure 2.10(ii).

Now the number 3.556 lies between 3.5 and 3.6 (figure 2.10(iii)) we magnify this portion by magnifying glass and again we divide the portion between 3.5 and 3.6 in 10 equal parts and mark them. Then the first mark on the right of 3.5 represents 3.51, the second 3.52,... as shown in figure 2.10(iii).

Now we know that number 3.556 lies between 3.55 and 3.56, we visualize this portion on the number line, we magnify it and again we divide it in 10 equal parts. The first mark nearer to 3.55 is 3.551, the second 3.552,... the sixth mark is 3.556 (figure 2.10(iv)). In this way we can locate 3.556 on the number line. This method of representing a real number on the number line by magnifying the portion with magnifying glass is known as the **process of successive magnification.**

Thus the real number with a terminating decimal expression can be represented on the number line by successive magnification.

Now we will take one example to visualize the portion of a real number with a non-terminating recurring decimal and will represent it on the number line by the process of successive magnification method.

Example 11 : Represent $4.2\overline{3}$ on the number line up to 4 decimal places, i.e. up to 4.2333.

Solution : We proceed by successive magnification. We know that $4.2\overline{3}$ lies between 4 and 5. Then we locate $4.2\overline{3}$ between 4.2 and 4.3 (figure 2.11(i)) on the number line. We divide this portion in 10 equal parts and we mark them. The first mark on the right of 4.2 represents 4.21, the second 4.22,... as shown in as shown in figure 2.11(ii).

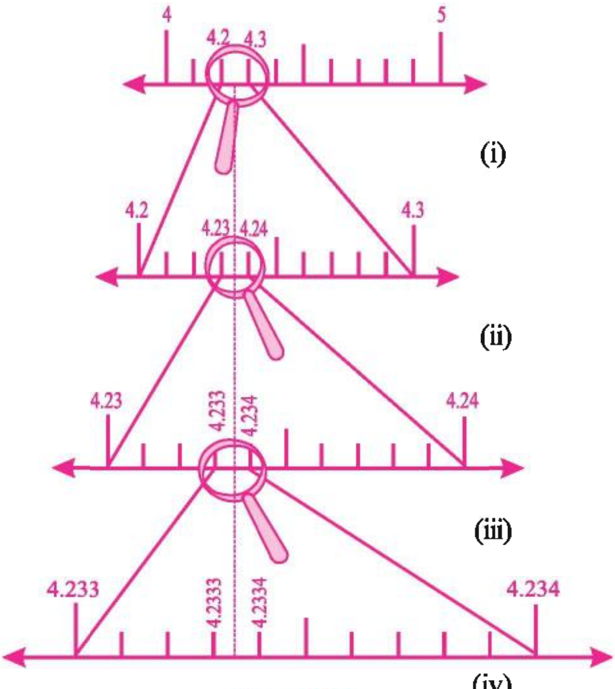


Figure 2.11

Now the number $4.2\overline{3}$ lies between 4.23 and 4.24 (figure 2.11(ii)). We magnify this portion by magnifying glass and again we divide the portion between 4.23 and 4.24 in 10 equal parts and will mark them. Then the first mark on the right of 4.23 represents 4.231, the second 4.232,... as shown in figure 2.11 (iii).

Now we know that number $4.2\overline{3}$ lies between 4.233 and 4.234, we visualize this portion on the number line, we magnify it and again we divide it in 10 equal parts. The first mark nearer to 4.233 is 4.2331, the second 4.2332, the third mark is 4.2333 (figure 2.11(iv)). In this way we can locate 4.2333 on the number line. We notice that $4.2\overline{3}$ is closer to 4.2333 than 4.2334.

We can proceed in this manner endlessly. As we proceed step by step, the length between two consecutive marks decreases and we can be closer and closer to the given number. We visualize it by the magnifying glass. Thus we can locate the number more accurately whose decimal expression is non-terminating recurring.

To visualize a real number on number line whose decimal expression is non-terminating non-recurring, we adopt the same procedure of successive magnification.

Thus, **corresponding to every real number we get a unique point on the number line and conversely corresponding to every point of the number line we get a unique real number. This line is called the real number line.**

Activity

1. Represent 4.572 on the number line, using successive magnification.
2. Represent $1.\overline{3}$ on the number line, using successive magnification, up to 3 decimal places.

*

2.5 Operation on Real Number

We have learnt in our earlier classes that for the set of rational numbers Q, the commutative laws, associative laws for addition and multiplication, distributive law of multiplication over addition are valid.

Now we note one important property of Q. The sum of rational numbers is a rational number. This property is called the closure property for addition for Q. It can also be said that Q is closed under addition. Similarly Q is also closed under subtraction, multiplication and division (when divisor is non-zero.)

Irrational numbers also satisfy commutative and associative laws for addition and multiplication and distributive law for multiplication over addition are valid. But the sum, difference, product and quotient of irrational numbers may not always be irrational.

For example : Results of $\sqrt{5}+(-\sqrt{5})$, $\sqrt{3}-\sqrt{3}$, $(2\sqrt{2})\times(3\sqrt{2})$, $\frac{5\sqrt{7}}{2\sqrt{7}}$ are rationals.

Let us think about addition of an irrational number and a rational number, e.g. $\sqrt{2}+2$ is irrational, and $\sqrt{2}\times 3=3\sqrt{2}$ is also an irrational. Because, decimal expression of $\sqrt{2}$ is non-terminating non-recurring. So $2+\sqrt{2}$ and $3\sqrt{2}$ have also non-terminating non-recurring decimal expression.

*n*th Root of Positive Real Numbers

We know that $2^3 = 8$ and the cube-root of 8 is 2. We also write $\sqrt[3]{8} = 2$. Similarly, $3^4 = 81$ and 3 is the fourth root of 81. This is written as $\sqrt[4]{81} = 3$.

Now, $81 = (-3)^4$ also. So while considering the fourth root of 81 the question arises whether to take 3 or -3 ? We also know that the square of any real number is non-negative. Thus -3 cannot be square of any real number. Thus there is no square-root of (-3). Thus the dilemma arises while defining $\sqrt[n]{a}$ where n is even. For positive a, there are two real nth roots of a. For negative a, there is no real nth root. In order to resolve this dilemma, we define the positive nth root of a positive real number a.

If a is a positive real number and $n \in$ N, then there is one and only one positive real number x, so that $x^n = a$ (we accept this). **This number x is called the positive nth root of a and it is expressed as $x = \sqrt[n]{a}$. Also $0^n = 0$ and we take $\sqrt[n]{0} = 0$.**

If $a > 0$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$. We note that $\sqrt[n]{a}$ denotes positive nth root of positive number a. Thus $\sqrt{36} = 6$ and not -6. Although $6^2 = 36$ and $(-6)^2 = 36$ and we say 36 has two square roots 6 and -6, we write $\sqrt{36} = 6$ when using symbols.

$(-4)^3 = -64$ and hence cube root of -64 is -4, but we can not write $\sqrt[3]{-64} = -4$ as the symbol $\sqrt[3]{a}$ is defined for $a > 0$ only.

Thus, $\sqrt{81} = 9$, $\sqrt[3]{27} = 3$, $\sqrt[4]{16} = 2$, $\sqrt[3]{-27}$ is not defined. But cube root of -27 is -3. Square roots of 81 are 9 and -9.

Explanation for $\sqrt{x^2}$: It is wrong to write $\sqrt{x^2} = \pm x$. According to definition, $\sqrt{x^2}$ is that unique non-negative number whose square is x^2.

If $x \geq 0$, x is the non-negative number whose square is x^2.

$\therefore$ If $x \geq 0$, then $\sqrt{x^2} = x = |x|$

If $x < 0$ then $-x > 0$. Also $(-x)^2 = [(-1) \cdot x]^2 = (-1)^2 \cdot x^2 = x^2$.

Thus, $-x$ is the non-negative number whose square is x^2.

$\therefore$ $\sqrt{x^2} = -x = |x|$ as $x < 0$.

$\therefore$ In any case for every $x \in$ R, $\sqrt{x^2} = |x|$.

Thus $\sqrt{7^2} = |7| = 7$ and $\sqrt{(-7)^2} = |-7| = 7$. We take $0^n = 0$. ($\sqrt[n]{0} = 0$)

Let x, $y \in$ R$^+$, then (R$^+$ is the set of all positive real numbers.)

(1) $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ (2) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

(3) $(\sqrt{a} + \sqrt{b})(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay} + \sqrt{bx} + \sqrt{by}$

(4) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$

(5) $(\sqrt{x} \pm \sqrt{y})^2 = x \pm 2\sqrt{xy} + y$

Let us solve examples using above identities for square roots.

Example 12 : Simplify.

(1) $(\sqrt{7} + \sqrt{3})(\sqrt{5} - \sqrt{3})$ (2) $(\sqrt{13} + \sqrt{5})(\sqrt{13} - \sqrt{5})$

(3) $(2 + \sqrt{3})(2 - \sqrt{3})$ (4) $(\sqrt{6} + \sqrt{3})^2$ (5) $(6 - \sqrt{7})^2$

Solution : (1) $(\sqrt{7} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = \sqrt{7 \times 5} - \sqrt{7 \times 3} + \sqrt{3 \times 5} - \sqrt{3 \times 3}$

$= \sqrt{35} - \sqrt{21} + \sqrt{15} - 3$

(2) $(\sqrt{13} + \sqrt{5})(\sqrt{13} - \sqrt{5}) = (\sqrt{13})^2 - (\sqrt{5})^2$

$= 13 - 5$

$= 8$

(3) $(2 + \sqrt{3})(2 - \sqrt{3}) = (2)^2 - (\sqrt{3})^2$

$= 4 - 3$

$= 1$

(4) $(\sqrt{6} + \sqrt{3})^2 = (\sqrt{6})^2 + 2\sqrt{6 \times 3} + (\sqrt{3})^2$

$= 6 + 2\sqrt{18} + 3$

$= 6 + 2\sqrt{9 \times 2} + 3$

$= 9 + 2(3)\sqrt{2} = 9 + 6\sqrt{2}$

(5) $(6 - \sqrt{7})^2 = (6)^2 - 2(6)(\sqrt{7}) + (\sqrt{7})^2$

$= 36 - 12\sqrt{7} + 7$

$= 43 - 12\sqrt{7}$

Example 13 : Add.

(1) $3\sqrt{5} + 2\sqrt{2}$ to $\sqrt{5} - \sqrt{2}$ (2) $3 - 2\sqrt{7}$ to $2 + 2\sqrt{7}$

Solution : (1) $(3\sqrt{5} + 2\sqrt{2}) + (\sqrt{5} - \sqrt{2})$

$= (3\sqrt{5} + \sqrt{5}) + (2\sqrt{2} - \sqrt{2})$

$= (3 + 1)\sqrt{5} + (2 - 1)\sqrt{2}$

$= 4\sqrt{5} + \sqrt{2}$

(2) $(2 + 2\sqrt{7}) + (3 - 2\sqrt{7}) = (2 + 3) + (2\sqrt{7} - 2\sqrt{7})$

$= 5 + 0$

$= 5$

Example 14 : Substract.

(1) $3\sqrt{5} + \sqrt{3}$ from $5\sqrt{5} - 2\sqrt{3}$ (2) $4 - 3\sqrt{3}$ from $2 - 3\sqrt{3}$

Solution : (1) $(5\sqrt{5} - 2\sqrt{3}) - (3\sqrt{5} + \sqrt{3})$

$= (5\sqrt{5} - 3\sqrt{5}) + (-2\sqrt{3} - \sqrt{3})$

$= 2\sqrt{5} - 3\sqrt{3}$

(2) $(2 - 3\sqrt{3}) - (4 - 3\sqrt{3}) = (2 - 4) + (-3\sqrt{3} + 3\sqrt{3})$

$= -2 + 0$

$= -2$

Example 15 : Multiply : (1) $(3 + \sqrt{7}) \times (4 - 2\sqrt{7})$ (2) $2\sqrt{7} \times 5\sqrt{7}$

(3) $(\sqrt{3} + \sqrt{2}) \times (\sqrt{3} - \sqrt{2})$

Solution : (1) $(3 + \sqrt{7}) \times (4 - 2\sqrt{7}) = 3(4 - 2\sqrt{7}) + \sqrt{7}(4 - 2\sqrt{7})$

$= (12 - 6\sqrt{7}) + (4\sqrt{7} - 14)$

$= (12 - 14) + (-6\sqrt{7} + 4\sqrt{7})$

$= -2 - 2\sqrt{7}$

$= -(2 + 2\sqrt{7})$

(2) $(2\sqrt{7}) \times (5\sqrt{7}) = 2 \times 5 \times \sqrt{7} \times \sqrt{7}$

$= 10 \times 7$

$= 70$

(3) $(\sqrt{3} + \sqrt{2}) \times (\sqrt{3} - \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2$

$= 3 - 2$

$= 1$

Example 16 : Divide : (1) $4\sqrt{21}$ by $2\sqrt{7}$ (2) $3\sqrt{11}$ by $6\sqrt{11}$

Solution : (1) $4\sqrt{21} \div 2\sqrt{7} = \frac{4\sqrt{3} \times \sqrt{7}}{2\sqrt{7}}$

$= 2\sqrt{3}$

(2) $3\sqrt{11} \div 6\sqrt{11} = \frac{3\sqrt{11}}{6\sqrt{11}}$

$= \frac{1}{2}$

Above examples lead to the following facts :

(1) The sum or difference or product of a rational number with an irrational number is irrational. (for product rational number must be non-zero.)

(2) Quotient of a non zero rational number by an irrational number is irrational.

(3) Sum, difference, product or division of two irrational numbers may be a rational number or an irrational number.

Now we will see how to represent square roots of real numbers on number line.

We have studied in section 2.2, how to represent $\sqrt{n}$ on the number line for $n \in$ R.

Now we show that, how to represent $\sqrt{a}$ geometrically, when a is a positive real number.

We do one geometric construction to represent $\sqrt{a}$, where $a \in R^+$. ($a > 1$)

Steps of Construction :

(1) Draw $\overrightarrow{AX}$.

(2) Mark B on $\overrightarrow{AX}$ in such a way that AB = a units.

(3) Mark C on $\overrightarrow{AX}$ in such a way that BC = 1 unit.

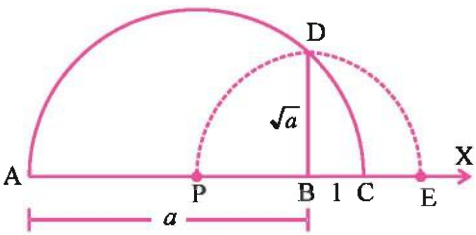


Figure 2.12

(4) Let P be the mid-point of $\overline{AC}$.

(5) Draw a semicircle with centre P and radius AP.

(6) Draw perpendicular at B to $\overline{AC}$ intersecting the semicircle in D.

(7) BD = $\sqrt{a}$

Justification : Radius of semicircle is $\frac{a+1}{2}$ units.

$\therefore$ PC = PD = PA = $\frac{a+1}{2}$ units **(Radii)**

$\therefore$ PB = PC − BC = $\frac{a+1}{2} - 1 = \frac{a-1}{2}$ units

In the right ΔPBD, by Pythagoras' theorem,

$$PD^2 = PB^2 + BD^2$$

$$\therefore \quad \left(\frac{a+1}{2}\right)^2 = \left(\frac{a-1}{2}\right)^2 + BD^2$$

$$\therefore \quad BD^2 = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 = a$$

$$\therefore \quad BD = \sqrt{a}$$

This construction shows us that for all $a \in R^+$, $\sqrt{a}$ exists.

Now to represent $\sqrt{a}$, $a \in R^+$, on the number line, take $\overleftrightarrow{AX}$ as number line and point B corresponding to zero. If we draw an arc with centre B and having radius BD, the arc will intersect $\overrightarrow{BX}$ at some point say E. Then E represents the number $\sqrt{a}$.

Example 17 : Find $\sqrt{2.5}$ geometrically and represent it on the number line.

Draw $\overrightarrow{AX}$. Mark B on $\overrightarrow{AX}$ such that AB = 2.5 units. Mark C on $\overrightarrow{BX}$ such that BC = 1 unit. Draw perpendicular bisector of $\overline{AC}$ which intersects $\overline{AC}$ at P. Draw a semicircle with centre P and radius AP. Draw perpendicular at B to $\overline{AC}$ intersecting the semicircle at D.

Thus BD = $\sqrt{2.5}$

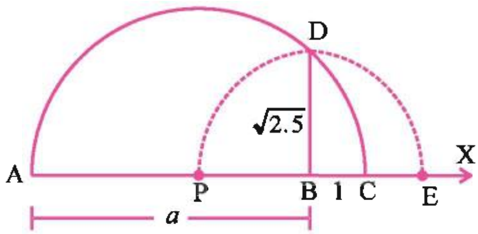


Figure 2.13

Now with B as centre and radius BD, draw an arc which intersects $\overrightarrow{BX}$ at E. Then E represents $\sqrt{2.5}$ on number line. (here B corresponds to zero.)

Now we will extend our idea of square roots to cube roots, fourth roots and so on. We extend it up to nth root of a positive real number, where $n \in N$.

In the end let us think of plotting $\frac{1}{\sqrt{3}}$ on the number line. Here the denominator is $\sqrt{3}$, an irrational number. So we can not divide unit length of number line into $\sqrt{3}$ number of equal parts. If the denominator is a rational number 3, then division process is possible. There is one process known as **rationalization** which can make this possible.

Rationalization : If an irrational number is multiplied by some suitable irrational number which can make the product a rational number, then such a process is known as rationalization.

The suitable multiplier irrational number is called a **rationalizing factor** and the given number is said to be **rationalized**. **Rationalization factor is not unique.**

$\sqrt{3} \cdot \sqrt{3} = 3$, $\sqrt{3}(2\sqrt{3}) = 6...$

So, a rationalizing factor of $\sqrt{3}$ could be $\sqrt{3}$, $2\sqrt{3}$, $3\sqrt{3}$, etc.

For example : $\frac{1}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}$

Here denominator $\sqrt{3}$ is rationalized and $\sqrt{3}$ (need not be the same every time) is a rationalizing factor.

Now to represent $\frac{1}{\sqrt{3}}$ on the number line, we represent $\frac{\sqrt{3}}{3}$ on the number line. This number is between 0 and $\sqrt{3}$.

We divide the portion between 0 and $\sqrt{3}$ in three equal parts, the first part near to 0 is the point on the number line which corresponds to number $\frac{1}{\sqrt{3}}$.

Example 18 : Rationalize the denominator of $\frac{4}{\sqrt{5}+\sqrt{2}}$.

Solution : Here we multiply and divide $\frac{4}{\sqrt{5}+\sqrt{2}}$ by $\sqrt{5} - \sqrt{2}$.

$$\frac{4}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = 4\frac{\sqrt{5}-\sqrt{2}}{5-2} = \frac{4}{3}(\sqrt{5} - \sqrt{2}).$$

Example 19 : Rationalize the denominator of $\frac{1}{2-\sqrt{7}}$.

Solution : We get $\frac{1}{2-\sqrt{7}} = \left(\frac{1}{2-\sqrt{7}}\right)\left(\frac{2+\sqrt{7}}{2+\sqrt{7}}\right)$

$$= \frac{2+\sqrt{7}}{4-7} = \frac{2+\sqrt{7}}{-3} = -\frac{1}{3}(2 + \sqrt{7})$$

Example 20 : Rationalize the denominator of $\frac{2}{3\sqrt{2}-2\sqrt{3}}$.

Solution : $\frac{2}{3\sqrt{2}-2\sqrt{3}} = \left(\frac{2}{3\sqrt{2}-2\sqrt{3}}\right)\left(\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}\right) = \frac{2(3\sqrt{2}+2\sqrt{3})}{18-12} = \frac{2(3\sqrt{2}+2\sqrt{3})}{6}$

$$= \frac{1}{3}(3\sqrt{2} + 2\sqrt{3})$$

EXERCISE 2.4

1. Classify the following numbers as rational or irrational :

(1) $3 + \sqrt{5}$ (2) $(5 - \sqrt{21}) + (3 + \sqrt{21})$ (3) $(4 - \sqrt{7}) + (4 + \sqrt{7})$

(4) $\frac{1}{\sqrt{2}-1}$ (5) $-\frac{\sqrt{48}}{\sqrt{27}}$ (6) $\frac{\pi+3}{2\pi}$

2. Simplify each of the following expressions :

(1) $(3 - \sqrt{7})(5 + \sqrt{3})$ (2) $(\sqrt{6} + \sqrt{3})^2$

(3) $(\sqrt{18} - \sqrt{5})(\sqrt{2} - \sqrt{15})$ (4) $(1 + \sqrt{8})(1 - 2\sqrt{2})$

(5) $(3 - \sqrt{5})^2$ (6) $(\sqrt{5})^3 - (\sqrt{2})^3$

3. Represent $\sqrt{4.2}$ on the number line.

4. Give a rationalizing factor of the following :

(1) $\frac{1}{\sqrt{8}}$ (2) $\frac{4}{\sqrt{5}}$ (3) $\frac{8+\sqrt{7}}{5}$ (4) $\frac{23}{-\sqrt{3}-\sqrt{2}}$ (5) $4-\sqrt{11}$

5. Rationalize the denominators of the following :

(1) $\frac{3}{\sqrt{15}}$ (2) $\frac{1}{4-\sqrt{7}}$ (3) $\frac{1}{-2-\sqrt{3}}$ (4) $\frac{1}{\sqrt{11}-1}$ (5) $\frac{1}{\sqrt{14}-\sqrt{7}}$

*

2.6 Laws of Exponents for Real Numbers

In earlier classes, we have learnt the following laws of exponents, when the base and exponents are natural numbers.

(1) $a^m \cdot a^n = a^{m+n}$

(2) (i) $\frac{a^m}{a^n} = a^{m-n}$; $m > n$ (ii) $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$; $m < n$ (iii) $\frac{a^m}{a^n} = 1$, $m = n$

(3) $(a^m)^n = a^{mn}$ (4) $(ab)^m = a^m \cdot b^m$ (5) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

We define $a^0 = 1$, $a^{-n} = \frac{1}{a^n}$, $a \in R - \{0\}$, $n \in Z$

Then we extend above laws for negative exponents also.

For example :

(1) $7^8 \times 7^{-11} = 7^{8-11} = 7^{-3}$

(2) (a) $\frac{5^4}{5^7} = 5^{4-7} = 5^{-3}$ (b) $\frac{5^{-4}}{5^7} = 5^{-4-7} = 5^{-11}$

(3) (a) $(3^{-4})^2 = 3^{-4(2)} = 3^{-8}$ (b) $(3^4)^{-2} = 3^{4(-2)} = 3^{-8}$

(4) $3^{-2} \times 7^{-2} = (3 \times 7)^{-2} = 21^{-2}$

(5) $\frac{9^{-4}}{11^{-4}} = \left(\frac{9}{11}\right)^{-4}$

Again we extend the laws of exponents when base is a positive real number and the exponents are rational numbers.

We defined $\sqrt[n]{a}$, where $a \in R^+$ and $n \in N$ as follows.

If $b^n = a$ then $\sqrt[n]{a} = b$, where $b \in R^+$. We will write $\sqrt[n]{a}$ as $a^{\frac{1}{n}}$.

We can perform calculations of $4^{\frac{3}{2}}$ type of examples in two ways.

$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$ or $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8$

We can define $a^{\frac{m}{n}}$ where $a \in R^+$, $m \in Z$, $n \in N$; m, n are co-prime.

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ (Both results seem to be different, but they are same)

If base is a positive real number and exponents are rational numbers, we can write the laws of exponents as follows :

For $a, b \in R^+$ and $p, q \in Q$, we have

(1) $a^p \cdot a^q = a^{p+q}$ (2) $\frac{a^p}{a^q} = a^{p-q}$ (3) $(a^p)^q = a^{pq}$

(4) $(ab)^p = a^p \cdot b^p$ (5) $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

Example 21 : Simplify.

(1) (a) $5^{\frac{3}{4}} \cdot 5^{\frac{1}{4}}$ (b) $7^{\frac{3}{5}} \cdot 7^{\frac{2}{3}}$ (2) (a) $\frac{11^{\frac{1}{2}}}{11^{\frac{5}{2}}}$ (b) $\frac{3^{\frac{4}{5}}}{3^{\frac{1}{3}}}$

(3) (a) $(2^3)^{\frac{1}{5}}$ (b) $(2^{\frac{3}{4}})^{\frac{2}{3}}$ (4) $17^{\frac{1}{7}} \cdot 5^{\frac{1}{7}}$

(5) (a) $\frac{14^{\frac{5}{6}}}{7^{\frac{5}{6}}}$ (b) $\frac{15^{\frac{3}{4}}}{20^{\frac{3}{4}}}$

Solution :

(1) (a) $5^{\frac{3}{4}} \cdot 5^{\frac{1}{4}} = 5^{\left(\frac{3}{4}+\frac{1}{4}\right)} = 5^{\frac{4}{4}} = 5^1 = 5$ (b) $7^{\frac{3}{5}} \cdot 7^{\frac{2}{3}} = 7^{\left(\frac{3}{5}+\frac{2}{3}\right)} = 7^{\frac{19}{15}}$

(2) (a) $\frac{11^{\frac{1}{2}}}{11^{\frac{5}{2}}} = 11^{\left(\frac{1}{2}-\frac{5}{2}\right)} = 11^{-2} = \frac{1}{11^2} = \frac{1}{121}$ (b) $\frac{3^{\frac{4}{5}}}{3^{\frac{1}{3}}} = 3^{\left(\frac{4}{5}-\frac{1}{3}\right)} = 3^{\frac{7}{15}}$

(3) (a) $(2^3)^{\frac{1}{5}} = 2^{3 \times \frac{1}{5}} = 2^{\frac{3}{5}}$ (b) $(2^{\frac{3}{4}})^{\frac{2}{3}} = 2^{\frac{3}{4} \times \frac{2}{3}} = 2^{\frac{1}{2}} = \sqrt{2}$

(4) $17^{\frac{1}{7}} \cdot 5^{\frac{1}{7}} = (17 \times 5)^{\frac{1}{7}} = 85^{\frac{1}{7}}$

(5) (a) $\frac{14^{\frac{5}{6}}}{7^{\frac{5}{6}}} = \left(\frac{14}{7}\right)^{\frac{5}{6}} = 2^{\frac{5}{6}}$ (b) $\frac{15^{\frac{3}{4}}}{20^{\frac{3}{4}}} = \left(\frac{15}{20}\right)^{\frac{3}{4}} = \left(\frac{3}{4}\right)^{\frac{3}{4}}$

EXERCISE 2.5

1. Find : (1) $(225)^{\frac{1}{2}}$ (2) $(81)^{\frac{1}{4}}$ (3) $(625)^{\frac{1}{2}}$ (4) $(64)^{\frac{1}{6}}$

2. Find : (1) $(9)^{\frac{5}{2}}$ (2) $(125)^{\frac{5}{3}}$ (3) $(16)^{-\frac{3}{4}}$ (4) $(243)^{-\frac{3}{5}}$

3. Simplify : (1) $3^{\frac{3}{2}} \cdot 3^{\frac{4}{5}}$ (2) $16^{\frac{4}{3}} \times 4^{\frac{2}{3}}$ (3) $3^{\frac{1}{2}} \cdot 12^{\frac{1}{2}}$ (4) $\frac{5^{\frac{4}{5}}}{(25)^{\frac{3}{2}}}$

EXERCISE 2

1. Find five rational numbers between $\frac{1}{5}$ and $\frac{3}{4}$.
2. Find four rational numbers between $-\frac{1}{7}$ and $-\frac{2}{5}$.
3. Represent $\sqrt{8}$ on the number line.
4. Represent $\sqrt{6}$ on the number line.
5. Convert $\frac{-3}{13}$ and $\frac{15}{4}$ in decimal form and state the kind of its decimal expression.
6. Express in the form of $\frac{p}{q}$; $p \in Z$, $q \in N$: (1) $0.3\overline{2}$ (2) $1.4\overline{73}$ (3) $0.\overline{271}$ (4) $0.\overline{35}$
7. Visualize 5.341 on the number line, using successive magnification. **(Activity)**
8. Visualize $2.\overline{7}$ on the number line using successive magnification up to 3 decimal places. **(Activity)**
9. Simplify :

   (1) $(\sqrt{3} - \sqrt{7})(3 + \sqrt{5})$ (2) $(\sqrt{15} - \sqrt{5})^2$ (3) $(\sqrt{7} + \sqrt{2})(\sqrt{14} - \sqrt{8})$
10. Rationalize the denominator of the following :

    (1) $\frac{1}{\sqrt{3} - \sqrt{15}}$ (2) $\frac{5}{3 + \sqrt{2}}$ (3) $\frac{3}{\sqrt{5} - 2}$ (4) $\frac{1}{-8 - \sqrt{6}}$
11. If $\sqrt[3]{a} \cdot \sqrt{b} = (x)^{\frac{1}{6}}$, then find x. $(a > 0, b > 0)$
12. Prove that $(\sqrt{x} + 1) \cdot (\sqrt[4]{x} + 1) \cdot (\sqrt[8]{x} + 1) \cdot (\sqrt[8]{x} - 1) = x - 1$, $(x \in R^+)$
13. Simplify : $(a^{\frac{1}{2}} \cdot b^{\frac{1}{3}})^{\frac{1}{4}} \cdot (a^{\frac{1}{3}} \cdot b^{\frac{1}{4}})^{\frac{1}{2}} \cdot (a^{\frac{1}{4}} \cdot b^{\frac{1}{2}})^{\frac{1}{3}}$
14. Find the value of : $\frac{(81)^{\frac{1}{4}}}{(625)^{\frac{1}{4}}} + \frac{(216)^{\frac{1}{3}}}{(8)^{\frac{1}{3}}} - (729)^{\frac{1}{6}}$
15. Simplify : (1) $\sqrt{\sqrt[4]{256}}$ (2) $\frac{1}{4}\sqrt[3]{128}$
16. Rationalize the denominator and simplify the following :

    (1) $\frac{7 + 3\sqrt{5}}{7 - 3\sqrt{5}}$ (2) $\frac{3\sqrt{2} - \sqrt{5}}{3\sqrt{3} + 2\sqrt{2}}$
17. Prove that $\frac{1}{\sqrt{9} - \sqrt{8}} - \frac{1}{\sqrt{8} - \sqrt{7}} + \frac{1}{\sqrt{7} - \sqrt{6}} - \frac{1}{\sqrt{6} - \sqrt{5}} + \frac{1}{\sqrt{5} - \sqrt{4}} - \frac{1}{\sqrt{4} - \sqrt{3}} + \frac{1}{\sqrt{3} - \sqrt{2}} - \frac{1}{\sqrt{2} - 1} = 2$
18. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

    (1) Set of all natural number is denoted by ...... ☐

    (a) N (b) W (c) Z (d) R

(2) Set of all whole number is denoted by ......

(a) N (b) W (c) Z (d) Q

(3) Set of all integers is denoted by ......

(a) N (b) W (c) Z (d) Q

(4) Set of all rational number is denoted by ......

(a) N (b) W (c) Z (d) Q

(5) ...... is a true statement.

(a) Every whole number is a natural number

(b) Every integer is a rational number

(c) Every rational number is an integer

(d) Every real number is an irrational number.

(6) The number $\frac{3}{4}$ is ......

(a) a natural number (b) an integer

(c) a whole number (d) a rational number

(7) A pair of equivalent rational numbers is ...

(a) $\frac{4}{7}$ and $\frac{104}{182}$ (b) $\frac{5}{2}$ and $\frac{155}{64}$ (c) $\frac{144}{169}$ and $\frac{169}{225}$ (d) $\frac{8}{27}$ and $\frac{125}{216}$

(8) .... is rational number between 10 and 11.

(a) $\frac{21}{4}$ (b) $\frac{87}{8}$ (c) $\frac{97}{8}$ (d) $\frac{47}{4}$

(9) $\sqrt{9}$ = ...

(a) 3 (b) −3

(c) 3 and −3 (d) all a, b, c are true

(10) There are ...... rational numbers between two given numbers.

(a) two (b) can't say (c) finitely many (d) infinitely many

(11) $\sqrt{2}$ belongs to ....

(a) the set of whole numbers (b) the set of rational numbers

(c) the set of irrational numbers (d) the set of natural numbers

(12) The collection of rational numbers and irrational numbers together is called .....

(a) the set of whole numbers (b) the set of real numbers

(c) the set of natural numbers (d) the set of integers

(13) $\sqrt{16}$ is not ......

(a) a natural number (b) a real number

(c) an irrational number (d) a whole number

(14) The decimal expansion of $\frac{7}{4}$ is ......

(a) terminating (b) non terminating recurring

(c) non-terminating non-recurring (d) infinite

(15) 44.7232323... can be written as ......

(a) $44.\overline{723}$ (b) $44.7\overline{23}$ (c) $4\overline{4.723}$ (d) $4\overline{4}.7\overline{23}$

(16) The number 0.235 is ......

(a) a natural number (b) an integer
(c) an irrational number (d) a rational number

(17) The $\frac{p}{q}$ form of $0.3\overline{5}$... is ......

(a) $\frac{16}{45}$ (b) $\frac{176}{495}$ (c) $\frac{35}{99}$ (d) $\frac{16}{495}$

(18) The $\frac{p}{q}$ form of $0.\overline{01}$ is ......

(a) $\frac{1}{99}$ (b) $\frac{10}{99}$ (c) $\frac{100}{99}$ (d) $\frac{101}{99}$

(19) ...... is an irrational number.

(a) 0.3786 (b) $\sqrt{225}$ (c) 1.010010001... (d) 0.2353535...

(20) If $\frac{2}{7} = 0.\overline{285714}$, then $\frac{6}{7}$ =......

(a) $0.\overline{571428}$ (b) $0.\overline{142857}$ (c) $0.\overline{857142}$ (d) $0.\overline{095235}$

(21) $(\sqrt{6}) + (-\sqrt{6})$ is ......

(a) a natural number (b) an irrational number
(c) a whole number (d) an infinite number

(22) $\left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)$ is ......

(a) an irrational number (b) a rational number
(c) a whole number (d) a natural number

(23) $\sqrt{3} \cdot \sqrt{6}$ is ......

(a) a whole number (b) a natural number
(c) an irrational number (d) a rational number

(24) $\sqrt{5} + 29$ is ......

(a) an integer (b) an irrational number
(c) a whole number (d) a rational number

(25) $\sqrt{3} + \sqrt{3}$ is ......

(a) an integer (b) an irrational number
(c) a rational number (d) a whole number

(26) $6\sqrt{5} \cdot 3\sqrt{5}$ is not ......

(a) a natural number (b) an irrational number
(c) a whole number (d) a rational number

(27) $8\sqrt{8} \div 3\sqrt{2}$ is ......

(a) an integer
(b) a rational number
(c) a whole number
(d) an irrational number

(28) $8\sqrt{15} \div 2\sqrt{5}$ is ......

(a) an irrational number
(b) an integer
(c) a whole number
(d) a rational number

(29) $(\sqrt{10} - \sqrt{3})(\sqrt{10} - \sqrt{3}) =$ ......

(a) 0 (b) $13 - 2\sqrt{30}$ (c) $7 - 2\sqrt{30}$ (d) $7 + 2\sqrt{30}$

(30) $(7 + \sqrt{7})(7 - \sqrt{7}) =$ ......

(a) 0 (b) $2\sqrt{7}$ (c) $7\sqrt{7}$ (d) 42

(31) $(\sqrt{5} - \sqrt{2})^2$ is ......

(a) a natural number
(b) an irrational number
(c) a whole number
(d) a rational number

(32) $\frac{3}{2 - \sqrt{5}}$ is rationalized by ......

(a) -3 (b) $2 - \sqrt{5}$ (c) $2 + \sqrt{5}$ (d) $-2 + \sqrt{5}$

(33) An equivalent expression of $\frac{5}{7 + 4\sqrt{5}}$ after rationalizing the denominator is .....

(a) $\frac{20\sqrt{5} - 35}{31}$ (b) $\frac{20\sqrt{5} - 35}{129}$ (c) $\frac{35 - 20\sqrt{5}}{31}$ (d) $\frac{35 - 20\sqrt{5}}{129}$

(34) If $\sqrt[n]{a^2} = b$, then $b^{2n} =$ ...... $(a,\ b > 0,\ n \in N)$

(a) a (b) $(a)^{\frac{n}{2}}$ (c) a^{2n} (d) a^4

(35) $\sqrt[3]{\sqrt{64}} =$ ......

(a) 8 (b) 4 (c) 2 (d) not possible

(36) $\frac{\pi}{4}$ is ......

(a) a natural number
(b) an irrational number
(c) a rational number
(d) a whole number

*

Summary

In this chapter we have studied the following points :

1. A number which can be written in the form $\frac{p}{q}$, where $p \in Z$, $q \in N$ is called a rational number. The set of rational numbers is denoted by Q.
2. A number which is not a rational is called an irrational number.
3. The collection of rational numbers and irrational numbers is the set of real numbers, denoted by R.
4. The decimal expression of rational number is either terminating or non-terminating recurring. Conversely, if the decimal expression is either terminating or non-terminating recurring, then the number is a rational number.
5. The decimal expression of an irrational number is non-terminating non-recurring. Conversely, if the decimal expression is non-terminating non-recurring, then the number is an irrational number.
6. A real number (however small) can be visualized on the number line by the process of successive magnification.
7. Each real number corresponds to a point on the number line and each point on the number line corresponds to a real number.
8. For $x, y \in R^+$, the following properties are true :

   (1) $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ (2) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

   (3) $(\sqrt{a} + \sqrt{b})(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay} + \sqrt{bx} + \sqrt{by}$

   (4) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$ (5) $(\sqrt{x} \pm \sqrt{y})^2 = x \pm 2\sqrt{xy} + y$
9. If $p \in Q$, $p \neq 0$ and q is an irrational, then $p + q$, $p - q$, $p \times q$ and $\frac{p}{q}$ are irrational numbers.
10. To rationalize denominator of a number like $\frac{1}{\sqrt{a} + \sqrt{b}}$ or $\frac{1}{\sqrt{a} + b}$ we have to multiply or divide by $\sqrt{a} - \sqrt{b}$ or $\sqrt{a} - b$ respectively.
11. Laws of exponents for real numbers :

    For $a, b \in R^+$ and $p, q \in Q$, we have

    (1) $a^p \cdot a^q = a^{p+q}$ (2) $\frac{a^p}{a^q} = a^{p-q}$ (3) $(a^p)^q = a^{pq}$

    (4) $(ab)^p = a^p \cdot b^p$ (5) $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

CHAPTER 3

POLYNOMIALS

3.1 Introduction

In earlier classes we have studied about algebraic operations on polynomials like addition, subtraction and multiplication of polynomials. They are like the operations on the integers. In this chapter, we shall learn about the zeroes and factorization of polynomials. We know about unknown numbers. On the basis of that we also have an idea of a term and polynomials.

3.2 Polynomials in One Variable

We begin with some terminology.

Variable : A symbol which takes different numerical values is called a variable and it is denoted by *x,y,z* etc.

Algebraic expressions : An algebraic expression is a combination of a variable and constants connected by the operations like addition, subtraction, multiplication and division. e.g. $2x + 3$, $5 - 7x$, $\frac{x}{2}$, etc. If we use a, b, c for constants, then algebraic expressions can be written as ax, bx,... etc.

Let us try to understand this by an example.

If we consider an equilateral triangle with sides of a unit length, then its perimeter is $1 + 1 + 1 = 3$ units. Look at the figure 3.1 where AB = BC = AC = 1 unit.

Thus the perimeter of ΔABC = AB + BC + AC.

Similarly if we consider the length of the sides of ΔABC as AB = c, BC = a and AC = b, then perimeter of ΔABC = BC + AC + AB = $a + b + c$.

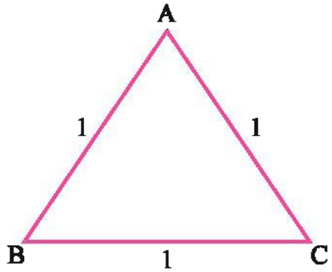


Figure 3.1

Let us think about the perimeter of a square. We know that for a square, length of each side is same. So if we take the length of a side of a square as x, then the perimeter of the square will be $x + x + x + x = 4x$. Here we say that $4x$ is an algebraic expression. If we consider the length of a rectangle as x and breadth as y, then the area of the rectangle is xy. It is also an algebraic expression. We are also familiar with other algebraic expressions like $x^3 - x^2 + x + 5$, $x^2 - 2x + 7$, etc. which are known as polynomials in one variable. Thus, we notice that the algebraic expressions which have only whole numbers as the exponents of the variable is a polynomial in one variable. So we can define a polynomial formally as follows.

Polynomial : An expression of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + .. + a_0$, $a_n \neq 0$, $a_0, a_1, a_2, ... a_n \in R$ is called a polynomial in variable x, where $n \in N \cup \{0\}$. Polynomials are denoted by $P(x)$, $Q(x)$, $p(x)$, $q(x)$,...etc.

$a_n x^n$, $a_{n-1} x^{n-1}$,..., a_0 are called **terms** of the polynomial.

Here, a_i $(i = 0, 1, 2,..., n)$ is the coefficient of x^i. Further x^n is the highest power of x in this polynomial. Thus **$a_n x^n$ is called the leading term of the polynomial and a_n is called the leading coefficient**. As such we can write the terms of a polynomial in any order but when the terms in a polynomial are written in order of decreasing power of x, then we say that it is written in the standard form. It can be seen that when a polynomial is written in the standard form, **the first term is the leading term and exponent of the variable in the leading term is called the degree of the polynomial.**

To determine degree of a polynomial, leading term and leading coefficient of a given polynomial, the terms of the polynomial have to be rearranged so as to write it in the standard form. Here **the coefficient a_0 of x^0 is called the constant term in the polynomial. The polynomial $P(x) = a_0$ is called a constant polynomial. The constant polynomial 0 is called the zero polynomial. If $a_0 \neq 0$, the constant polynomial a_0 has degree 0. The zero polynomial has no degree.**

The polynomial having only one term is called a monomial. If a polynomial has two terms, it is called a binomial and a polynomial having three terms is called a trinomial.

A polynomial can have any finite number of terms. In general, a polynomial of degree n has at most $(n + 1)$ terms with non-zero coefficients. For example $x^{2010} + x^{2009} + x^{2008} + ... + x^2 + x + 1$ is a polynomial of degree 2010 and it has 2011 terms.

The variable x can take any real value. Taking $x = \alpha$ in a polynomial $p(x)$, we get $p(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2} + ... + a_0$, called the value of the polynomial $p(x)$ at $x = \alpha$.

Example 1 : Are $\frac{x^2}{3} - 2x + 5$ and $x^{\frac{3}{2}} + 5x^2 - 7x + \frac{3}{4}$ polynomials in variable x ?

Solution : Here $\frac{x^2}{3} - 2x + 5$ is a polynomial in variable x, but $x^{\frac{3}{2}} + 5x^2 - 7x + \frac{3}{4}$ is not a polynomial in variable x because the exponent of x in the term $x^{\frac{3}{2}}$ is $\frac{3}{2}$ which is not a non-negative integer. In case of $\frac{x^2}{3} - 2x + 5$ variable x occurs to exponent 2, 1, 0 in different terms which are non-negative integers. $(5 = 5x^0)$

Example 2 : Find the degree of the polynomial.

(a) $2x + 7$ (b) $-3x^2 + 7x + 6$ (c) $3x^4 + x^5 - 7x^3 + x - 1$

Solution : Polynomial in (a) has degree 1, because index of x in the term of the highest power of x is 1.

Polynomial in (b) has degree 2, because the exponent of x in the term of the highest power of x is 2.

In case of (c) rewriting the polynomial in standard form $x^5 + 3x^4 - 7x^3 + x - 1$, the term of the highest power in it is x^5. Therefore the degree of polynomial is 5.

Linear polynomial : A polynomial of degree 1 is called a linear polynomial. So we can write the general form of a linear polynomial in one variable as $p(x) = ax + b$ where $a \neq 0$ and $a, b \in$ R.

$3x - 7$, $x + \sqrt[3]{7}$, $7x$ have degree 1. Hence they are linear polynomials.

Quadratic polynomial : A polynomial having degree 2 is known as a quadratic polynomial So, the general form of a quadratic polynomial in one variable is $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in$ R.

Now, $3x^2 - 11$, $\frac{3}{7}x - x^2$, $\sqrt{3}x^2 + 9$ have degree two and they are quadratic polynomials.

Cubic polynomial : A polynomial having degree 3 is known as a cubic polynomial. So, the general form of a cubic polynomial in one variable is $p(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, $a, b, c, d \in$ R.

The polynomials like $x^3 - 4x^2 + 7$, $\sqrt{8}x^2 - 3x^3 + 15x + 17$, $x^3 + x^2 + x + 1$, have degree 3 and hence they are cubic polynomials.

Thus, if we continue this process upto degree n of variable we get,

$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$

where, $a_n \neq 0$. $a_0, a_1, a_2 \ldots a_n$ are constants i.e. $a_i \in$ R $(i = 0, 1, 2, 3, \ldots, n)$

This is a polynomial of degree n in variable x.

EXERCISE 3.1

1. Write the degree of the following polynomials.

(1) $p(x) = 3x^7 - 6x^5 + 4x^3 - x^2 - 5$

(2) $p(x) = x^{100} - (x^{10})^{20} + 3x^{50} + x^{25} + x^5 - 7$

(3) $p(x) = 7x - 3x^2 + 4x^3 + x^4$

(4) $p(x) = 3.14x^2 + 1.57x + 1$

2. Write the coefficient of x^3 in the following polynomials.

(1) $p(x) = 4x^3 + 3x^2 + 2x + 1$ (2) $p(x) = x^2 + 2x + 1$

(3) $p(x) = x^2 - \sqrt{3}x^3 + 4x^7 + 6$

3. Classify the following polynomials as linear, quadratic or cubic.

(1) $p(x) = x^2 + 27$ (2) $p(x) = 2010x + 2009$

(3) $p(x) = 4x^2 + 7x^3 + 3$ (4) $p(x) = (x - 1)(x + 1)$

4. Verify whether following algebraic expressions are polynomial or not.

(1) $p(x) = x^7 + 10x^5 + 4x^3 + 3x + 1$ (2) $p(x) = x^{\frac{-5}{2}} + 10x + 4$

(3) $p(x) = x + \frac{1}{x}$

5. Give an example of each of a monomial of degree 10, a binomial of degree 20 and a trinomial of degree 25.

*

3.3 Zeros of a Polynomial

Let us consider the polynomial $p(x) = x^3 - 5x^2 + 6x$. If we take $x = 0$, we get $p(0) = (0)^3 - 5(0)^2 + 6(0) = 0$. If we take $x = 1$, we get $p(1) = (1)^3 - 5(1)^2 + 6(1) = 2$. Similarly for $x = 2$, we get $p(2) = 0$. For $x = 3$, $p(3) = 0$. For $x = 4$, $p(4) = 8$. Here we observe that the value of given polynomial is 0 at $x = 0$, 2 and 3. These values $x = 0, 2, 3$ where $p(x) = 0$ are called zeros of the polynomial $p(x)$. **Thus if for some $x \in R$, $p(x) = 0$, then x is called a zero of the polynomial $p(x)$. The zeros of a polynomial $p(x)$, if they exist, are called roots (solution) of the polynomial equation $p(x) = 0$.** For example, for no real number x, $x^2 + 1 = 0$. Hence the polynomial $p(x) = x^2 + 1$ has no zero, i.e. polynomial equation $x^2 + 1 = 0$ has no real roots.

Now, if we consider a constant polynomial $c(c \neq 0)$, then what can we say about its zeroes ? It is clear that it has no zeros because replacing x with any number in cx^0 we get the result c, where c is a non-zero constant.

Example 3 : Find the value of each of the following polynomials at the values of variable mentioned.

(1) $p(x) = 3x^2 - 7x + 5$ at $x = 2$

(2) $p(x) = (x^2 - 9)(x^3 + 7)$ at $x = 3, -1$

(3) $p(x) = 3x^4 - 2x^3 + 7x^2 - 5x + 9$ at $x = 3$

Solution : (1) $p(x) = 3x^2 - 7x + 5$

Let $x = 2$.

Hence we get $p(2) = 3(2)^2 - 7(2) + 5$

$= 12 - 14 + 5 = 3$

$\therefore \quad p(2) = 3$

(2) $p(x) = (x^2 - 9)(x^3 + 7)$

$= x^2(x^3 + 7) - 9(x^3 + 7)$

$= x^5 + 7x^2 - 9x^3 - 63$

$= x^5 - 9x^3 + 7x^2 - 63$

$\therefore \quad p(x)$ is a polynomial.

Let $x = -1$.

$\therefore \quad p(-1) = -1 - 9(-1) + 7 - 63$

$= -1 + 9 + 7 - 63$

$= -48$

$\therefore \quad p(-1) = -48$

Now if we consider $x = 3$, in $x^5 - 9x^3 + 7x^2 - 63$

$\therefore \quad p(3) = 3^5 - 9 \cdot 3^3 + 7 \cdot 3^2 - 63$

$= 3^5 - 3^5 + 63 - 63$

$= 0$

$\therefore \quad p(3) = 0$

So we can say that 3 is a zero of $p(x)$.

(3) $p(x) = 3x^4 - 2x^3 + 7x^2 - 5x + 9$

$p(3) = 3(3)^4 - 2(3)^3 + 7(3)^2 - 5(3) + 9$

$= 3(81) - 2(27) + 7(9) - 15 + 9$

$= 243 - 54 + 63 - 15 + 9 = 246$

Example 4 : For $p(x) = x^2 - 4$, find the value of $p(x)$ for $x = 1, 2, 3, 4$. From these what information do we get about the zeros of $p(x)$?

Solution : $p(1) = 1^2 - 4 = -3$ $\quad$ $p(2) = 4 - 4 = 0$

$p(3) = 9 - 4 = 5$ $\quad$ $p(4) = 16 - 4 = 12$

As $p(2) = 0$, $x = 2$ is a zero of $p(x)$.

Example 5 : Find the zeros of linear polynomial $p(x) = ax + b$, $a \neq 0$, $a, b \in$ R.

Solution : Let $p(x) = ax + b = 0$

$\therefore \quad ax + b = 0$

$\therefore \quad ax = -b$

$\therefore \quad x = \frac{-b}{a}$ $\qquad (a \neq 0)$

$\frac{-b}{a}$ is the zero of the linear polynomial $p(x) = ax + b$.

$\therefore$ We can say that a linear polynomial in one variable has one and only one zero.

e.g. $x + 3$ has zero -3 and $2x + 7$ has zero $\frac{-7}{2}$.

Now some important results for the zeros of polynomial are as follows :

(1) 0 may be a zero of a polynomial but a zero of polynomial need not be 0

(2) A linear polynomial has one and only one zero.

(3) A polynomial can have more than one zero.

EXERCISE 3.2

1. Verify whether 3 and 0 are the zeros of $p(x) = x^3 - x$.

2. Find the value of the following polynomials at values of x specified.

(1) $p(x) = x^4 + 2x^3 - x + 5$, $\quad$ at $x = 2$

(2) $p(x) = 3x^3 - 5x^2 + 6x - 9$, $\quad$ at $x = 0, -1$

(3) $p(x) = 5x^3 + 11x^2 + 10$, $\quad$ at $x = -2$

3. Find $p(0)$, $p(1)$, $p(2)$ for each of the following polynomials.

(1) $p(x) = x^7$ $\qquad$ (2) $p(x) = (x - 1)(x + 3)$

(3) $p(x) = x^2 - 2x$

4. Find the zeros of the following polynomials.

(1) $p(x) = 3x + 2$ $\qquad$ (2) $p(x) = 5x - 3$

(3) $p(x) = 3$

*

3.4 Remainder Theorem

Let us consider two numbers 21 and 8. If we divide 21 by 8, we get the quotient 2 and remainder 5. This fact can be expressed as.

$21 = (8 \times 2) + 5$

Here, 21 is called the dividend, 8 is called the divisor, 2 is the quotient and 5 is the remainder.

We observe that the remainder 5 is less than the divisor 8.

If we divide 16 by 8, then we get $16 = (8 \times 2) + 0$.

Here, the remainder is 0 and we say that 8 is a factor of 16 or 16 is a multiple of 8.

As seen above, we write dividend = (divisor)(quotient) + remainder for integers

Thus, when an integer a is divided by a non-zero integer b, we get an expression like $a = bq + r$ (where $|r| < |b|$), q is called the quotient and r is called the remainder, a is called the dividend, b is called the divisor and if $r = 0$, then b is called a factor of a or a divisor of a.

In analogy with this, if a polynomial $p(x)$ is divided by a non-zero polynomial $d(x)$, we get $p(x) = d(x) \cdot q(x) + r(x)$.

Here, $p(x)$ = dividend polynomial, $d(x)$ = divisor polynomial, $q(x)$ = quotient polynomial and $r(x)$ = remainder polynomial. Thus,

Dividend polynomial =

Divisor polynomial × quotient polynomial + remainder polynomial

Here degree of the remainder polynomial is less than the degree of the divisor polynomial or the remainder is zero.

Now consider the polynomial $p(x) = x^2 + x + 1$

$= x(x + 1) + 1$

If we regard x as a divisor then $(x + 1)$ is the quotient and 1 is the remainder. Degree of constant polynomial 1 is zero and degree of divisor polynomial x is 1.

To understand the division of a polynomial $p(x)$ by a polynomial $d(x)$, we take the following example.

Example 6 : Divide $p(x) = x^3 + 13x^2 + 32x + 20$ by $d(x) = x + 2$.

Solution : First we understand, how to solve this example using the following steps.

Step 1 : Rewrite the polynomial $p(x)$, $d(x)$ in standard form.

Step 2 : Divide the leading term of $p(x)$ by the leading term of $d(x)$. In this case we divide x^3 by x, which gives us the first term of the quotient $q(x)$.

i.e. $\frac{x^3}{x} = x^2$ = first term of quotient.

Step 3 : Multiply this first term of the quotient with the divisor and subtract this result from the dividend. i.e. multiply x^2 with $(x + 2)$. Hence we get $x^3 + 2x^2$. We subtract this result from $x^3 + 13x^2 + 32x + 20$. So we get $11x^2 + 32x + 20$.

$$\begin{array}{r|l} & x^2 \\ \hline x + 2 & x^3 + 13x^2 + 32x + 20 \\ & x^3 + \ 2x^2 \\ & - \quad - \\ \hline & \qquad 11x^2 + 32x + 20 \end{array}$$

Thus, we get $x^3 + 13x^2 + 32x + 20 = x^2 (x + 2) + 11x^2 + 32x + 20$.

Since the degree of $11x^2 + 32x + 20$ is more than the degree of $x + 2$, we proceed in a similar manner as in step 2, taking $11x^2 + 32x + 20$ as 'new dividend'.

Step 4 : To find the next term of quotient, we divide the first term of new dividend by the first term of the same divisor i.e. $\frac{11x^2}{x} = 11x$ = second term of quotient.

Step 5 : Now we multiply second term $11x$ with the divisor and subtract the result from new dividend. Here we multiply $11x$ with $(x + 2)$ we get $11x^2 + 22x$. Subtract it from $11x^2 + 32x + 20$ and we get new dividend i.e. $10x + 20$.

$$\begin{array}{r|l} & 11x \\ \hline x + 2 & 11x^2 + 32x + 20 \\ & 11x^2 + 22x \\ & - \quad - \\ \hline & \qquad 10x + 20 \end{array}$$

This process is continued till the remainder is 0 or degree of 'new dividend' is less than the degree of the divisor.

Step 6 : Now here, $10x + 20$ is the new dividend, first term of which $10x$ is to be divided by the first term of divisor i.e. x. Hence we get 10. Now we multiply this divisor by 10 and subtract the result from $10x + 20$ and we get the remainder 0.

$$\begin{array}{r|l} & 10 \\ \hline x + 2 & 10x + 20 \\ & 10x + 20 \\ & - \quad - \\ \hline & \qquad 0 \end{array}$$

Finally we get $x^3 + 13x^2 + 32x + 20 = (x^2 + 11x + 10)(x + 2) + 0$

Hence, in this $q(x) = x^2 + 11x + 10$ and $r(x) = 0$.

This result can be written as...

Dividend = (Divisor) (quotient) + remainder i.e. $p(x) = d(x) \cdot q(x) + r(x)$

Here remainder is $r(x)$ and $r(x) = 0$ or degree of $r(x)$ is less than degree of $d(x)$ and $d(x)$ is the divisor, $q(x)$ is the quotient, $p(x)$ is the dividend.

This process can be written as long division also as illustrated below.

Thus, the above example, can be solved as follows : (by long division)

$$
\begin{array}{r|l}
 & +x^2 + 11x + 10 \\
\hline
x + 2 & +x^3 + 13x^2 + 32x + 20 \\
 & +x^3 + 12x^2 \\
 & - \quad - \\
\hline
 & \quad 11x^2 + 32x + 20 \\
 & \quad 11x^2 + 22x \\
 & \quad - \quad - \\
\hline
 & \quad\quad 10x + 20 \\
 & \quad\quad 10x + 20 \\
 & \quad\quad - \quad - \\
\hline
 & \quad\quad\quad 0
\end{array}
$$

Here $x^3 + 13x^2 + 32x + 20 = (x + 2)\ (x^2 + 11x + 10)$ and the remainder is zero. In this case, we say $x + 2$ is a factor of $x^3 + 13x^2 + 32x + 20$. In general if $r(x) = 0$. $d(x)$ is a factor of $p(x)$

Let us take some more examples :

Example 7 : Divide $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $(x - 3)$ using long division.

Solution :

$$
\begin{array}{r|l}
 & +x^3 + x^2 - 4x - 4 \\
\hline
x - 3 & +x^4 - 2x^3 - 7x^2 + 8x + 12 \\
 & +x^4 - 3x^3 \\
 & - \quad + \\
\hline
 & \quad x^3 - 7x^2 + 8x + 12 \\
 & \quad 2x^3 - 3x^2 \\
 & \quad - \quad + \\
\hline
 & \quad\quad - 4x^2 + 8x + 12 \\
 & \quad\quad - 4x^2 + 12x \\
 & \quad\quad + \quad - \\
\hline
 & \quad\quad\quad - 4x + 12 \\
 & \quad\quad\quad - 4x + 12 \\
 & \quad\quad\quad + \quad - \\
\hline
 & \quad\quad\quad\quad 0
\end{array}
$$

$\therefore \quad (x^4 - 2x^3 - 7x^2 + 8x + 12) = (x - 3)\,(x^3 + x^2 - 4x - 4)$

Here also $(x - 3)$ is a factor of $x^4 - 2x^3 - 7x^2 + 8x + 12$.

Example 8 : Divide $x^3 + 3x^2 + 4x + 7$ by $x^2 + x$

Solution :

$$
\begin{array}{r|l}
 & x + 2 \\
\hline
x^2 + x & x^3 + 3x^2 + 4x + 7 \\
 & x^3 + x^2 \\
 & - \quad - \\
\hline
 & 2x^2 + 4x \\
 & 2x^2 + 2x \\
 & - \quad - \\
\hline
 & 2x + 7
\end{array}
$$

Here degree of $2x + 7$ is less than the degree of $x^2 + x$. Hence $2x + 7$ is the remainder and $x + 2$ is the quotient. ($2x + 7$ has degree 1 and $x^2 + x$ has degree 2.)

Example 9 : Divide $4x^3 - 7x + 10$ by $x - 2$ and find the remainder.

Solution :

$$
\begin{array}{r|l}
 & 4x^2 + 8x + 9 \\
\hline
x - 2 & 4x^3 - 7x + 10 \\
 & 4x^3 - 8x^2 \\
 & - \quad + \\
\hline
 & 8x^2 - 7x + 10 \\
 & 8x^2 - 16x \\
 & - \quad + \\
\hline
 & 9x + 10 \\
 & 9x - 18 \\
 & - \quad + \\
\hline
 & 28
\end{array}
$$

If we divide $p(x) = 4x^3 - 7x + 10$ by the linear polynomial $x - 2$, we get quotient polynomial $4x^2 + 8x + 9$ and remainder is 28.

$\therefore \quad p(x) = (x - 2)(4x^2 + 8x + 9) + 28$

Here also degree of constant polynomial 28 is zero and it is less than degree of divisor $x - 2$.

Now, if we find $p(2)$, then $p(2) = 4(2)^3 - 7(2) + 10 = 32 - 14 + 10 = 28$

$\therefore \quad p(2) = 28$

Here $r(x) = p(2) = 28$

If we divide the polynomial $p(x)$ by linear polynomial $x - a$ then we have the remainder $p(a)$. The proof of this statement will now be given. This theorem is known as **remainder theorem**.

Remainder Theorem : If a polynomial $p(x)$ of degree greater than or equal to 1 is divided by linear polynomial $x - a$, the remainder is $p(a)$. ($a \in R$)

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1 Suppose the dividend $p(x)$ is divided by the divisor $(x - a)$. Let the quotient be $q(x)$ and remainder be $r(x)$. So we get,

$$p(x) = d(x)\,q(x) + r(x)$$
$$= (x - a)\,q(x) + r(x)$$

Degree of divisor $x - a$ is 1.

Degree of $r(x)$ < degree of divisor or $r(x) = 0$.

$\therefore$ degree of $r(x) < 1$ or $r(x) = 0$

$\therefore$ degree of $r(x) = 0$ or $r(x) = 0$

$\therefore$ $r(x) = r$, a non-zero constant or $r(x) = 0$.

$\therefore$ $r(x) = r$, a real number not depending upon value of x.

$\therefore$ $p(x) = (x - a)\,q(x) + r$.

In particular if $x = a$, this identity becomes,

$$p(a) = (a - a)\,q(a) + r$$
$$= r$$

$\therefore$ The remainder r is $p(a)$.

This proves the theorem. Let us use this theorem in the following example.

Example 10 : Find the remainder when $y^3 - 2y^2 - 29y - 42$ is divided by $(y - 3)$.

Solution :

Here $p(y) = y^3 - 2y^2 - 29y - 42$ and the divisor is $y - 3$. So $a = 3$.

$$p(3) = (3)^3 - 2(3)^2 - 29(3) - 42$$
$$= 27 - 18 - 87 - 42$$
$$= -120$$

So, by the remainder theorem, -120 is the remainder when $p(y)$ divided by $y - 3$.

It means that if any polynomial is to be divided by a linear polynomial $(x - a)$, then find the zero of the linear polynomial a and substitute its value in $p(y)$. Then we get the remainder $p(a)$ directly.

Example 11 : Find the value of c, if $y - 1$ is a factor of $p(y) = 4y^3 + 3y^2 - 4y + c$.

Solution : As $y - 1$ is a factor of $p(y)$, the remainder $p(1) = 0$

$\therefore$ $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + c$

$\therefore$ $0 = 4 + 3 - 4 + c$

$\therefore$ $c = -3$

Example 12 : Verify whether the polynomial $2x^4 + x^3 - 14x^2 - 19x - 6$ is divisible by $x + 1$ or not.

Solution : Now, here we consider the zero of linear polynomial (i.e. divisor) $x + 1$. So we get $x = -1$. Substitute this value in the given polynomial.

$\therefore$ $p(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$

$= 2 - 1 - 14 + 19 - 6$

$= 21 - 21$

$= 0$

$\therefore$ The remainder when $2x^4 + x^3 - 14x^2 - 19x - 6$ is divided by $x + 1$ is zero.

$\therefore$ $p(x)$ is divisible by $(x + 1)$

EXERCISE 3.3

1. Divide the following polynomials by $(x - 1)$ and find the quotient and remainder.
   (1) $p(x) = x^5 - 1$
   (2) $p(x) = x^4 + 4x^3 - 3x^2 - x + 1$
2. Find the remainder when the polynomial $p(t) = 2t^4 - 7t^3 - 13t^2 + 63t - 45$ is divided by the following polynomials.
   (1) $(t - 1)$ (2) $t - 3$ (3) $2t - 5$ (4) $t + 3$ (5) $2t + 3$
3. What is the remainder when $p(x) = x^4 - 4x^3 + 3x - 1$ is divided by $d(x) = x + 2$?
4. What should be added to $p(y) = 12y^3 - 39y^2 + 50y + 97$ so that the resulting polynomial is divisible by $y + 1$?
5. What should be subtracted from $p(x) = x^4 + 85$ so that the resulting polynomial is divisible by $x + 3$?
6. The product of two polynomial is $x^3 - 8x - 12 + x^2$. If one of the polynomial is $x + 2$, then find the other $\left[\textbf{Hint : Other polynomial} = \dfrac{\textbf{Product of two polynomials}}{\textbf{One polynomial}}\right]$
7. Carry out the following division and find the remainder :
   (1) $(x^4 + x^3 + 3x^2 + 2x + 2) \div (x^2 + 2)$
   (2) $(x^3 - 15x^2 - 54x + 23) \div (x^2 + 3x)$
   (3) $(x^4 + 4x^3 + 10x^2 + 12x + 15) \div (x^2 + 2x + 3)$
8. If the polynomial $ax^5 - 23x^3 + 47x + 1$ is divided by $x - 2$, then remainder is 7. Find the value of a.

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3.5 Factorization of Polynomials

When discussing zeros of a polynomial, we have already seen that if $p(a) = 0$, then the polynomial $p(x)$ is exactly divisible by divisor $(x - a)$ and remainder is zero. In such a case the divisor is considered as its factor. The same procedure is shown in the example 12, where the given polynomial is divisible by $(x + 1)$. Here we say that $(x + 1)$ is a factor of the polynomial. We have the following theorem, which is known as **Factor Theorem**.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and $a \in$ R then

(1) If $p(a)$ is zero, then $x - a$ is a factor of $p(x)$ and

(2) If $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Proof : By the remainder theorem, we have $p(x) = (x - a)\, q(x) + p(a)$

So, (1) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$ which shows that $(x - a)$ is a factor of $p(x)$

(2) Since $(x - a)$ is a factor of $p(x)$, we have $p(x) = (x - a)\, g(x)$ for some polynomial $g(x)$.

Hence $p(a) = (a - a)\, g(a) = 0.\ g(a) = 0$.

Let us understand the above theorem by following example.

Example 13 : Examine whether $x^2 - 3x + 2$ has the factor $x - 2$.

Solution : For divisor polynomial $x - 2$, $a = 2$.

Subsititute $x = 2$ in the given polynomial $p(x)$

$$p(2) = (2)^2 - 3(2) + 2$$
$$= 4 - 6 + 2 = 0.$$

$\therefore$ By factor theorem, $p(a) = p(2) = 0$. Hence we can say that $x - 2$ is a factor of given polynomial $p(x)$.

In the above example, given polynomial has degree 2. So this polynomial is in the form of a quadratic polynomial like $ax^2 + bx + c$. We can factorize this polynomial as a product of linear polynomials. So, this polynomial can be factorized as $(px + l)(qx + m)$. Thus,

$$ax^2 + bx + c = (px + l)(qx + m) = pqx^2 + pmx + qlx + ml$$
$$= pqx^2 + (pm + lq)x + ml$$

Hence, $a = pq$, $b = pm + lq$, $c = ml$.

This shows that b is the sum of two numbers pm and lq and product of them $= (pm)\ (lq) = (pq)\ (lm) = ac$

$\therefore$ To factorize $ax^2 + bx + c$, we have to write b as the sum of two numbers product of which is ac. We can see this procedure in the following example.

Note : For the factorization, we assume that a, b, c are integers.

i.e. we factorize the polynomials on the set of integers.

$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. We do not take this type of factors.

We also do not think about the factorization of the polynomials like

$x^2 - (\sqrt{2} + \sqrt{3})x + \sqrt{6} = (x - \sqrt{2})(x - \sqrt{3})$.

Example 14 : Factorize $5x^2 + 9x + 4$ by splitting the middle term and by using factor theorem.

Solution : Here, given quadratic polynomial is $5x^2 + 9x + 4$. By splitting middle term, we have to find two numbers whose sum is 9 and product is 20. Such numbers are 5 and 4.

Here 5 and 4 are two numbers such that $5 + 4 = 9$ and $5 \times 4 = 20$

$$5x^2 + 9x + 4 = 5x^2 + 5x + 4x + 4$$

$$= 5x(x + 1) + 4(x + 1)$$

$\therefore$ $5x^2 + 9x + 4 = (x + 1)(5x + 4)$

Example 15 : If $3x^3 - x^2 - 27x + k$ has a factor $3x - 1$, then find the constant k.

Solution : $3x - 1$ is a factor of given polynomial $p(x) = 3x^3 - x^2 - 27x + k$. Hence, by considering $3x - 1 = 0$, we get $x = \frac{1}{3}$. Thus $p\left(\frac{1}{3}\right) = 0$ **(by the factor theorem).** substitute $x = \frac{1}{3}$ in the given polynomial and we get,

$$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 27\left(\frac{1}{3}\right) + k = 0$$

$\therefore$ $3\left(\frac{1}{27}\right) - \left(\frac{1}{9}\right) - 9 + k = 0$

$\therefore$ $\frac{1}{9} - \frac{1}{9} - 9 + k = 0$

$\therefore$ $k = 9$

Example 16 : Verify that $(x - 1)$ is a factor of $15x^3 - 20x^2 + 13x - 8$ and hence factorize $15x^3 - 20x^2 + 13x - 8$.

Solution : Let $p(x) = 15x^3 - 20x^2 + 13x - 8$ be given. Since $(x - 1)$ is a factor, $p(1)$ should be zero.

$p(1) = 15 - 20 + 13 - 8 = 0$

$\therefore$ $(x - 1)$ is a factor of $p(x)$

Let, $15x^3 - 20x^2 + 13x - 8$ be divided by $(x - 1)$

$$
\begin{array}{r|l}
 & 15x^2 - 5x + 8 \\
\cline{2-2}
x - 1 & 15x^3 - 20x^2 + 13x - 8 \\
 & 15x^3 - 15x^2 \\
 & - \quad + \\
\cline{2-2}
 & \quad - 5x^2 + 13x - 8 \\
 & \quad - 5x^2 + 5x \\
 & \quad + \quad - \\
\cline{2-2}
 & \qquad 8x - 8 \\
 & \qquad 8x - 8 \\
 & \qquad - \quad + \\
\cline{2-2}
 & \qquad\quad 0
\end{array}
$$

$\therefore \quad 15x^3 - 20x^2 + 13x - 8 = (x - 1)(15x^2 - 5x + 8)$

So, $p(x) = (x - 1)$ g(x), where g$(x) = 15x^2 - 5x + 8$. Here $a = 15$, $b = -5$, $c = 8$

Now we have to find two numbers sum of which is -5 and product is 120 which is not possible. Hence the factors of $p(x)$ are $(x - 1)$ and $(15x^2 - 5x + 8)$.

Infact $120 = 2 \times 60 = 3 \times 40 = 4 \times 30 = 5 \times 24 = 6 \times 20 = 8 \times 15 = 10 \times 12$

Hence in any pair of factors, the sum is at least 22 in absolute value.

According to factor theorem how can we know whether $(x - 1)$ or $(x + 1)$ are factors of the given polynomial $p(x)$. Thus to understand this we take the example of a cubic polynomial. These results are true for any polynomials to have $(x - 1)$ as a factor.

(1) Criterion for $x - 1$ to be a factor of $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in$ R, $a \neq 0$

By the remainder theorem, $(x - 1)$ is a factor of $p(x)$ if and only if $p(1) = 0$.

$p(x) = ax^3 + bx^2 + cx + d$

Now, $p(1) = a(1)^3 + b(1)^2 + c(1) + d$

$= a + b + c + d$

Thus, $p(1) = a + b + c + d = 0$ if and only if $(x - 1)$ is a factor.

Now, $a + b + c + d$ is the sum of the coefficients of $p(x)$.

Thus, **$(x - 1)$ is a factor of $p(x)$ if and only if the sum of all the coefficients of $p(x)$ is zero.**

[**Note** : In general, let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$

Then $p(a) = a_n + a_{n-1} + \ldots + a_0$

= sum of the coefficients of $p(x)$

$\therefore \quad (x - 1)$ is a factor of $p(x)$ if and only if $p(1) = 0$.

i.e. sum of coefficients is zero.]

Let us understand this by the following example :

Example 17 : Factorize $p(x) = x^3 + 5x^2 + 2x - 8$.

Solution : The sum of the coefficients of $p(x) = 1 + 5 + 2 - 8 = 0$

$\therefore$ $(x - 1)$ is a factor of $p(x)$.

Now we shall factorise $p(x)$ by considering $(x - 1)$ as a factor.

$p(x) = x^3 + 5x^2 + 2x - 8$

$= x^3 - x^2 + 6x^2 - 6x + 8x - 8$

$= x^2(x - 1) + 6x(x - 1) + 8(x - 1)$

$= (x - 1)(x^2 + 6x + 8)$

$= (x - 1)(x^2 + 4x + 2x + 8)$ **(factors of 8 whose sum is 6)**

$= (x - 1)[x(x + 4) + 2(x + 4)]$

$= (x - 1)(x + 2)(x + 4)$

(2) Criterion for $x + 1$ to be a factor of $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$, $a \neq 0$

$(x + 1)$ is a factor of $p(x)$ if and only if $p(-1) = 0$.

Substitute $x = -1$ in $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = a(-1)^3 + b(-1)^2 + c(-1) + d$

$= -a + b - c + d$

$p(-1) = 0$ if and only if $-a + b - c + d = 0$.

$\therefore$ $a + c = b + d$

Here $a + c$ is the sum of the coefficients of odd powers of x in $p(x)$ and $b + d$ is the sum of the coefficients of even powers of x in $p(x)$.

Thus, $(x + 1)$ is a factor of $p(x)$ if and only if the sum of the coefficients of odd powers of x in $p(x)$ is equal to the sum of the coefficients of even powers of x in $p(x)$. This is true in general for a polynomial of any degree greater than or equal to 1 also.

Let us understand this by the following example :

Example 18 : Factorize $x^3 + 4x^2 + 4x + 1$.

Solution : Let $p(x) = x^3 + 4x^2 + 4x + 1$

The sum of the coefficients of $p(x) = 1 + 4 + 4 + 1 = 10 \neq 0$

$\therefore$ $(x - 1)$ is not a factor of $p(x)$.

The sum of the coefficients of odd powers of x in $p(x) = 1 + 4 = 5$

and the sum of the coefficients of even powers of x in $p(x) = 4 + 1 = 5$

Since both are equal, $(x + 1)$ is a factor of $p(x)$.

$p(x) = x^3 + 4x^2 + 4x + 1$
$= x^3 + x^2 + 3x^2 + 3x + x + 1$
$= x^2(x + 1) + 3x(x + 1) + 1(x + 1)$
$= (x + 1)(x^2 + 3x + 1)$

(Here $x^2 + 3x + 1$ can not be factored further as there is no pair of factors of 1 whose sum is 3).

EXERCISE 3.4

1. From the following polynomials, find out which of them has $(x - 1)$ as a factor :

(1) $2x^3 - 3x^2 + 3x - 2$ (2) $4x^3 + x^4 - x + 1$
(3) $5x^4 - 4x^3 - 2x + 1$ (4) $3x^3 + x^2 + x + 11$

2. By using factor theorem, find the other factor of the given polynomial $p(x)$, where $d(x)$ is a given factor.

(1) $p(x) = 21x^2 + 26x + 8$, $d(x) = 3x + 2$
(2) $p(x) = x^3 + 10x^2 + 23x + 14$, $d(x) = x + 1$
(3) $p(x) = x^3 - 9x^2 + 20x - 12$, $d(x) = x - 6$

3. If $p(x) = ax^3 + 3x^2 + 7x + 13$ is divided by $(x + 3)$, then the remainder is -8. Find the value of a.

4. Factorise the following polynomials :

(1) $3x^2 + 7x + 4$ (2) $15x^2 + 16x + 4$ (3) $-21x^2 + 16x + 5$

5. From the following polynomials, decide which has $(x + 1)$ or $(x - 1)$ as a factor.

(1) $p(x) = 3x^3 - 7x^2 + 5x - 1$ (2) $p(x) = 21x^3 + 16x^2 + 4x + 9$
(3) $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 2$ (4) $p(x) = x^3 + 13x^2 + 32x + 20$

6. If $x - 4$ is a factor of $p(x) = ax^4 - 7x^3 - 3x^2 - 2x - 8$, then find the value of a.

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3.6 Algebraic Identities

In previous classes, we have already seen the following algebraic identities. An algebraic identity is true for all values of the variables occuring in it. These identities are as follows. R.H.S. is the expansion of the expression on L.H.S.

(1) $(a + b)^2 = a^2 + 2ab + b^2$
(2) $(a - b)^2 = a^2 - 2ab + b^2$
(3) $(a - b)(a + b) = a^2 - b^2$
(4) $(x + a)(x + b) = x^2 + (a + b)x + ab$
(5) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

More identities can be obtained from the above identities as follows :

$$(a + b)^3 = (a + b) \cdot (a + b)^2$$
$$= (a + b) \cdot (a^2 + 2ab + b^2)$$
$$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$\therefore$ $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ **(6)**

Similarly, $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ **(7)**

The above two identities can also be writen as follows :

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ **(8)**

$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$ **(9)**

$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) = a^3 + b^3 + c^3 - 3abc$ **(10)**

Identity (10) can be proved by multiplication.

R.H.S. in the above 10 identities are called the expansions of the algebraic expressions on L.H.S. Let us understand above identities by following examples.

Example 19 : Find : $(3x - 4y)^2$

Solution : Here $a = 3x$, $b = 4y$. By using expansion of $(a - b)^2$,

we get $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$
$$= 9x^2 - 2(12xy) + 16y^2$$
$$= 9x^2 - 24xy + 16y^2$$

Example 20 : Find : 105×95 by using an appropriate identity.

Solution : Here appropriate identity is $(a + b)(a - b) = a^2 - b^2$

Taking $a = 100$, $b = 5$

$\therefore$ $105 \times 95 = (100 + 5)(100 - 5)$
$$= (100)^2 - (5)^2$$
$$= 10000 - 25$$
$$= 9975$$

Example 21 : Find : 107×102 by using an appropriate identity.

Solution : Here appropriate identity is $(x + a)(x + b) = x^2 + (a + b)\,x + ab$

Here $x = 100$, $a = 7$, $b = 2$

$\therefore$ $107 \times 102 = (100 + 7)(100 + 2)$
$$= (100)^2 + (7 + 2)(100) + (7)(2)$$
$$= 10000 + 9(100) + 14$$
$$= 10000 + 900 + 14$$
$$= 10914$$

Example 22 : Find : $(2x - 3y + 4z)^2$

Solution : By using the expansion of $(a + b + c)^2$, where $a = 2x$, $b = -3y$, $c = 4z$, we get,

$$(2x - 3y + 4z)^2 = (2x)^2 + (-3y)^2 + (4z)^2 + 2(2x)(-3y) + 2(-3y)(4z) + 2(2x)(4z)$$
$$= 4x^2 + 9y^2 + 16z^2 + 2(-6xy) + 2(-12yz) + 2(8zx)$$

$$\therefore \quad (2x - 3y + 4z)^2 = 4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$$

Example 23 : Find : $\left(\frac{2x}{3}+\frac{y}{5}\right)^2$

Solution : Let $a = \frac{2x}{3}$, $b = \frac{y}{5}$ in the expansion of $(a + b)^2$.

$$\left(\frac{2x}{3}+\frac{y}{5}\right)^2 = \left(\frac{2x}{3}\right)^2 + 2\left(\frac{2x}{3}\right)\left(\frac{y}{5}\right) + \left(\frac{y}{5}\right)^2$$
$$= \frac{4x^2}{9} + 2\left(\frac{2xy}{15}\right) + \frac{y^2}{25}$$
$$= \frac{4x^2}{9} + \frac{4xy}{15} + \frac{y^2}{25}$$

Example 24 : Find : $\left(\frac{x}{3}-\frac{2y}{5}\right)^3$

Solution : Let $a = \frac{x}{3}$, $b = \frac{2y}{5}$ in the expansion of $(a - b)^3$

We get, $$\left(\frac{x}{3}-\frac{2y}{5}\right)^3 = \left(\frac{x}{3}\right)^3 - 3\left(\frac{x}{3}\right)\left(\frac{2y}{5}\right)\left(\frac{x}{3}-\frac{2y}{5}\right) - \left(\frac{2y}{5}\right)^3$$
$$= \frac{x^3}{27} - \frac{3(2xy)}{15}\left(\frac{x}{3}-\frac{2y}{5}\right) - \frac{8y^3}{125}$$
$$= \frac{x^3}{27} - \frac{2xy}{5}\left(\frac{x}{3}-\frac{2y}{5}\right) - \frac{8y^3}{125}$$
$$= \frac{x^3}{27} - \frac{2x^2y}{15} + \frac{4xy^2}{25} - \frac{8y^3}{125}$$

Example 25 : Find the value of $(110)^3$

Solution : $(110)^3 = (100 + 10)^3$

Let $a = 100$, $b = 10$, in the expansion of $(a + b)^3$

$$(110)^3 = (100)^3 + 3(100)(10)(100 + 10) + (10)^3$$
$$= 1000000 + 3000(110) + 1000$$
$$= 1000000 + 330000 + 1000$$
$$= 1331000$$

Example 26 : Find the value of $(997)^3$

Solution : $997 = 1000 - 3$

$\therefore \quad (997)^3 = (1000 - 3)^3$

Let, $a = 1000$, $b = 3$ in the expansion of $(a - b)^3$

$(997)^3 = (1000)^3 - 3(1000)(3)(1000 - 3) - (3)^3$

$= 1000000000 - 9000000 + 27000 - 27$

$= 991026973$

The above identities are in expansion form. If we interchange their L.H.S. and R.H.S. then they are considered to be in factor form as follows :

Expression on L.H.S. represents expansion of expression on R.H.S. and expression on R.H.S. represents factors of expression on L.H.S.

(1) $a^2 + 2ab + b^2 = (a + b)^2$

(2) $a^2 - 2ab + b^2 = (a - b)^2$

(3) $a^2 - b^2 = (a - b)(a + b)$

(4) $x^2 + (a + b)x + ab = (x + a)(x + b)$

(5) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2$

(6) $a^3 + b^3 + 3a^2b + 3ab^2 = (a + b)^3$

(7) $a^3 - b^3 - 3a^2b + 3ab^2 = (a - b)^3$

(8) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Hence if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

Example 27 : Factorize : $27x^3 + 189x^2y + 441xy^2 + 343y^3$

Solution : This expression can be writen as

$(3x)^3 + (7y)^3 + 3(3x)^2 (7y) + 3(3x)(7y)^2$

$= (3x)^3 + (7y)^3 + 3(3x)(7y)(3x + 7y)$

$= (3x + 7y)^3$

Example 28 : Factorize : $x^3 + 8y^3 - 27z^3 + 18xyz$

Solution : $x^3 + 8y^3 - 27z^3 + 18xyz = (x)^3 + (2y)^3 + (-3z)^3 - 3(x)(2y)(-3z)$

$= (x + 2y - 3z)[(x)^2 + (2y)^2 + (-3z)^2 - (x)(2y) - (2y)(-3z) - (x)(-3z)]$

$= (x + 2y - 3z)(x^2 + 4y^2 + 9z^2 - 2xy + 6yz + 3zx)$

Example 29 : Factorize : $9x^2 - 30xy + 25y^2$

Solution : $9x^2 - 30xy + 25y^2$

$= (3x)^2 - 2(3x)(5y) + (5y)^2$

$= (3x - 5y)^2$

Example 30 : Factorize : $a^4 - 81b^4$

Solution : $a^4 - 81b^4 = (a^2)^2 - (9b^2)^2$

$= (a^2 - 9b^2)(a^2 + 9b^2)$

$= [(a)^2 - (3b)^2](a^2 + 9b^2)$

$= (a - 3b)(a + 3b)(a^2 + 9b^2)$

Example 31 : Factorize : $x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6zx$

Solution : $x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6zx$

$= (x)^2 + (2y)^2 + (3z)^2 + 2(x)(2y) + 2(2y)(3z) + 2(x)(3z)$

$= (x + 2y + 3z)^2$

EXERCISE 3.5

1. Use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$ to find the value of the following product :

   (1) $(x - 7)(x - 12)$ (2) $(5 - 4x)(7 - 4x)$

   (3) $\left(x+\frac{3}{2}\right)\left(2x+\frac{5}{3}\right)$ (4) $\left(3x+\frac{3}{2}\right)\left(3x+\frac{5}{2}\right)$

2. Evaluate by using $(a^2 - b^2) = (a - b)(a + b)$

   (1) 97×103 (2) 57×63 (3) 34×26

3. Factorize the following by using appropriate identities.

   (1) $16x^2 - 40xy + 25y^2$ (2) $\frac{x^2}{9} + \frac{4xy}{15} + \frac{4y^2}{25}$

   (3) $9a^2 + 25b^2 + 49c^2 - 30ab + 70bc - 42ac$

   (4) $16a^4 - 625b^4$ (5) $\frac{8x^3}{27} + \frac{27y^3}{64} + \frac{64z^3}{125} - \frac{6}{5}xyz$

   (6) $125a^3 + 600a^2b + 960ab^2 + 512b^3$ (7) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

4. Evaluate by using the identities :

   (1) 105×102 (2) $(92)^2$ (3) $(8)^3 - (4)^3$

5. If $a + b + c = 0$, by using the identity $a^3 + b^3 + c^3 = 3abc$, find the value of $(-28)^3 + (15)^3 + (13)^3$.

EXERCISE 3

1. If for $p(x) = x^3 + kx^2 - 4x + 5$, $p(3) = 0$, then find the value of k.

2. Divide the following polynomials by $(x + 2)$ and find the quotient and the remainder :

   (1) $p(x) = x^4 + 2x^3 + 7x^2 + x - 5$ (2) $p(x) = 2x^3 - 5x^2 + 11x + 19$

   (3) $p(x) = 5x^3 + 9x^2 + 8x + 20$

3. If a student of std. IX - A, distributes equal number of chocolates from $x^4 - 3x^3 + 5x^2 + 8x + 5$ chocolates to all of his friends, then each friend gets $(x^2 - 1)$ chocolates and remain 26 chocolates there with him for his teachers. Find how many chocolates did the boy have ? How many chocolates does each friend get ? How many friends did the boy have ?

4. In a class ₹ $(2x + 3)$ were collected from each student for relief fund. If the total sum collected was ₹ $(2x^3 + x^2 - 5x - 3)$, find the number of students in the class.

5. The product of two polynomials is $x^4 - 3x^3 + 8x^2 - 9x + 15$. If one of the polynomials is $(x^2 - 3x + 5)$, then find the other.

6. If $x - 4$ is a factor of $x^3 - 6x^2 + 4x + 16$, then find the other factor.

7. Evaluate $(107)^2$ by using appropriate identity.

8. Find the value of $(-7)^3 + (12)^3 + (-5)^3$ by using appropriate identity.

9. Factorize : $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20zx$

10. Find the quotient and the remainder when following divisions are carried out

(1) $(2x^3 + x^2 + 9x + 17) \div (x - 2)$

(2) $(x^5 + 1) \div (x + 1)$

(3) $(3x^4 + 7x^3 - 6x^2 + 5x - 9) \div (x - 1)$

(4) $(7x^3 - 11x^2 + 3x - 49) \div (x^2 + x + 3)$

11. If $a + b + c = 6$ and $a^2 + b^2 + c^2 = 60$, then find $ab + bc + ca$ and $a^3 + b^3 + c^3 - 3abc$.

12. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If $p(3) = 0$, then a factor of $p(x)$ is ...... ☐

(a) $(x - 3)$ (b) $(x - 2)$ (c) $(x + 3)$ (d) $(x + 2)$

(2) If $x^3 + 2x^2 - 6x + 9$ is divided by $x - 2$, then ...... is the remainder. ☐

(a) -13 (b) 13 (c) 9 (d) -16

(3) The degree of the polynomial $x^5 + 3x^3 - 7x^2 + 9x + 11$ is .... ☐

(a) 1 (b) 2 (c) 3 (d) 5

(4) If $x - 2$ is a factor of $3x^4 - 2x^3 + 7x^2 - 21x + k$, then the value of k is ...... ☐

(a) 2 (b) 9 (c) 18 (d) -18

(5) The zero of $7x - 3$ is ...... ☐

(a) $\frac{-3}{7}$ (b) $\frac{3}{7}$ (c) $\frac{7}{3}$ (d) $\frac{-7}{3}$

(6) If $x^2 + 6x + 7$ is divided by $x + 1$, then the remainder is ......

(a) 1 (b) 2 (c) 5 (d) 7

(7) Factors of $y^2 + 10y + 21$ are ......

(a) $(y + 3)$ and $(y - 7)$ (b) $(y - 3)$ and $(y + 7)$

(c) $(y - 3)$ and $(y - 7)$ (d) $(y + 3)$ and $(y + 7)$

(8) If $a - b = 2$ and $ab = 3$, then $a^3 - b^3 =$ ......

(a) 8 (b) 27 (c) 26 (d) 6

(9) If $a = b = c$ then $a^3 + b^3 + c^3 - 3abc =$ ......

(a) a^3 (b) $2a^3$ (c) $3a^3$ (d) 0

(10) If one factor of the polynomial $x^3 + 4x^2 - 3x - 18$ is $x + 3$, then the other factor is ......

(a) $x^2 + x$ (b) $x^2 + x + 6$ (c) $x^2 + x - 6$ (d) $x^2 - x + 6$

(11) If $(x^3 + 28)$ is divided by $(x + 3)$, then the remainder is ......

(a) 0 (b) 1 (c) -1 (d) 2

(12) ...... should be added to $x^3 - 76$ so that the resulting polynomial is divisible by $x - 4$.

(a) 5 (b) -5 (c) 12 (d) -12

(13) If $25x^2 - 49y^2$ has one factor $(5x - 7y)$, then the other factor is ......

(a) $7x + 5y$ (b) $-7x - 5y$ (c) $5x + 7y$ (d) $-5x + 7y$

(14) If $p(x) = x^3 - 2x^2 + 7x - 6$, then a zero of $p(x)$ is ......

(a) 0 (b) 1 (c) 2 (d) 3

(15) If the cost of one mathematics textbook is ₹ $(x + 4)$, then ...... textbook can be purchased by ₹ $(x^3 + 64)$.

(a) $x^2 + 8x + 16$ (b) $x^2 - 8x - 16$ (c) $x^2 - 4x + 16$ (d) $x^2 - 4x - 16$

(16) $(4x - 7y)^3 =$ ......

(a) $4x^3 - 7y^3 + 84xy$ (b) $16x^3 + 49y^3 + 84xy$

(c) $64x^3 - 343y^3 - 336x^2y + 588xy^2$ (d) $64x^3 + 343y^3 + 336x^2y - 588xy^2$

*

Summary

1. An expression of form $a_nx^n + a_{n-1}x^{n-1} + ... + a_0$, where $a_0, a_1, a_2, ... a_n \in R$; $a_n \neq 0$ is called a polynomial in variable x. ($n \in N \cup \{0\}$). n is called the degree of the polynomial.
2. If a polynomial has degree zero, then it is a constant polynomial.
3. Constant polynomial 0 is known as zero polynomial.
4. According as a polynomial has one, two or three terms, then it is known as a monomial, a binomial or a trinomial respectively.
5. If the degree of a polynomial is 1, it is linear polynomial.
6. According as a polynomial has the degree 2 or 3, it is known as a quadratic or a cubic polynomial respectively.
7. If $a \in R$ and $p(a) = 0$ then a is a zero of polynomial $p(x)$. a is also called the root of polynomial equation $p(x) = 0$.
8. **Remainder Theorem :** If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.
9. **Factor Theorem :** If $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$ and if $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$.
10. $(x - 1)$ is a factor of a polynomial, if the sum of its coefficients is zero.
11. $(x + 1)$ is a factor of a polynomial, if the sum of coefficients of odd power of x equals, sum of coefficients of even power of x.
12. $(a + b)^2 = a^2 + 2ab + b^2$
13. $(a - b)^2 = a^2 - 2ab + b^2$
14. $(a + b)(a - b) = a^2 - b^2$
15. $(x + a)(x + b) = x^2 + (a + b)x + ab$.
16. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
17. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)$
18. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)$
19. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
20. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

CHAPTER 4

COORDINATE GEOMETRY

'I think, therefore I am.' - **René Déscartes**

4.1 Introduction

A very beautiful and important branch of mathematics known as **Coordinate Geometry**, was initially developed by the French philosopher and mathematician **René Déscartes** (1596-1650).

We know how to describe the position of a point on the real line. Every real number is represented by a unique point on the number line and also, every point on the number line represents a unique real number. In other words there is a one-to-one correspondence between the points on the line and the set of all real numbers.

The real (number) line is given here :

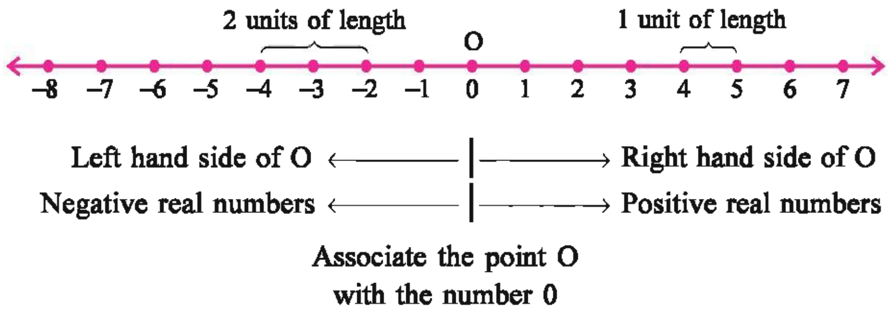


Figure 4.1

In a plane to describe the exact position of a point we need the reference of more than one line. For example, consider the following situation.

In figure 4.2, there is a main road running in the East-West direction and streets with numbering from West to East and house numbers from 1 to 5 are marked on each street.

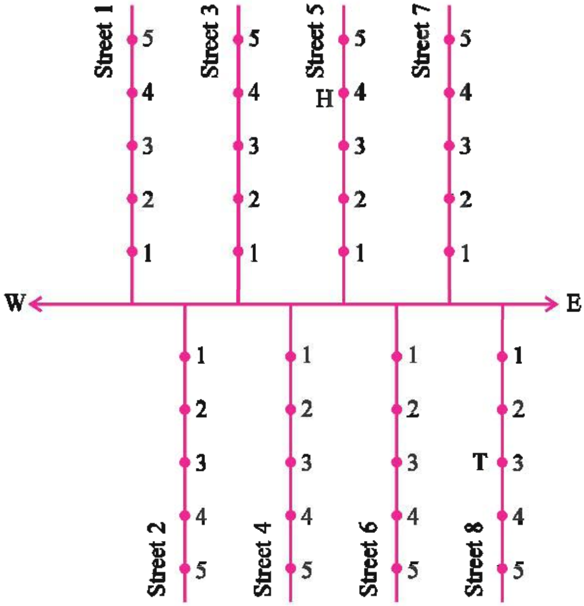


Figure 4.2

To look for a friend's house here, we need to know two points of information about it, namely the number of the street on which it is situated and the house number. If we want to reach the house, which is situated in the 5th street and has the number 4, first of all we would identify the 5th street and then the house numbered 4 on it. In figure. 4.2., H shows the location of the house. Similarly, T shows the location of the house corresponding to street number 8 and house number 3.

Two points are to be noted from this illustration :

To reach a definite place (1) one should start from a fixed point, and (2) the house number must be given on a specific perpendicular street.

Thus, we observe that to describe the position of any object lying in a plane, we need two perpendicular lines. This simple idea has given rise to a very important branch of mathematics known as Coordinate Geometry.


René Dēscartes
(1596-1650)

René Déscartes was born on 31st March, 1596 in LaHaye in South of France, the great French Mathematician of the seventeenth century, liked to lie in bed and think ! One day, when resting in bed, he solved the problem of describing the position of a point in a plane. His method was development of the older idea of latitude and longitude. He is credited as the father of analytic geometry. In honour of René Déscartes, the system used for describing the position of a point in a plane is also known as the *Cartesian system.* He died on 11th February, 1650, Stockholm, Sweden.

In honour of René Dēscartes, the system used for describing the position of a point in a plane is known as the cartesian coordinate system. In this chapter, we shall learn about it.

4.2 Cartesian Coordinate System

If A and B are non-empty subsets of the set R of real numbers, then the Cartesian product of non-empty sets A and B, symbolically represented by $A \times B$ (to be read : A cross B) is the set of all ordered pairs (a,b), where $a \in A$, $b \in B$. We write, $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Here, $A \times B$, $B \times A$, $A \times A$ and $B \times B$ all are subsets of $R \times R$. Also, they can be represented by a graph. This type of representation depends on a graph drawn in a plane. So, first of all we shall study the method of sketching a graph in a plane.

Draw two perpendicular lines in the plane; one horizontal and another vertical. The point of their intersection O is called the **origin**.

The horizontal line is called the X-axis and the vertical line is called the Y-axis. Both the axes are called the coordinate axes. The plane is called the **co-ordinate plane** or the **cartesian plane**, or the **XY-plane**.

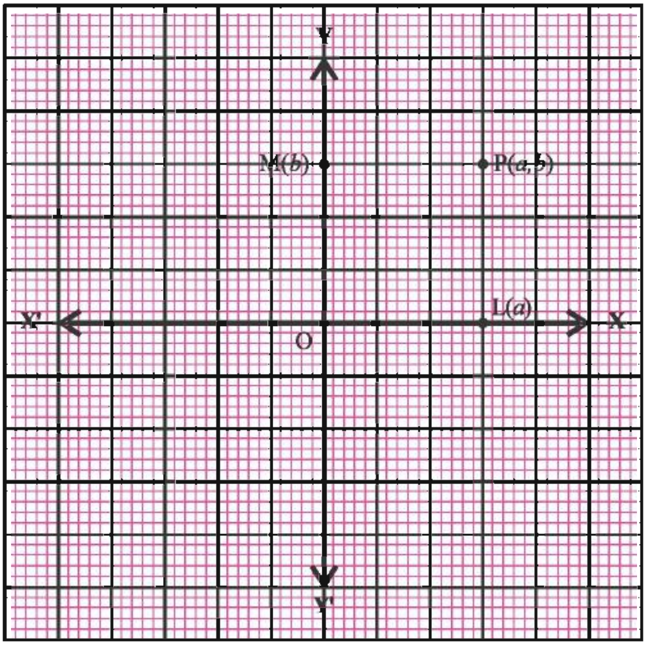


Figure 4.3

We know that there is a one-to-one correspondence between a line and the set of all real numbers.

Associate the point O on the X-axis with the number 0 (zero). Associate the points on the right hand side of O on the X-axis with positive real numbers. Associate the points on the left hand side of O on the X-axis with negative real numbers. Thus, corresponding to each point on the X-axis, there is a unique real number and conversely, corresponding to each real number, there is a unique point on the X-axis.

Similarly O on Y-axis corresponds to zero and points on Y-axis above semi plane of X-axis, correspond to positive real numbers and points on Y-axis below semi plane of X-axis correspond to negative real numbers. Thus, to every real number corresponds a point on Y-axis and conversely.

Now corresponding to each ordered pair (a, b) of $R \times R$, we get points $L(a)$ and $M(b)$ on X-axis and Y-axis respectively. If the lines perpendicular to X-axis from L and perpendicular to Y-axis from M intersect in P, then P is called the point corresponding to the ordered pair (a,b). It is denoted as $P(a,b)$. Conversely, for each point P of the plane, we get $L(a)$ and $M(b)$ on the coordinate axes (by drawing perpendiculars). (See figure 4.3)

Thus, corresponding to each ordered pair (a, b) there is a unique point P in the plane and corresponding to each point P in the plane, there is a unique ordered pair (a, b) in $R \times R$. a is called the **x-coordinate or abscissa** of P and b is called the **y-coordinate or ordinate** of P.

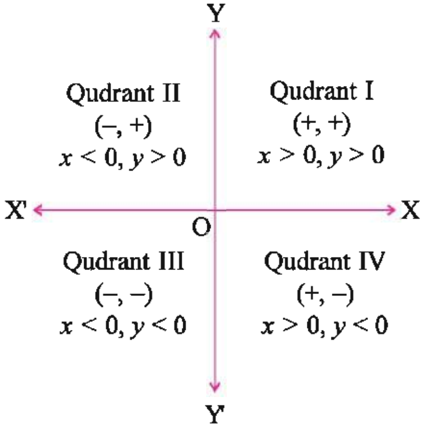


Figure 4.4

Quadrant : In the cartesian coordinate system, the perpendicular axes (i.e. coordinate axes) partition the plane into four parts. The plane is the union of points on axes and points in each subset (part). Each part is known as a quadrant. Each quadrant is named by numbers I, II, III and IV in anti-clockwise direction starting from $\overrightarrow{OX}$ (see Fig. 4.4)

The plane is the union set of the axes and four quadrants.

Considering the association of real numbers with coordinate axes, we get the following property of each quadrant :

Quadrant	part	x-co-ordinate	y-co-ordinate
First (I)	Interior of ∠XOY	+	+
Second (II)	Interior of ∠YOX'	–	+
Third (III)	Interior of ∠X'OY'	–	–
Fourth (IV)	Interior of ∠Y'OX	+	–

An ordered pair corresponding to a point in the plane : Suppose P is a point in the plane. The feet of perpendiculars from P to X-axis and Y-axis are M and N respectively. The unique real numbers associated with M and N are 3 and 4 respectively. So, the x-coordinate of P is 3 and the y-coordinate is 4. Thus, corresponding to P, there is a unique ordered pair (3, 4) of real numbers. (see figure 4.5)

Similarly by considering another point Q in the plane, the unique ordered pair corresponding to Q is (–3, –4). (See figure 4.5)

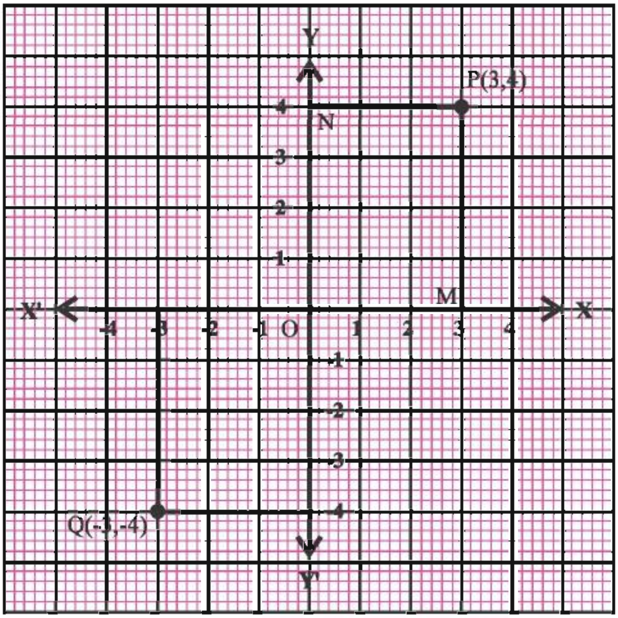


Figure 4.5

From this illustration, we can say that corresponding to each point in the plane there is a unique ordered pair of real numbers.

Coordinates of the origin are (0, 0). For a point on the X-axis with corresponding number *a* (called *x* coordinate or abscissa) *y*-coordinate is always 0. Thus, on the X-axis the points are of the form (*a*, 0); *a* is a real number. Similarly, the points on the Y-axis are of the form (0, *b*). Here, *b* is called *y*-coordinate or ordinate. *b* is a real number.

Example 1 : (a) Write the *x*-coordinate (abscissa) and the *y*-coordinate (ordinate) of the points A, B, C, D, P, Q, R, U, V and W in the figure 4.6. (b) Write the coordinates of all the points T, M, N and S from the figure 4.6

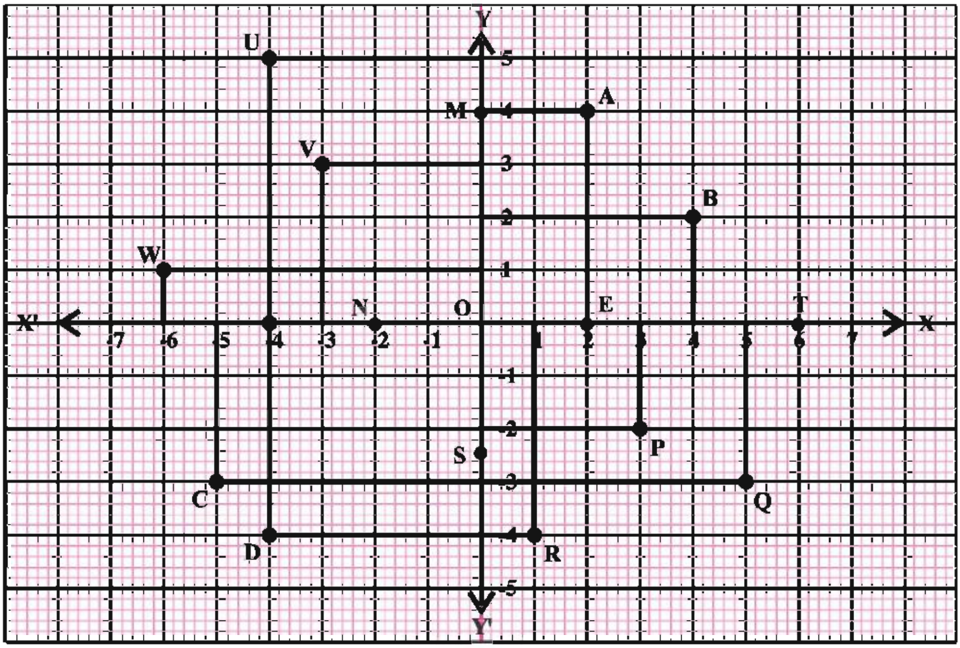


Figure 4.6

Solution : (a) The feet of perpendiculars from the point A to the Y-axis and X-axis are respectively M and E. The unique real numbers 2 and 4 are associated with E and M respectively.

∴ The *x*-coordinate of the point A is 2 and the y-coordinate of the point A is 4. So we write A(2,4). For the point B; the foot of perpendicular from the point B to the Y-axis is associated with the unique real number 2 and the foot of perpendicular from the point B to the X-axis is associated with the unique real number 4.

∴ 4 is the x-coordinate of the point B and 2 is the *y*-coordinate of the point B i.e. B(4,2). Now from the point U the feet of perpendiculars to the X-axis and to the

Y-axis are associated with the unique real numbers −4 and 5 respectively. Hence, for the point U, the x-coordinate is −4 and the y-coordinate is 5.

For the point D, we get unique numbers −4 on the X-axis and −4 on the Y-axis associated with the feet of perpendiculars from the point D. So the coordinates of D are (−4, −4), i.e. D(−4, −4). Similarly for the point P the *x*-coordinate of P is 3 and the *y*-coordinate of P is −2. i.e. P(3, −2). Similarly W(−6, 1), C(−5, −3), Q(5, −3), R(1, −4) and V(−3, 3).

(b) Similarly for the given points the coordinates are T(6, 0), M(0, 4), N(−2, 0) and $S\left(0, \frac{-5}{2}\right)$

EXERCISE 4.1

1. Answer as directed :
   (1) Write the name of the axes in the cartesian plane.
   (2) Give the name of each subset (part) of the plane partitioned by the axes.
   (3) Do the axes intersect ? If yes, give the name of the point of intersection and also write its coordinates.
2. See the figure 4.7 and answer the questions that follows :

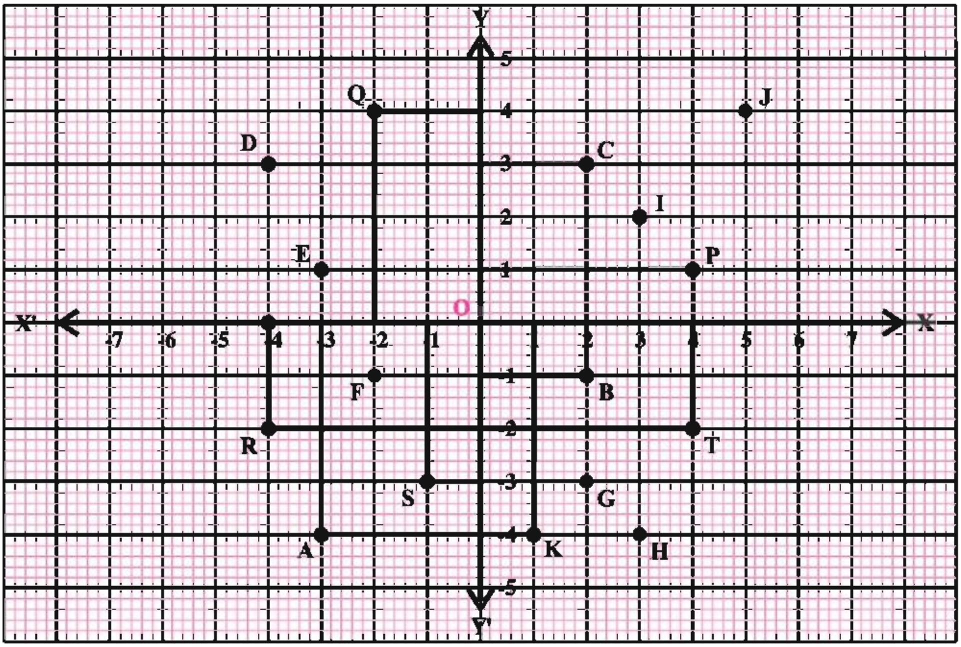


Figure. 4.7

(1) The coordinates of P and Q
(2) Point identified as (−2, 4).
(3) The abscissa of the point A.
(4) The ordinate of the point R.
(5) Write the coordinates of the points A, B, C, D, E, F, G, H, I, J, R, S and T.

*

4.3 Plotting a Point in the Plane if its Coordinates are Given

Let us obtain a point in the plane corresponding to the ordered pair (2, 3). The x-coordinate and y-coordinate are positive. On X-axis on the right side of O, there is a unique point M corresponding to 2. On Y-axis, in the upper half-plane there will be a unique point N corresponding to 3. Draw lines from M and N, perpendicular to X-axis and to Y-axis respectively. The unique point P of their intersection is the point in the plane corresponding to (2, 3).

Now let us represent graphically the point corresponding to the ordered pair (−2, −3) in the plane. Both the coordinates of (−2, −3) are negative. On X-axis, on the left hand side of O, there is a unique point A corresponding to −2 and on Y-axis, in the lower half plane of the X-axis, there is a unique point B corresponding to −3. Draw lines perpendicular to X-axis from A and to Y-axis from B respectively. Their point of intersection, the unique point Q, is the point in the plane corresponding to (−2, −3). Similarly (−2, 3) and (2, −3) are represented as points R and S respectively (See figure 4.8).

We have seen that, a point x-coordinate of which is zero lies on the Y-axis and a point y-coordinate of which is zero lies on the X-axis. T represents (1, 0) and F represents (0, 2)

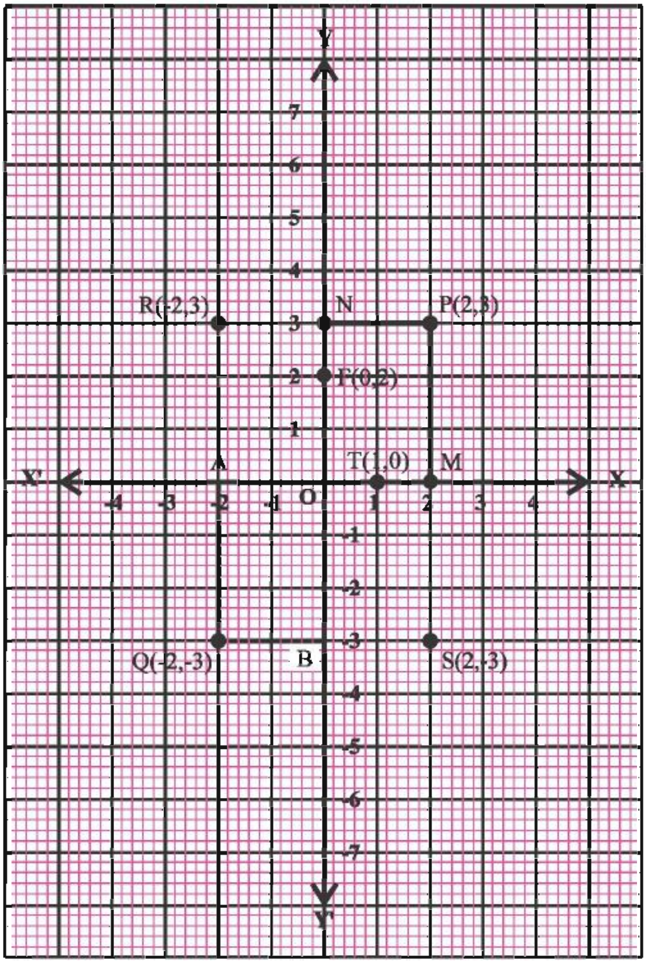


Figure. 4.8

From this illustration, we can say that to each ordered pair of real numbers, a unique point of the plane is associated. **(i)**

We have also seen that corresponding to each point in the plane there is a unique ordered pair of real numbers. **(ii)**

From (i) and (ii) we can say that there is a one-one correspondence between the plane and R × R and if a point P of the plane and the ordered pair (x, y) correspond to each other, then we write P (x, y).

P is called the representation of (x, y) in the plane and x and y are called cartesian co-ordinates of P. x is called the x-coordinate and y is called the y-coordinate of P. In fact, we identify P and (x, y) and say that (x, y) (like P) is a point of the plane.

By drawing graph of a set A × B we mean plotting of points of A × B in the Cartesian plane.

Example 2 : Locate the point corresponding to ordered pairs.

(−3, 4), (−3,−1), (4, 0), (0, 5), (1, −2) and (2, 3) in the Cartesian plane.

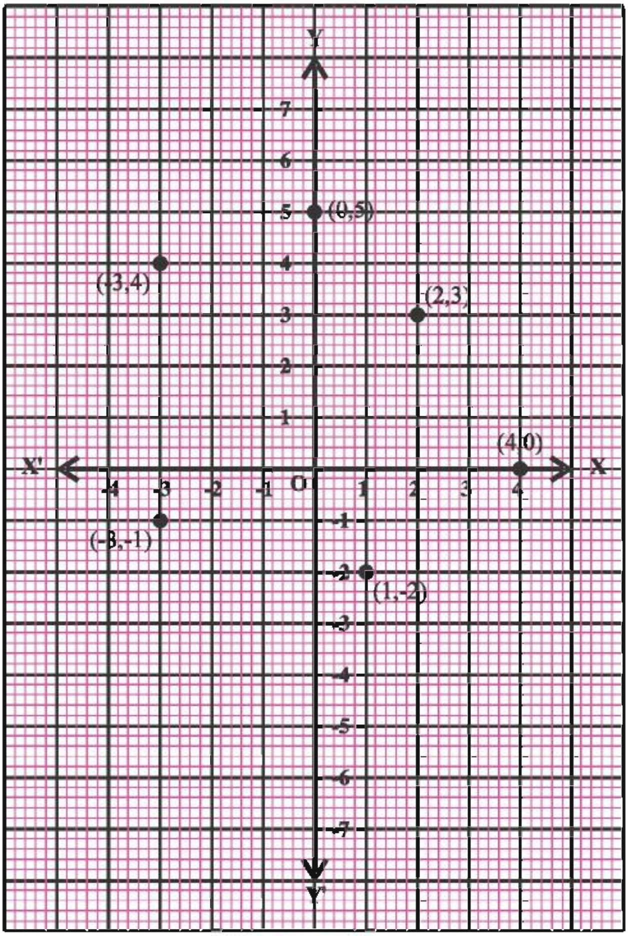


Figure. 4.9

Solution : Taking the scale 1 cm = 1 unit on the axes draw the X-axis and Y-axis on the graph paper. The positions of the points are shown by dots in the Fig. 4.9

Note : For the ordered pairs (a, b) and (p, q), $(a, b) = (p, q)$ if and only if $a = p$ and $b = q$. For example, let us find x and y, if $(5, 4y - 1) = (3x - 4, 7)$

Here, $(3x - 4, 7) = (5, 4y - 1)$

$\therefore$ $3x - 4 = 5$ and $4y - 1 = 7$

$\therefore$ $3x = 5 + 4$ and $4y = 7 + 1$

$\therefore$ $3x = 9$ and $4y = 8$

$\therefore$ $x = \frac{9}{3}$ and $y = \frac{8}{4}$

$\therefore$ $x = 3$ and $y = 2$

EXERCISE 4.2

1. Plot the following ordered pairs (x, y) in the plane :
   $(-4, -3)$, $(-3, 5)$, $(-2, -4)$, $(-1, 6)$, $(0, 2)$, $(1, -3.5)$, $(2, 3)$, $(4, -2)$.
2. Plot the points (x, y) in the cartesian plane obtained by taking values of x in the polynomial $y = 3x - 2$, $x = -3, -2, -1, 0, 1, 2, 3, 4$.
3. If $P = \{0, 1, -1\}$ and $Q = \{-3, 2\}$, then draw the graph of $P \times Q$ and $Q \times P$.
4. If $A = \{-2, 3\}$ and $B = \{-1, 1, 4\}$, then draw the graphs of
   (1) $A \times B$ (2) $B \times A$ (3) $A \times A$ (4) $B \times B$
5. Plot the points A(4, 5), B(−2, −1), C(−3, 6) and D(5, −2). From the graph, find the midpoints of $\overline{AB}$ and $\overline{CD}$.
6. Represent the points M(3, 4), N(−3, −2), P(−2, 5) and Q(4, −1) in the plane. Draw $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. From the graph, find their point of intersection.
7. Examine the validity of the following statements :
   (1) Point (4, 0) lies on the X-axis.
   (2) P(−2, 3) is a point in the third quadrant.
   (3) For the point A, if the abscissa is 4 and the ordinate is −3, then A lies in the fourth quadrant.
   (4) The point of intersection of the axes has co-ordinates (0, 0).
   (5) In the plane the position of (y, x) is the same as the position of (x, y), where $x \neq y$.
   (6) B(0, −9) is a point on $\overrightarrow{OY}$.
   (7) For $x = 3$, $y = 2$, $u = -7$, $v = 11$ the point $(x - u, y - v)$ lies in the 1st quadrant.
   (8) Point (4, −5) lies in the lower half-plane of the X-axis and to the right hand side of Y-axis.

(9) If $(2x - 1, 1)$ and $(-3, 3y - 2)$ are two equal ordered pairs of $R \times R$, then $x = -1$ and $y = 1$.

EXERCISE 4

1. In which quadrant do the following points lie ?

(1) $(2, 4)$ (2) $(3, -5)$ (3) $(-5, -\frac{3}{2})$ (4) $(-5, 2)$ (5) $\left(\frac{1}{2}, \frac{-1}{2}\right)$ (6) $(1, 3)$

2. Plot the following points on a graph paper :

(1) $(3, -2)$ (2) $(-2, 3)$ (3) $(-5, 4)$ (4) $(-4, -3)$ (5) $(0, 4)$ (6) $(3, 0)$

3. If $A = \{-3, 2, 4\}$ and $B = \{-1, 1\}$, then draw the graphs of

(1) $A \times B$ (2) $A \times A$ (3) $B \times B$ (4) $B \times A$

4. Write the coordinates of each of the following points marked in the graph paper (figure 4.10).

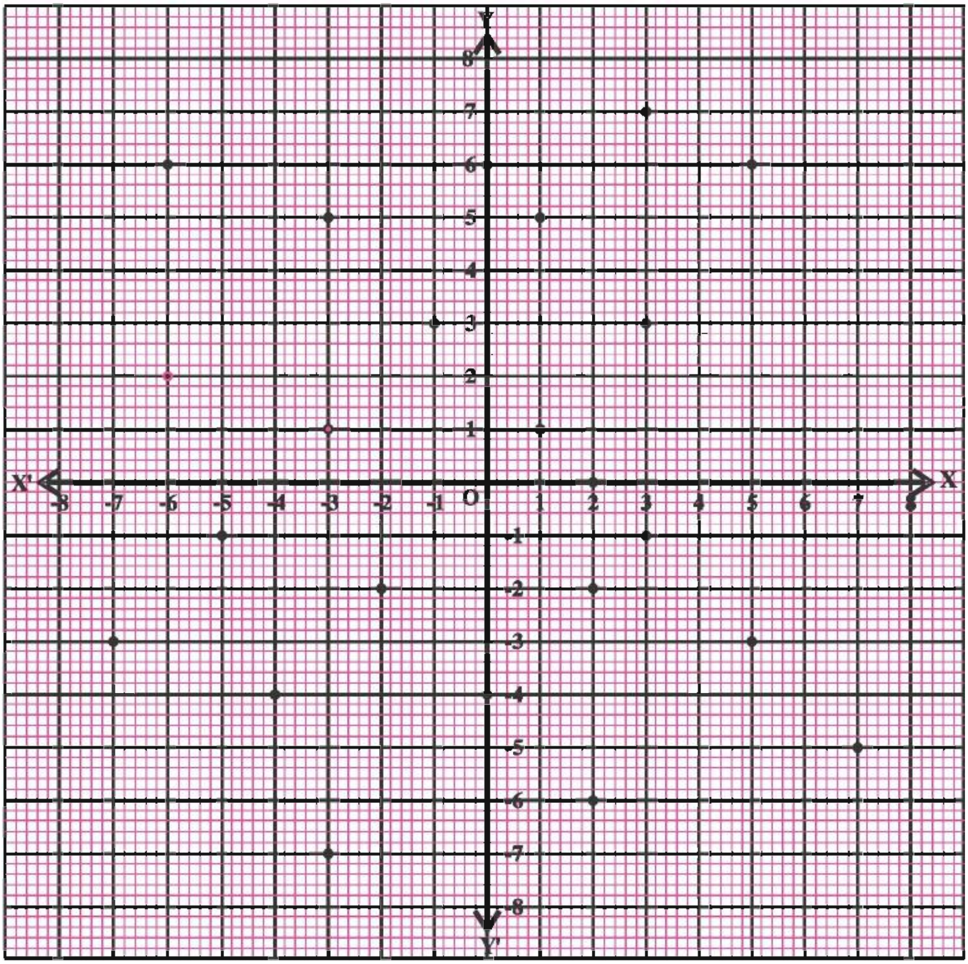


Figure 4.10

5. If $x = -1$, $y = 5$, $z = 3$, $w = -4$, then in which quadrants do the points $(x + y, z + w)$, $(y - z, w + x)$ and $(x - w, y + z)$ lie ?

6. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Point (4, 0) lies on ......

(a) $\overrightarrow{OX'}$ (b) $\overrightarrow{OY}$ (c) $\overrightarrow{OX}$ (d) $\overrightarrow{OY'}$

(2) For a point, if the abscissa is −3 and the ordinate is 5, then it lies in the ...... quadrant.

(a) I (b) II (c) III (d) IV

(3) The point of intersection of the axes has co-ordinates ....

(a) (0, 1) (b) (1, 0) (c) (0, 0) (d) (0, −1)

(4) The point (−2, 0) lies on ......

(a) $\overrightarrow{OY}$ (b) $\overrightarrow{OX'}$ (c) 1^{st} quadrant (d) $\overrightarrow{OX}$

(5) Point (5, −2) lies in the ...... quadrant.

(a) I (b) II (c) III (d) IV

(6) For the point (7, −4), the abscissa is ......

(a) −4 (b) −7 (c) 4 (d) 7

(7) For the point (3, −5), the ordinate is ......

(a) 3 (d) 5 (c) −3 (d) −5

(8) For the origin O, abscissa and ordinate are both ......

(a) 1 (b) −1 (c) 0 (d) 0.5

(9) The 3rd quadrant is the interior of ......

(a) ∠YOX' (b) ∠X'OY' (c) ∠Y'OX (d) ∠XOY

(10) The coordinates of any point on the Y-axis are of the form $(0, b)$, where $|b|$ is the distance of the point from the ......

(a) Y-axis (b) X-axis (c) (0, 1) (d) (1, 0)

(11) The measure of the angle between the $\overleftrightarrow{X'X}$ and $\overleftrightarrow{Y'Y}$ is ......

(a) 90 (b) 0 (c) 180 (d) 60

(12) For $x = 3$, $y = 2$, $u = -9$, $v = 13$ the point $(x + y, u + v)$ lies in the ...... quadrant.

(a) III (b) II (c) IV (d) I

(13) In the plane, $(x, y) = (y, x)$ if ......

(a) $x = 3, y = 3$ (b) $x = 3, y = 2$ (c) $x = 2, y = 3$ (d) $x = 1, y = 0$

(14) If the co-ordinates of the points are of the same sign (both positive or both negative), then points lie in the ...... quadrants.

(a) I and II (b) I and III (c) I and IV (d) II and IV

(15) The point having coordinates of the opposite signs lies in ......

(a) I and II (b) I and III (c) I and IV (d) II and IV

(16) Any point on the X-axis is of the type ......

(a) $(0, x)$ (b) $(0, y)$ (c) $(0, 1)$ (d) $(a, 0)$

(17) The coordinate axes divide plane into ...... parts called quadrants.

(a) two (b) five (c) four (d) six

(18) X-axis is a horizontal line passing through ......

(a) Point (0, 1) (b) origin (c) Point (0, −1) (d) quadrant I

(19) The vertical line through the origin is called the ......

(a) X-axis (b) XY-plane (c) Y-axis (d) $\overleftrightarrow{Y'Y}$

(20) The ...... quadrant is bounded by the $\overrightarrow{OX'}$ and the $\overrightarrow{OY}$.

(a) 1st (b) 3rd (c) 2nd (d) 4th

(21) In the plane origin O (0,0) lies on the ......

(a) X-axis only (b) Y-axis only

(c) 1st quadrant (d) X-axis and Y-axis both

(22) The point (0, 3) lies on the ......

(a) X-axis (b) $\overleftrightarrow{Y'Y}$ (c) 1st quadrant (d) 2nd quadrant

(23) The point (−4, 0) lies on the ......

(a) 2nd quadrant (b) $\overrightarrow{OX}$ (c) 3rd quadrant (d) $\overrightarrow{OX'}$

(24) The point (0, −2) lies on the ......

(a) Y-axis (b) X-axis

(c) 1st and 4th quadrant (d) 3rd quadrant

(25) The point (−3, 4) lies in the ......

(a) 1st quadrant (b) 3rd quadrant

(c) interior of $\angle YOX'$ (d) interior of $\angle Y'OX$

*

Summary

In this chapter, you have studied the following points :

1. To locate the position of an ordered pair or a point in a plane, we require coordinate axes, namely, X-axis (the horizontal line) and Y-axis (the vertical line).
2. The plane is called the cartesian plane or coordinate plane or cartesian coordinate plane.
3. The point of intersection of the axes is called the origin O (0, 0).
4. If the x-coordinate (the abscissa) is a and the y-coordinate (the ordinate) is b, then a and b are called the coordinates of the point.
5. The coordinate axes divide the plane into four parts called quadrants.
6. On the X-axis every point is of the form $(x, 0)$ and on the Y-axis every point is of the form $(0, y)$. x and y are real numbers.
7. If $x \neq y$, then $(x, y) \neq (y, x)$ and $(x, y) = (y, x)$, if $x = y$

Birth Place	:	La Haye en Touraine, Touraine (present-day Descartes, Indre-et-Loire), France
Era	:	17th-century philosophy
Region	:	Western Philosophy
School	:	Cartesianism, Rationalism, Foundationalism
Main interests	:	Metaphysics, Epistemology, Mathematics
Notable ideas	:	Cogito ergo sum, method of doubt, Cartesian coordinate system, Cartesian dualism, ontological argument for the existence of Christian God; Folium of Descartes
Signature	:	

René Déscartes

CHAPTER 5

LINEAR EQUATIONS IN TWO VARIABLES

5.1 Introduction

In earlier classes, we have studied linear equations in one variable of the form $ax + b = c$ (where a, b, c are constants and $a \neq 0$). Such equations have a unique (i.e one and only one) solution.

Here, $ax + b = c$

So, $ax = c - b$

$\therefore$ Since $a \neq 0$, $x = \dfrac{c-b}{a}$ is the solution of the equation $ax + b = c$.

For example $x - 2 = 0$, $x + \sqrt{3} = 0$ and $\sqrt{5}x - \sqrt{7} = 0$ are linear equations in one variable. These equations have a unique solution.

Linear equations in one variable of the form $ax + b = cx + d$

(where $a, b, c, d \in R$, $a \neq c$) have a unique solution.

Linear equations in one variable of the form $\dfrac{ax+b}{cx+d} = k$; k is constant (where, $cx + d \neq 0$, $a, b, c, d \in R$, $a \neq kc$) have a unique solution.

Now, let us consider a practical problem related to the linear equation given by the statement. "The sum of the ages of two friends is 27 years and the difference of their ages is 3 years." Find their ages.

It can be translated into an equation form as follows. Suppose the older friend has age x years, then the other has age $(x - 3)$ years. Since sum of ages of both is 27.

$\therefore$ $x + (x - 3) = 27$

$\therefore$ $2x - 3 = 27$. This is a linear equation in one variable.

In this chapter, we shall recall the knowledge of linear equations in one variable and will extend it to linear equations in two variables. We shall also discuss whether the solution of a linear equation in two variables is unique or not and how the solution can be represented in the cartesian coordinate plane.

5.2 Linear Equations

Consider the linear equation $2x - 7 = 0$ in one variable.

Its solution (i.e. root of the equation) is $x = \frac{7}{2}$ and it can be represented on the number line as shown below :

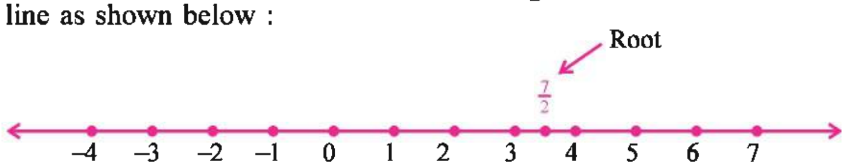


Figure 5.1

Linear Equations in Two Variables : "The sum of two numbers is 6", we shall express this statement in the form of an equation. Let these numbers be x and y.

Then the statement is translated in symbols as $x + y = 6$.

This is an equation having two variables x and y. Since, **the exponent of each variable is 1 and there is no product term xy, such an equation is called a linear equation in two variables.** It is customary to denote the variables in such equations by x and y, but other letters like p and q, r and s, u and v etc. may also be used. Some examples of linear equations in two variables are :

$3p + 2q = 12$, $2.5r + 4s = 7$, $6u + \pi v = 9$ and $\sqrt{3}x - 5y = 4$

The equations $x + y = 6$, $3x + 2y + 4 = 0$, $x - 2y + 3 = 0$ and $5x + \frac{3}{2}y = 4$ are linear equations in two variables x and y.

In these equations there are two variables and each variable occurs to index 1. The linear equation $3x - 2 = 5$ is an equation in one variable. It can also be expressed as an equation $3x + 0y - 7 = 0$ in two variables x and y.

Similarly, $4y - 3 = 0$ can be expressed as $0x + 4y - 3 = 0$; $5x = 0$ as $5x + 0y = 0$ and $3y = 0$ as $0x + 3y = 0$.

Thus, **each linear equation in one variable can be expressed in the form of a linear equation in two variables. The standard form of a linear equation in two variables is $ax + by + c = 0$, where $a, b, c \in$ R, a and b are not zero simultaneously, i.e. $a^2 + b^2 \neq 0$.**

Note : $a^2 \geq 0$ and $b^2 \geq 0$. So $a^2 + b^2 \geq 0$. If $a^2 + b^2 = 0$, then $a = b = 0$. Hence $a^2 + b^2 \neq 0$ means a and b are not simultaneously zero.

Now onwards whenever we consider a linear equation $ax + by + c = 0$ in two variables, we shall accept that $a, b, c \in R$ and a, b are not simultaneously zero, even if it is not explicitly mentioned.

Example 1 : State which of the following equations are linear equations in two variables and indicate the values of a, b and c in each case.

(1) $4x + 7y = 5$ (2) $3x = 6$ (3) $2 = 3x - 5y$ (4) $2x^2 = 3y$

(5) $6y = 0$ (6) $xy = 3$ (7) $x^2 + 3x + 2 = 0$

(8) $y - 3 = \sqrt{2}x$ (9) $7x = 8y$ (10) $4x + \frac{7}{3}y = \frac{11}{3}$

Solution : The standard of a linear equation in two variables is $ax + by + c = 0$.

(1) $4x + 7y = 5$ is a linear equation in two variables.
The standard form is $4x + 7y - 5 = 0$ where, $a = 4$, $b = 7$, $c = -5$.

(2) $3x = 6$ is a linear equation in two variables.
It can be written as $3x + 0y - 6 = 0$ in the standard form.
Here, $a = 3$, $b = 0$, $c = -6$.

(3) $2 = 3x - 5y$ is a linear equation in two variables.
It can be written as $3x - 5y - 2 = 0$ in the standard form.
Here, $a = 3$, $b = -5$, $c = -2$.

(4) $2x^2 = 3y$ is not a linear equation, as exponent of variable x is 2.

(5) $6y = 0$ is a linear equation in two variables.
It can be written as $0x + 6y + 0 = 0$ in the standard form.
Here, $a = 0$, $b = 6$, $c = 0$

(6) $xy = 3$ is not a linear equation in two variables as it is not in standard form $ax + by + c = 0$. Note that indices of x and y are 1 and xy has index $1 + 1 = 2$.

(7) $x^2 + 3x + 2 = 0$ is not a linear equation because in x^2, the index of variable x is 2.

(8) $y - 3 = \sqrt{2}x$ is a linear equation in two variables.
It can be written as $\sqrt{2}x - y + 3 = 0$ in the standard form.
Here, $a = \sqrt{2}$, $b = -1$, $c = 3$.

(9) $7x = 8y$ is a linear equation in two variables.
It can be written as $7x - 8y + 0 = 0$ in the standard form.
Here, $a = 7$, $b = -8$, $c = 0$

(10) $4x + \frac{7}{3}y = \frac{11}{3}$ is a linear equation in two variables.
It can be written as $12x + 7y - 11 = 0$ in the standard form.
Here, $a = 12$, $b = 7$, $c = -11$.
or $4x + \frac{7}{3}y - \frac{11}{3} = 0$, where $a = 4$, $b = \frac{7}{3}$, $c = -\frac{11}{3}$.

EXERCISE 5.1

1. "The cost of a notebook is twice the cost of a pen." Represent this statement as a linear equation in two variables.
2. State which of the following equations are linear equations in two variables and express them in the standard form $ax + by + c = 0$ and indicate the values of a, b and c in each case :

   (1) $5x = 6y$ (2) $y^2 = 3x$ (3) $7x = 0$ (4) $6y - 4x = 3$

   (5) $3x + 4.5y = 8.2$ (6) $y - \frac{x}{4} - 3 = 0$ (7) $9x = 3$ (8) $3x = 2y - 4$

   (9) $y = 2x + 5$ (10) $\frac{3}{2}x + \frac{7}{2}y = 1$ (11) $3y^2 + 2x = 2$ (12) $\frac{x}{3} - \frac{4}{y} = 2$

*

5.3 Solution of a Linear Equation in Two Variables

We have seen that every linear equation in one variable has a unique solution. What can be said about the solution of a linear equation in two variables ? As there are two variables in the equation, a solution means a pair of values, one for x and one for y which will satisfy the given equation. Let us consider the equation $x + y = 6$.

Taking $x = 4$ and $y = 2$, we get

$\therefore \quad x + y = 6$

Thus, $x = 4$ and $y = 2$ is a solution of the equation $x + y = 6$ or (4, 2) is a solution of this equation. Similarly, ordered pairs (3, 3), (1, 5), (2, 4), (5, 1), (–1, 7), (–2, 8) etc. also satisfy the equation $x + y = 6$. All these ordered pairs are solutions of this equation. It is not necessary that the solutions are in integers.

If we take $x = \frac{3}{2}$, $y = \frac{9}{2}$, then $x + y = \frac{3}{2} + \frac{9}{2} = \frac{3+9}{2} = \frac{12}{2} = 6$.

Similarly, the ordered pairs $\left(\frac{5}{3}, \frac{13}{3}\right)$, $\left(\frac{1}{2}, \frac{11}{2}\right)$, $\left(\frac{1}{3}, \frac{17}{3}\right)$, $\left(\frac{2}{3}, \frac{16}{3}\right)$, $\left(\frac{1}{4}, \frac{23}{4}\right)$ are also solutions of $x + y = 6$.

Further, taking $x = \frac{\sqrt{3}}{2}$, $y = \frac{12-\sqrt{3}}{2}$ we get

$$x + y = \frac{\sqrt{3}}{2} + \frac{12-\sqrt{3}}{2} = \frac{\sqrt{3}+12-\sqrt{3}}{2} = \frac{12}{2} = 6$$

So, $\left(\frac{\sqrt{3}}{2}, \frac{12-\sqrt{3}}{2}\right)$ is also a solution.

Further $(\sqrt{3}, 6 - \sqrt{3})$, $(\sqrt{5}, 6 - \sqrt{5})$, $(\pi, 6 - \pi)$ are also solutions of the equation $x + y = 6$

Thus, ordered pairs (6, 0), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (0, 6), (–1, 7), $\left(\frac{9}{2}, \frac{3}{2}\right)$, $\left(\frac{17}{3}, \frac{1}{3}\right)$, (7, –1), $(\sqrt{5}, 6 - \sqrt{5})$, $(\sqrt{3}, 6 - \sqrt{3})$, $(\pi, 6 - \pi)$ etc. are solutions of $x + y = 6$. So, there is no end to different solutions of a linear equation in two variables.

Hence, we can say that a linear equation in two variables has infinite number of solutions. (or infinitely many solutions).

We have studied in the chapter of coordinate geometry how to plot a point on a graph paper. Let us plot the solutions (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) of $x + y = 6$ on a graph paper.

These points are collinear. If we join them by a straight edge, we get the graph of $x + y = 6$. Thus we see that the graph of $x + y = 6$ is a line (see figure 5.2).

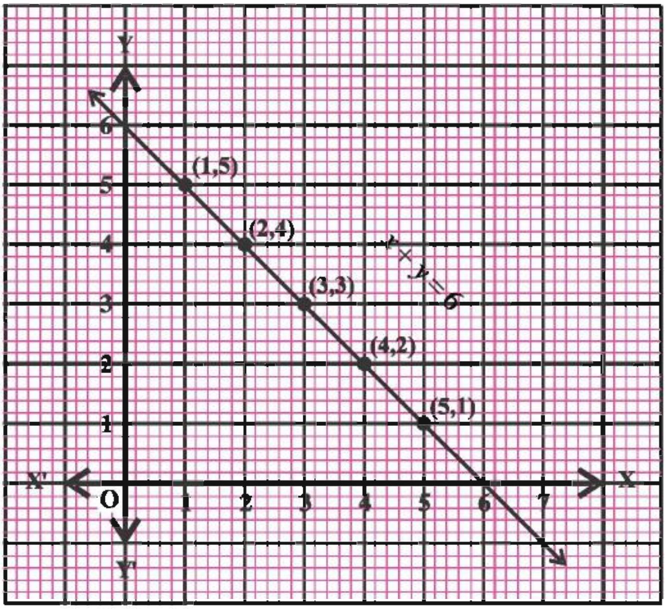


Figure 5.2

So, that is why the equation $x + y = 6$ is called a linear equation.

For real numbers x and y, if (x, y) satisfies the equation $ax + by + c = 0$, then (x, y) is called a solution of $ax + by + c = 0$. $a, b, c \in R$ and $a^2 + b^2 \neq 0$.

Therefore, **the set $\{(x, y) \mid ax + by + c = 0;\ x, y \in R\}$ is a solution set of the linear equation $ax + by + c = 0$ in two variables.**

Example 2 : Find any three elements of the solution set of the equation $4x + 3y = 12$.

Solution : The equation $4x + 3y = 12$ gives,

$\therefore \quad 3y = 12 - 4x$

$\therefore \quad y = \frac{12 - 4x}{3}$

For $x = 0$, $y = \frac{12 - 4(0)}{3} = \frac{12 - 0}{3} = \frac{12}{3} = 4$. So, (0, 4) is a solution of the equation.

For $x = 1$, $y = \frac{12 - 4(1)}{3} = \frac{8}{3}$. So, $\left(1, \frac{8}{3}\right)$ is a solution of the equation.

For $x = 3$: $y = \frac{12 - 4(3)}{3} = \frac{12 - 12}{3} = \frac{0}{3} = 0$. So, another solution is (3, 0).

Thus, (0, 4), $\left(1, \frac{8}{3}\right)$ and (3, 0) are three elements of the solution set of the equation $4x + 3y = 12$.

Remark : Note that an easy way of getting a solution is to take $x = 0$ and get the corresponding value of y. Similarly, we can put $y = 0$ and obtain the corresponding value of x.

Example 3 : Find four different solutions of the equation $2x + y = 6$.

Solution : The given equation is $2x + y = 6$

$2x + y = 6$

$\therefore \quad y = 6 - 2x$

For $x = 0, y = 6 - 2\,(0) = 6 - 0 = 6$

For $x = -1, y = 6 - 2\,(-1) = 6 + 2 = 8$

For $x = \frac{1}{2}, y = 6 - 2\left(\frac{1}{2}\right) = 6 - 1 = 5$

For $x = 3, y = 6 - 2\,(3) = 6 - 6 = 0$

Thus , (0, 6), (–1, 8), $\left(\frac{1}{2}, 5\right)$ and (3, 0) are four different solutions of the equation $2x + y = 6$.

Example 4 : Show that $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}-5}{3}\right)$ are solutions of the equation $2x - 3y - 5 = 0$.

Solution : For the point $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$, taking $x = \sqrt{3}$, $y = \frac{2\sqrt{3}-5}{3}$ we have

$$2x - 3y - 5 = 2\sqrt{3} - 3\left(\frac{2\sqrt{3}-5}{3}\right) - 5$$

$$= 2\sqrt{3} - (2\sqrt{3} - 5) - 5$$

$$= 2\sqrt{3} - 2\sqrt{3} + 5 - 5 = 0$$

Thus, the equation $2x - 3y - 5 = 0$ is verified for $x = \sqrt{3}$, $y = \frac{2\sqrt{3}-5}{3}$

So, $\left(\sqrt{3}, \frac{2\sqrt{3}-5}{3}\right)$ is a solution of the linear equation $2x - 3y - 5 = 0$.

Substituting $x = \frac{\sqrt{3}}{2}$ and $y = \frac{\sqrt{3}-5}{3}$, we have

$$2x - 3y - 5 = 2 \cdot \left(\frac{\sqrt{3}}{2}\right) - 3 \cdot \left(\frac{\sqrt{3}-5}{3}\right) - 5$$

$$= \sqrt{3} - (\sqrt{3} - 5) - 5$$

$$= \sqrt{3} - \sqrt{3} + 5 - 5$$

$$= 0$$

So, $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}-5}{3}\right)$ is a solution of the linear equation $2x - 3y - 5 = 0$

EXERCISE 5.2

1. Find five different solutions of each of the equations.

   (1) $2x = 3y + 5$ (2) $6y = 9$

2. Find two solutions for each of the equations.

   (1) $3x + 4y = 12$ (2) $5x - 2y = 0$ (3) $3x + 5 = 0$ (4) $\pi x + y = 4$

3. Find three elements of the solution set of the following equations.

   (1) $3x - 2y = 3$ (2) $2x = 4$ (3) $6y = 15$ (4) $5x + 3y = 0$

   (5) $3x + 4y = 6$ (6) $x + y = 0$ (7) $x - y = 0$ (8) $6x + 3y = 9$

4. Which one of the following options is true and why ?

   $3y = 2x + 7$ has...

   (1) a unique solution (2) only two solutions (3) infinitely many solutions

5. Examine which of the following points are solutions of the equation $2x - y = 5$ and which are not :

   (1) $(3, 1)$ (2) $(-2, -9)$ (3) $(0, 5)$ (4) $(5, 0)$ (5) $(0, -5)$ (6) $(4, 2)$

   (7) $(2, 1)$ (8) $\left(\frac{-1}{2}, \frac{-11}{2}\right)$ (9) $(1 + \sqrt{2}, -3 + 2\sqrt{2})$ (10) $(1, -6)$

6. Find the value of k in each of the following questions :

   (1) $x = 1, y = 2$ is a solution of the equation $3x - 2y = k$

   (2) $x = 1, y = 3$ is a solution of the equation $3x + ky = 9$

   (3) $kx + 5y = 11$ has a solution $(4, -1)$

   (4) $(2, 5)$ is a solution of the equation $4x + ky = 13k$

*

5.4 Graph of a Linear Equation in Two Variables

We know that a solution (x, y) of the linear equation $ax + by + c = 0$ in two variables is a point of the coordinate plane. If all the solutions are plotted in the plane, then they are collinear. Joining them by a straight edge, we get a line. This line is the graph of linear equation in two variables.

Since there are infinitely many solutions of a linear equation in two variables, it is not possible to plot all the solutions. As the graph of a linear equation in two variables is a line, it is enough to plot two ordered pairs of the solutions, then by joining them with a straight edge we can get the graph. We know that two points determine a line, however we shall plot at least three elements and prepare a graph using these points.

Remark : $ax + by + c = 0$ is a polynomial equation of degree one in two variables x and y. The equation $ax + by + c = 0$ is called a linear equation, simply because, its geometrical representation is a straight line.

Example 5 : Draw the graph of $2x - 3y = 0$

Solution : Here, $2x - 3y = 0$

$\therefore \quad 3y = 2x$

$\therefore \quad y = \frac{2x}{3}$

For $x = 0$, $y = \frac{2(0)}{3} = \frac{0}{3} = 0$

For $x = 3$, $y = \frac{2(3)}{3} = \frac{6}{3} = 2$

For $x = -3$, $y = \frac{2(-3)}{3} = \frac{-6}{3} = -2$

Three elements of the solution set of $2x - 3y = 0$ are

x	-3	0	3
y	-2	0	2

Plot the points $(-3, -2)$, $(0, 0)$, $(3, 2)$ on a graph paper.

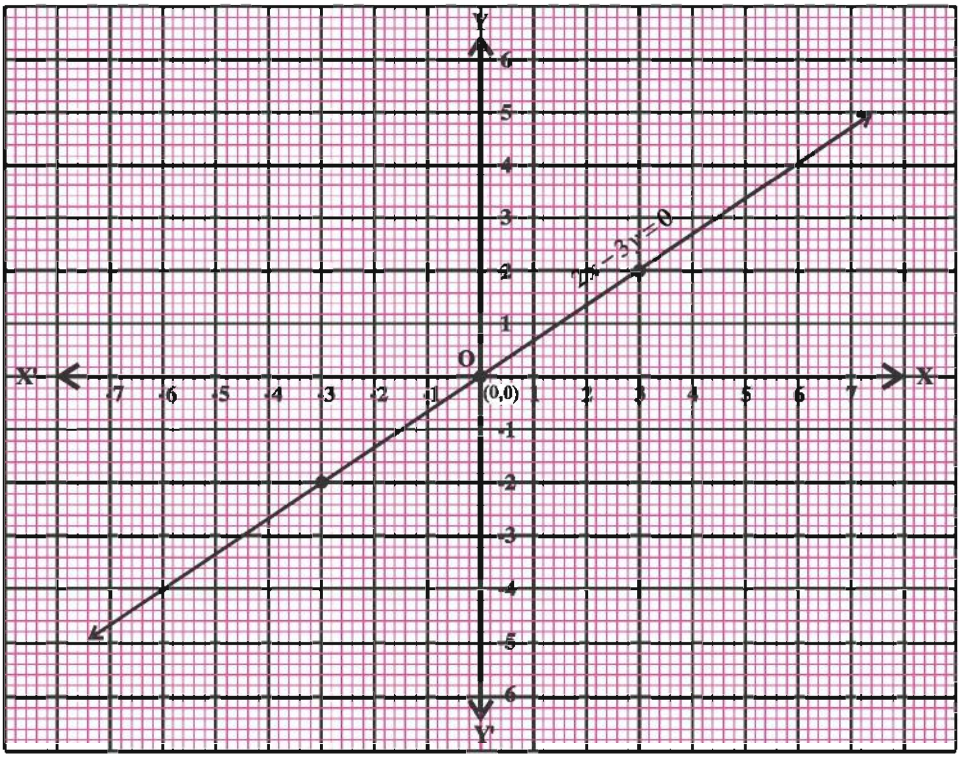


Figure 5.3

These points are collinear. Joining them by a straight edge, we get a line (see figure 5.3).

Observe that the graph of $2x - 3y = 0$ is a line passing through the origin O. If the constant term c is zero then the graph of the equation $ax + by + c = 0$ is always a line passing through the origin.

Example 6 : Draw the graph of $x = 0$

Solution : The equation $x = 0$ can be written as $x + 0y = 0$. In this equation coefficient of y is zero.

So, for any value of y we get $x = 0$

$\therefore$ (0, 1), (0, 4) and (0, −2) are three elements of the solution set. Plotting these points on the graph paper, it can be seen that all the three points are on the Y-axis.

i.e. the graph of $x = 0$ is the Y-axis (see Fig. 5.4)

The line $x = 0$ is the Y-axis.

Similarly, the equation $y = 0$

i.e. $0x + y = 0$ has solutions (1, 0), (2, 0), (3, 0), (−1, 0), (−3, 0), $\left(\frac{3}{2}, 0\right)$, $\left(\frac{5}{3}, 0\right)$,...

Plotting these points on a graph paper, we can see that the graph of the equation $y = 0$ is the X-axis (see figure 5.4). **Thus, the graph of the equation $x = 0$ is the Y-axis and the graph of the equation $y = 0$ is the X-axis.**

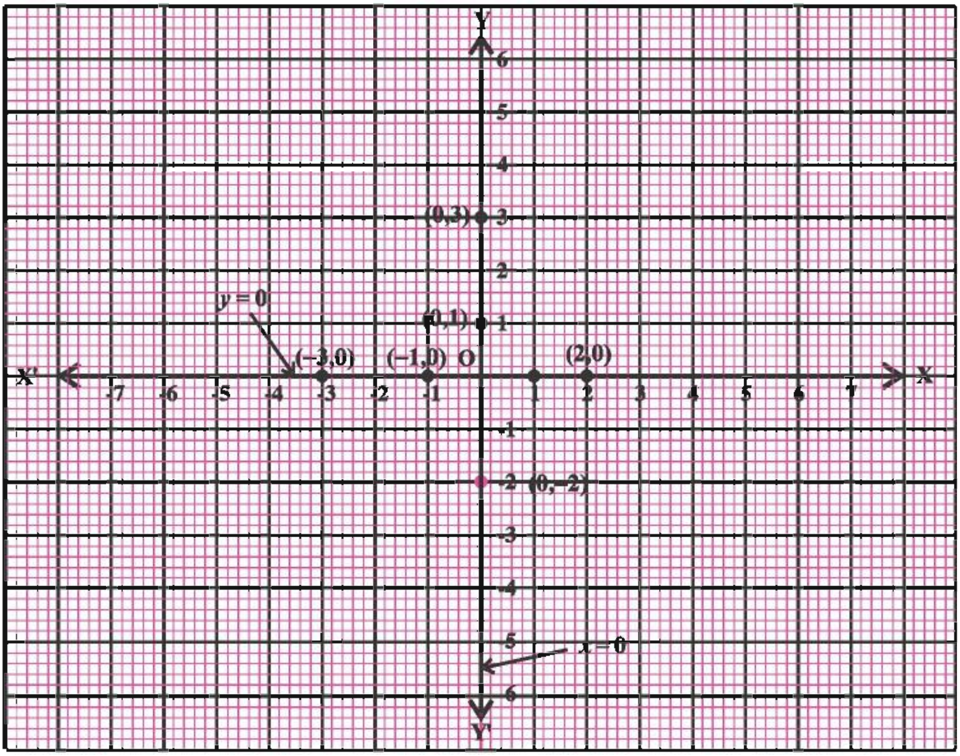


Figure 5.4

Example 7 : Draw the graph of $3x + 3y = 12$

Solution : Here the equation $3x + 3y = 12$ is given.

$\therefore \quad x + y = 4$ (Dividing by 3)

$\therefore \quad y = 4 - x$

For $x = 0,\ y = 4 - 0 = 4$

For $x = 4,\ y = 4 - 4 = 0$

For $x = 2,\ y = 4 - 2 = 2$

Three elements of the solution set of the given equation are (0, 4), (4, 0) and (2, 2). Plotting these points on the graph paper and joining them by a straight edge, we get the graph of $x + y = 4$, which is a line, intersecting both X-axis and Y-axis (see figure 5.5)

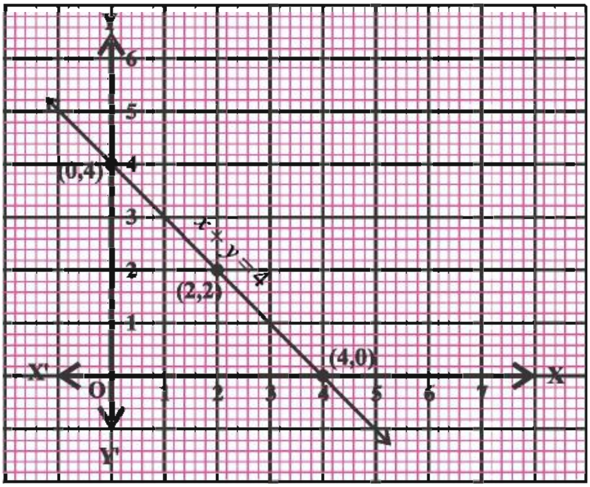


Figure 5.5

Note : If $a \neq 0$, $b \neq 0$, $c \neq 0$ then the graph of the equation $ax + by + c = 0$ is a line, intersecting both X-axis and Y-axis in distinct points.

Here, the graph of $x + y = 4$ intersects X-axis in (4, 0) and Y-axis in (0, 4)

Example 8 : For each of the graph in figure 5.6, 5.7, 5.8 select the correct equation graph of which is from the choices given below.

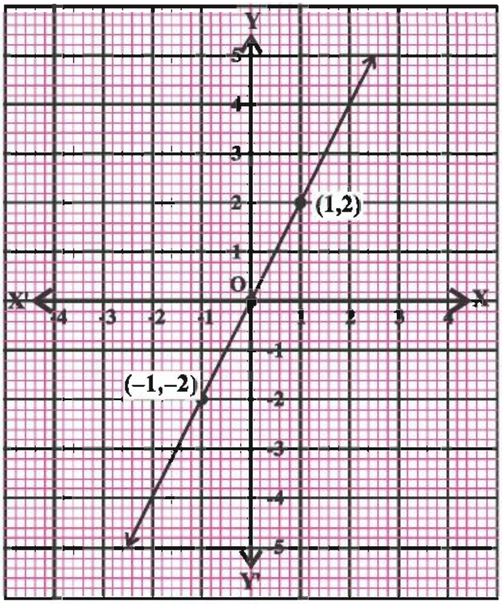


Figure 5.6

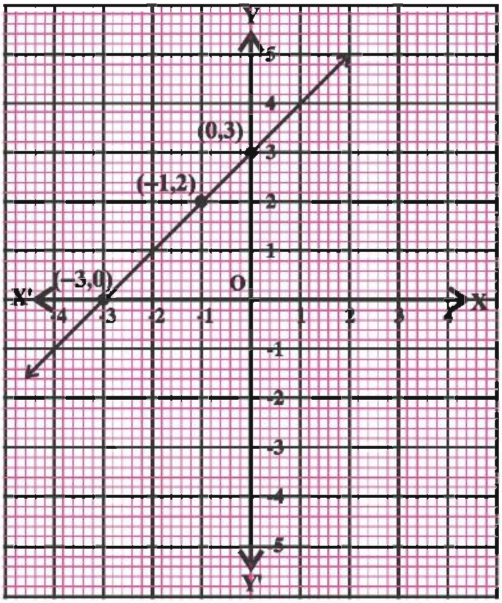


Figure 5.7

(a) For figure 5.6
(1) $x - y = 0$ (2) $y = -x$
(3) $y = 2x$ (4) $y = 3x + 1$

(b) For figure 5.7
(1) $x + 2y = 0$ (2) $y = -3x$
(3) $x - y + 3 = 0$ (4) $2y = 4x + 5$

(c) For figure 5.8
(1) $y = 2x + 1$ (2) $3x - 4y = 12$
(3) $2y = 3x + 2$ (4) $4x - 3y = 12$

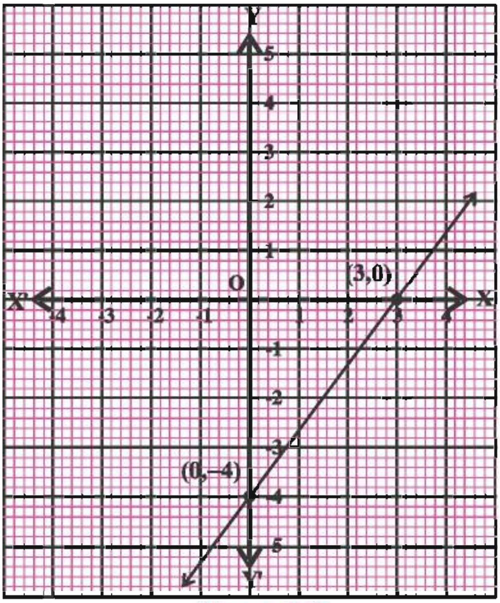


Figure 5.8

Solution :

(a) In figure 5.6, points $(-1, -2)$, $(0, 0)$, $(1, 2)$ are on the line, these points satisfy the equation $y = 2x$.

$\therefore$ $y = 2x$ is the equation corresponding to this graph.

(b) In figure 5.7, the points $(0, 3)$, $(-3, 0)$, $(-1, 2)$ lie on the line and satisfy the equation $x - y + 3 = 0$.

(c) In figure 5.8, obviously $(3, 0)$ and $(0, -4)$ satisfy $4x - 3y = 12$. The equation is $4x - 3y - 12 = 0$.

EXERCISE 5.3

1. Draw the graph of each of the following linear equations in two variables :
(1) $x + y = 6$ (2) $x - y = 2$ (3) $x - 2y = 6$
(4) $y = 3x$ (5) $y = x + 1$ (6) $3x + y = 2$

2. If the point $(2, 3)$ lies on the graph of the equation $2y = ax + 10$, find the value of a.

3. Given the point $(2, 3)$, find the equations of four distinct lines on which it lies. Draw the graph of each line. Also write the point of intersection of each line with the coordinate axes.

4. Consider the linear equation that converts Fahrenheit (F) to Celsius (C) :

$$F = \left(\frac{9}{5}\right)C + 32$$

(1) Draw the graph of this linear equation using Celsius for X-axis and Fahrenheit for Y-axis.

(2) If the temperature is 30° C, what is the corresponding temperature in Fahrenheit ?

(3) If the temperature is 95° F, what is the corresponding temperature in Celsius ?

(4) If the temperature is 0° C, what is the corresponding temperature in Fahrenheit and if the temperature is 0° F, what is the corresponding temperature in Celsius ?

(5) Is there a temperature which is numerically the same in both Fahrenheit and Celsius ? If yes, find it.

*

5.5 Equation of Lines Perpendicular to the X-axis and Y-axis

Consider the equation $3x = 6$, i.e. $x = \frac{6}{3}$. So, $x = 2$ or $x - 2 = 0$

Here, $x - 2 = 0$, if it is treated as an equation in one variable x only, then it has the unique solution $x = 2$, which is a point on the number line (see figure 5.9).

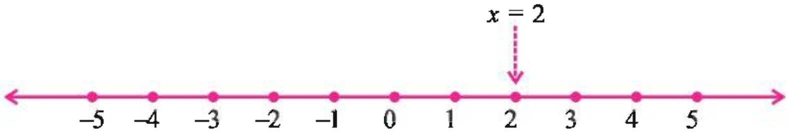


Figure 5.9

Now, in two variables the equation $x - 2 = 0$ can be expressed as $x + 0y - 2 = 0$

Since, the coefficient of y is zero, i.e. for any value of y we get $x = 2$. i.e. an equation $x + 0y - 2 = 0$ has infinitely many ordered pairs as solution. Thus all the solutions of this equation are of the form.

(2, r), where r is any real number (2, 0), (2, 1), (2, 2) are three of the solutions of the equation $x + 0y = 2$

Drawing the graph by plotting these points, we get a line parallel to Y-axis.

Thus the graph of $x = 2$ is a line perpendicular to X-axis (see figure 5.10)

If the coefficient of y is 0 in a linear equation in two variables, then its graph is a line perpendicular to X-axis.

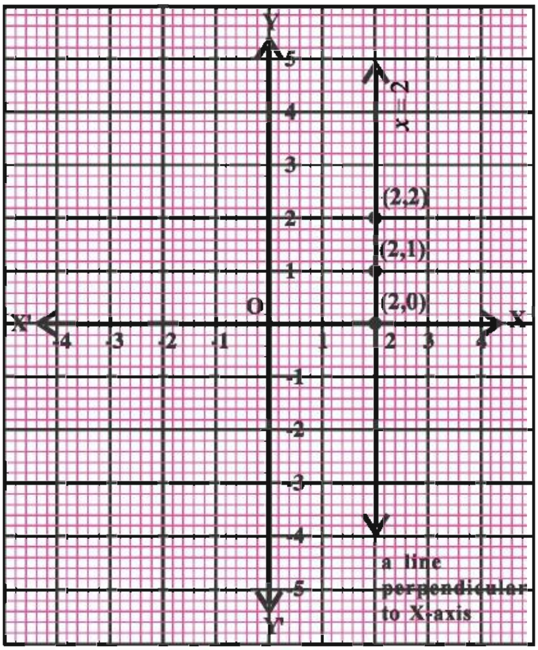


Figure 5.10

Similarly the equation $y = 3$, in two variables can be written as $0x + y = 3$

As discussed above, it has infinitely many solutions of the type (r, 3), where r is any real number. (–2, 3), (0, 3), (2, 3) are three of the solutions of the equation $y = 3$. Plotting these points, we can see that the graph of $y = 3$ is a line perpendicular to Y-axis. (See figure 5.11)

If the coefficient of x is 0 in a linear equation in two variables, then its graph is a line perpendicular to Y-axis.

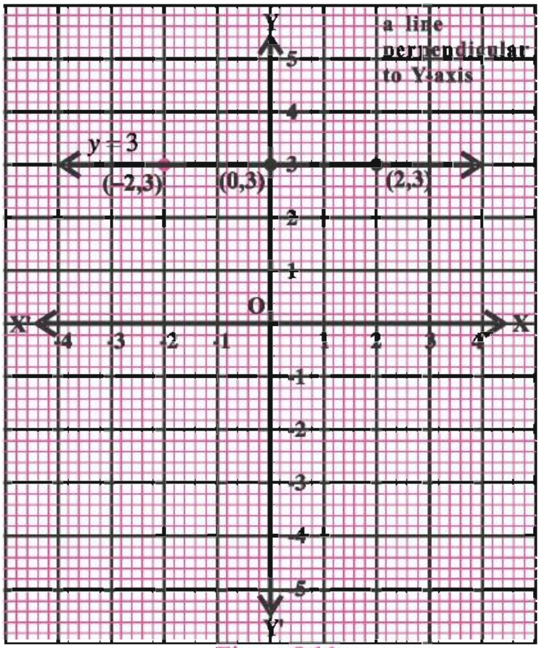


Figure 5.11

Conclusions :

From the above illustrations we have following facts :

For the graph of a linear equation $ax + by + c = 0$ in two variables,

(1) If $a = 0$, $c = 0$, i.e. the graph of the equation $y = 0$ is the X-axis.

(2) If $b = 0$, $c = 0$, i.e. the graph of the equation $x = 0$ is the Y-axis.

(3) If $a = 0$, $c \neq 0$, the equation is $by = -c$ or $y = p$; $p = \left(\frac{-c}{b}\right)$ then the graph of the equation is a line perpendicular to the Y-axis.

(4) If $b = 0$, $c \neq 0$, the equation is $ax = -c$ or $x = q$; $q = \frac{-c}{a}$ then the graph of the equation is a line perpendicular to the X-axis.

(5) If $a \neq 0$, $b \neq 0$, $c = 0$, then the graph is a line passing through the origin O.

EXERCISE 5.4

1. Give the geometric representations of the equations (1) $y = -4$ (2) $2x + 9 = 0$ in one variable and in two variables.

2. Draw the graph of the following linear equations in two variables

(1) $3y = 6$ (2) $x = 4$ (3) $2y = 10$ (4) $5x + 10 = 0$

3. Solve the equations (1) $2x + 1 = x - 2$ (2) $y - 1 = 2y - 5$, and represent their solutions on the (i) number line (ii) Cartesian plane.

4. Draw the graphs of $y = x + 1$ and $x + y - 3 = 0$ on the same graph paper and observe that these lines intersect at the point (......, ...).

EXERCISE 5

1. Examine whether the following expressions are linear equations in two variables or not.

(1) $\frac{7}{3}x + \frac{5}{2}y + \frac{1}{2} = 0$ (2) $\frac{3}{x} + 2y - 1 = 0$

(3) $\frac{x}{2} + \frac{y}{2} = 3$ (4) $\frac{2}{y} + \frac{2}{x} = \frac{1}{3}$

(5) $y + 3 = 0$ (6) $2x - 5 = 0$

2. Find three distinct solutions of each of the following equations :

(1) $\frac{x}{2} + y = 6$ (2) $x + \frac{y}{3} = 9$

(3) $x + y - 1 = 0$ (4) $x - y + 1 = 0$

(5) $2x + 3y = 6$ (6) $3x - 5y - 15 = 0$

3. If $a = 2k$, $b = 5k$, $c = 7k$; $k \neq 0$ and $k \in$ R, then find the value of k in the following cases : If

(1) $(b - a, c - b)$ is a solution of the linear equation $2x + 3y = 10$.

(2) $(c - 3a, 3b - 2c)$ is a solution of the equation $x + y - 3 = 0$.

(3) $(c + b - 5a, c - b - a)$ is a solution of the equation $2x + y - 8 = 0$.

(4) $(a + b, c + 1)$ is a solution of the equation $y = 2x$.

(5) $(2a + b - c - 2, 3b + 2a - 3c + 2)$ is the point of intersection of the coordinate axes.

4. Draw the graph of each of the following linear equations in two variables; also find their points of intersection with the axes :

(1) $x + y = 0$ (2) $x - y = 0$

(3) $x + y = 2$ (4) $x - y = 3$

(5) $3x + 4y + 12 = 0$ (6) $3x - 2y - 6 = 0$

(7) $3x + 2y - 6 = 0$ (8) $3x - 4y + 12 = 0$

(9) $2x + 5 = 0$ (10) $4y - 8 = 0$

5. Represent geometrically the solutions of the following equations :

(1) $3x + 2 = -x + 10$ (2) $4y - 3 = y + 6$

(3) $2x + 3 = x - 1$ (4) $3y + 2 = 2y - 3$

on the (i) same number line, (ii) same cartesian plane.

6. Draw the graphs of following in R^2.

(1) $x = 4$ (2) $y = 4$ (3) $x = -4$

(4) $y = -4$ (5) $y = x$ (6) $y = -x$

on the same graph paper and write the points where these lines intersect each other.

7. Draw the graphs of (1) $x + 3y - 6 = 0$ and (2) $2x - y - 5 = 0$, on the same graph paper and write the point, where these two lines intersect each other.

8. Draw the graphs of (1) $3x + 2y = 9$ and (2) $x + 4y = 8$, on the same graph paper and observe that the graphs intersect each other at the point $\left(2, \frac{3}{2}\right)$.

9. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Graph of the equation $y = x$ passes through the ...... quadrants and origin. ☐

(a) I and II (b) II and III (c) I and III (d) III and IV

(2) Line $x + y = 2$ passes through the ...... quadrants. ☐

(a) 1st and 3rd both (b) 2nd and 3rd

(c) 3rd and 4th both (d) 1st, 2nd and 4th all

(3) $x + y = 0$ passes through ...... quadrants. ☐

(a) I and II (b) I and II (c) II and IV (d) III and IV

(4) $ax + by = c$; $a^2 + b^2 \neq 0$, passes through origin, if ☐

(a) $a = 0$, $c \neq 0$ (b) $b = 0$, $c \neq 0$ (c) $c = 0$ (d) $a \neq 0$, $c \neq 0$

(5) The linear equation $4x - y + 8 = 0$ has ...... ☐

(a) no solution (b) unique solution

(c) only two solutions (d) infinitely many solutions

(6) If $x = 2$, $y = 5$ is a solution of the equation $5x + 7y - k = 0$, then the value of k is ☐

(a) 12 (b) 35 (c) 45 (d) −45

(7) If the equation is $F = \left(\frac{9}{5}\right)C + 32$, then C = ... ☐

(a) 5F − 160 (b) $\frac{1}{9}(5F - 160)$ (c) $\frac{5}{9}F - 32$ (d) $\frac{5}{9}(F - 32)$

(8) In the equation $F = \left(\frac{9}{5}\right)C + 32$, F = C ...... ☐

(a) is impossible (b) if C = 40 (c) if C = −40 (d) if F = 32

(9) If $F = \left(\frac{9}{5}\right)C + 32$, and F = −274, then C = ... ☐

(a) −338 (b) −274 (c) −170 (d) 170

(10) In the plane, the equation $y = mx$ represents ...... for different values of m. ☐

(a) perpendicular lines (b) parallel lines

(c) lines through origin (d) lines through the point other than origin.

(11) Line $y = 4$ is ...... ☐

(a) parallel to Y-axis (b) intersects both the axis

(c) parallel to X-axis (d) passing through the origin.

(12) Line $x = -2$ is ...... ☐

(a) parallel to X-axis (b) parallel to Y-axis

(c) passing through the origin. (d) intersecting both the axis

(13) One of the solutions of the linear equation $2x + 3y = 7$ is ......

(a) (1, 2) (b) (−1, 3) (c) (−2, 5) (d) (−2, 4)

(14) The graph of the equation ...... is a line parallel to Y-axis

(a) $x - 3 = 0$ (b) $x - y = 1$ (c) $y = 1$ (d) $x + y = 1$

(15) The graph of the equation ...... is a line passing through the origin.

(a) $x + y = 0$ (b) $x + y = 1$ (c) $2y - 3 = 0$ (d) $2x - 2y = 1$

*

Summary

In this chapter, you have studied the following points :

1. $ax + by + c = 0$ is a linear equation where a, b, c are real numbers; $a^2 + b^2 \neq 0$.
2. A linear equation in two variables has infinite number of solutions.
3. The graph of every linear equation in two variables is a straight line.
4. An equation of the type $y = mx$ represents a line passing through the origin.
5. The equation of Y-axis is $x = 0$ and the equation of X-axis is $y = 0$
6. The graph of $x = a$ is a straight line perpendicular to X-axis, i.e. $x = a$ is a vertical line.
7. The graph of $y = b$ is a straight line perpendicular to Y-axis i.e. $y = b$ is a horizontal line

•

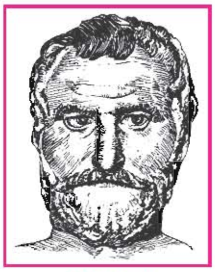
Chapter 6

STRUCTURE OF GEOMETRY

'A multitude of words is no proof of a prudent mind.' - ***Thales***
'Hope is the poor man's bread.' - ***Gary Herbert***
'The past is certain the future obscure.' - ***Thales***

6.1 Introduction

The word 'geometry' comes from the combination of two Greek words 'geo' meaning the 'earth' and 'metrein' meaning to 'measure'. Geometry appears to have originated from the need for measuring land. This branch was studied in various forms in every ancient civilization, like India, Greece etc.



Thales
(Born : 624-625 BC
Died : 546-547 BC)

It is believed that the knowledge of geometry passed on from Egyptians to Greeks. A Greek mathematician Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected by its diameter. Thales is considered to be the pioneer of geometry, because he used the word geometry for the first time. Pythagoras was one of most popular student of Thales. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent.

In the Indian subcontinent the excavations of Harappa and Mohen-Jo-Daro show that Indus Valley civilization made extensive use of geometry about 300 BC. The cities were well planned and organised. The roads were parallel to each other and the drainage system was underground. A house had rooms of different shapes and the bricks used had the ratio of 4:2:1 for length, breadth and height respectively.

Some of the Theorems of Thales :

(1) A circle is bisected by its diameter.

(2) Angles at the base of any isosceles triangle are equal.

(3) If two straight lines intersect, the opposite angles formed are equal.

(4) If one triangle has two angles and one side equal to those of another triangle the two triangles are equal in all respects.

(5) 'Any angle inscribed in semicircle is a right angle' is known as Thales theorem.

Sulbasutras provided literature for constructions using geometry. *Sulbasutras* were created between 800 BC to 500 BC. *Bodhayan Sulbasutra* is the oldest. Implicitly their construction implies knowledge of proof of Pythagoras' principle. Altars for public worship contained combinations of rectangles, triangles and trapaziums. Thus Indians knew Pythagoras' theorem before his birth. Aryabhatt, Brahmgupta and Bhaskaracharya contributed to the development of geometry.

Sacred fire found locations according to definite instructions about shapes and sizes. Squares and circular altars were used for residential rituals. *Sriyantra* consists of nine interwoven isosceles triangles arranged, so as to produce 43 subsidiary triangles.

Geometry was being developed in unorganized manner. Egyptians gave only statements of results. Egyptians and Babylonians used geometry solely for application and practical utility. But Greeks laid the basis of deductive reasoning.

6.2 Euclid's Approach

Euclid was a teacher of mathematics at Alexandria. He collected all the known results and compiled them in a series of a thirteen chapters each called a **'Book'**.

Euclid
(325 BC to 265 BC)

The compilation was named **'Elements'**. This **'Elements'** greatly influenced world's notion of geometry for years to come. The notions of point, line, plane, surface were derived from surrounding objects. An abstract geometrical concept of a solid object was created and developed from studies of space. Boundaries of a solid are surfaces partitioning space. 'Surfaces' have no thickness and their boundaries are curves or straight lines.

The lines end in points. Gradually proceeding from solids to point we lose 'dimension'.

Geometric quantity	Dimension
Solid	3
Surface	2
Line	1
Point	None

Euclid gave 23 definitions in Chaper 1 (called book) of Elements. A few of which are listed below :

(1) A point is that which has no part.
(2) A line is breadthless length.
(3) The ends of a line are points.
(4) A straight line is a line which lies evenly with the points on itself.
(5) A surface is that which has length and breadth only.
(6) The edges of a surface are lines.
(7) A plane surface is a surface which lies evenly with the straight lines on itself.

Now what is a 'part' ? If 'part' is some thing occupying 'area', what is 'area' ? So we get a chain of terms. So some of the terms are left as undefined terms. Even though we represent a point as a dot, a dot has some dimension. So we have intuitive feeling of the geometrical concept of a point. Similarly breadth and length are not defined. So **point, line, plane are taken as undefined terms** and we represent them using some imaginary physical models.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually obvious universal truths. He divided them into two types : axioms and postulates. The postulates are the assumptions that were specific to the geometry. Common notions often called axioms on the other were assumptions used throughout mathematics and not specifically linked to geometry.

Some of Euclid's axioms not in this order, are given below :

(1) Things which are equal to the same thing are equal to one another.
(2) If equals are added to equals, the wholes are equal.
(3) If equals are subtracted from equals, the remainders are equal.
(4) Things which coincide with one another are equal to one another.
(5) The whole is greater than a part.
(6) Things which are double of the same things are equal to one another.
(7) Things which are halves of the same things are equal to one another.

These common notions refer to magnitudes of same kind. Magnitude of the same kind can be compared and added, but magnitudes of different kinds can not be compared. For example a line cannot be added to a rectangle, nor can an angle be compared to a pentagon.

The fourth axiom stated above says identical things are equal. Everything is equal to itself.

This is the principle of superpositions. The fifth axiom gives definition of 'greater than' or 'less than'. If Q is a part of P, then we can write $P = Q + R$ for some quantity R. Thus $P > Q$ means $P = Q + R$ for some R.

Euclid's five postulate are as given below :

Postulate 1 : A straight line may be drawn from any one point to any other point.

This shows that there is one straight line passing through two distinct points, but there is no certainty that this line is unique. Although, Euclid has frequently used this fact without clarification. Thus we get the following axiom.

Axiom : Given two distinct points, there is a unique line that passes through them. How many lines passing through P also pass through Q ? Only one !

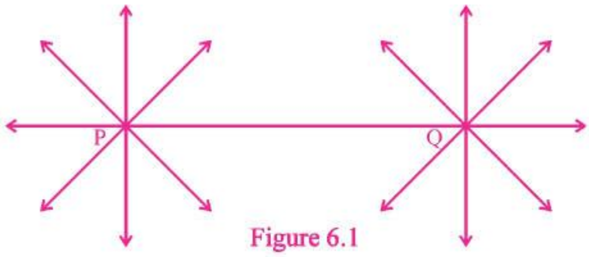


Figure 6.1

Postulate 2 : A terminated line can be produced indefinitly.

We call a segment what is called a terminated line by Euclid.

Postulate 3 : A circle can be drawn with any centre and any radius.

Postulate 4 : All right angles are equal to one another.

Postulate 5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely meet on that side on which the sum of angles is less than two right angles.

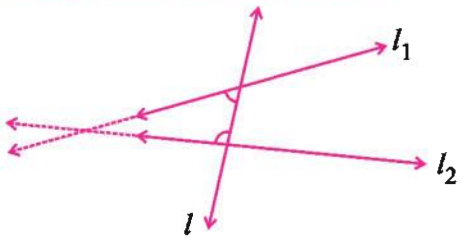


Figure 6.2

Postulate 5 is very complex in nature. Infact postulate is a verb. When we say 'let us postulate' it means 'let us make some statement' on an observed phenomenon in the universe. Its truth is examined later on and if found true, then it is termed a postulate.

A system of axioms is consistant, if it is not possible to deduce a statement contradicting any axiom or statements proved based on them.

Using these postulates and axioms, Euclid moved to prove results. He proved some more results using deductive logic. These proved statements are known as propositions or theorems. Euclid deducted 465 propositions in logical chain using his axioms, definitions and postulates.

Let us see the result in the following examples to understand how Euclid used his axioms and postulates for proving some of the results.

Example 1 : D, E and F are three points on the same line and E lies between D and F as shown in figure 6.3. Then prove that DE + EF = DF.

Solution : In the figure DF coinsides with DE + EF. Therefore using Euclid fourth axiom that the things which coincide with one another are equal to one another, we conclude that DE + EF = DF.

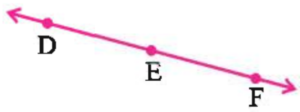


Figure 6.3

In this solution it has been assumed that there is a unique line passing through two distinct points.

Example 2 : Prove that an equilateral triangle can be constructed on any given line segment.

Solution : Let a line segment of given length say DE be given. (figure 6.4(i)).

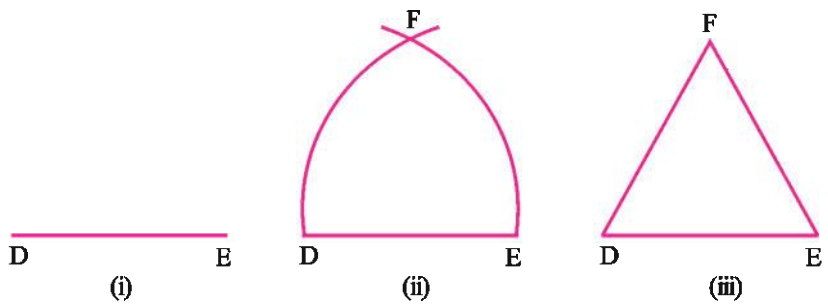


Figure 6.4

We use a construction using Euclids third postulate that "a circle can be drawn with any centre and any radius". Draw a circle with centre D and radius DE (Fig. 6.4(ii)). Similarly draw another circle with point E as a centre and DE as the radius. Let two circles intersect at the point F.

Now draw line segment DF and EF to form ΔDEF (figure 6.4(iii)). We have to prove that DE = EF = DF.

DE = EF **(radius of the same circle) (i)**

DE = DF **(radius of the same circle) (ii)**

From (i) and (ii) we observe that, DE = EF = DF. So ΔDEF is an equilateral triangle.

6.3 Equivalent Versions of Euclid's Fifth Postulate

Fifth postulate of Euclid is very significant in mathematics. We see that it implies no intersection of lines will take place when the sum of the measures of the interior angles on the same side of the falling line is exactly 180. There are several equivalent versions of this postulate. One of them is 'Playfair's Axiom' as given below :

For every line l and every point P not lying on l, there exists a unique line m passing through P and parallel to l.

It means **two distinct intersecting lines can not be parallel to the same line** (Fig. 6.5).

Here out of **all line passing through the point P only line m is parallel to l.**

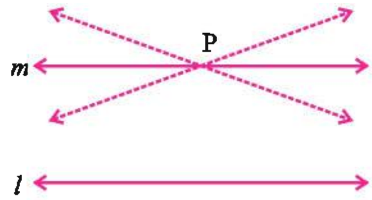


Figure 6.5

Euclid did not require his fifth postulate to prove his first 28 theorems. Many mathematicians including him, were convinced that the fifth postulate is actually a theorem that can be proved using just the first four postulates and other axioms. However all attempts to prove the fifth postulate as a theorem have failed. But these efforts have led to a great achievement, namely the creation of several other geometries.

These geometries quite different from Euclidean geometry, are called *non-Euclidean geometries*. Their creation is considered as a landmark in the history of thought because till then every one had believed that Euclid's was the only geometry and the world itself was Euclidean.

Now the geometry of the universe we live in has been demonstrated to be a *non-Euclidean geometry*. In fact it is called spherical geometry. In spherical geometry lines are not straight. They are parts of great circles.

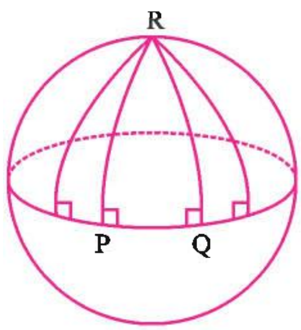


Figure 6.6

In figure 6.6 lines PR and QR which are parts of great circle of the sphere are perpendicular to the same line PQ. But they are meeting each other though the sum of angles on the same side of line PQ is not less than two right angles. In fact sum of measures of angles is 90 + 90 = 180. Also note that the sum of measures of the angles of the triangle RPQ is greater than 180. Thus Euclidean geometry is valid only for the figures in the plane. On the curved surfaces it is not satisfied.

EXERCISE 6.1

1. If a point Q lies between two points P and R such that PQ = QR, then prove that PQ = $\frac{1}{2}$PR. Explain by drawing the figure.

2. In figure 6.7 if PR = QS, then prove that PQ = RS.

Figure 6.7

3. Only one line can pass through a single point. Is this true or false ? Give reason for your answer.

4. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The three steps from solid to point are : ☐

(a) Solid - Surface - Line - Point (b) Line - Point - Surface - Solid

(c) Surface - Point - Line - Solid (d) Point - Surface - Line - Solid

(2) The number of dimensions a point has is ...... ☐

(a) 1 (b) 4 (c) 0 (d) 2

(3) The number of dimensions a surface has is ...... ☐

(a) 3 (b) 1 (c) 0 (d) 2

(4) Euclid divided his famous treatise 'the elements' into : ☐

(a) 12 chapters (b) 13 chapters (c) 9 chapters (d) 11 chapters

(5) Pythagoras was a student of : ☐

(a) Euclid (b) Thales (c) Ramanujan (d) Bhaskaracharya

(6) Which of the following needs a proof ? ☐

(a) Axiom (b) Postulate (c) Definition (d) Theorem

(7) Euclid stated that all right angles are equal to each other in the form of : ☐

(a) a proof (b) a definition (c) a postulate (d) an axiom

(8) 'Lines are parallel to each other if they do not intersect' is stated in the form of : ☐

(a) a definition (b) an axiom (c) a postulate (d) a proof

*

6.4 Requirement of Construction of Logical Geometry

Results based on measure of figures are sometimes misleading. Also they can not be universally applied. Hence it is necessary to develop logical concepts to study geometry.

Euclid's geometry was lacking in logic. Later Devid Hilbert modified the approach of Euclidean geometry and made it more logical and abstract. The structure of modern geometry is based on this formulation by Hilbert.

Among the Indian mathematicians Aryabhatta, Brahmagupta and Bhaskaracharya are the chief architects of geometry.

6.5 Special Phrases

There are some special phrases which are used in study of geometry such as (1) at least (2) at most (3) one and only one.

(1) At least : There are at least two points on a line. It means there are two or more than two points on a line but not less than two points on a line.

At least three means three or more than three. At least five means five or more than five.

(2) At most : x is at most five. It means x can be 5 or x can be less than 5, but x can never be more than 5.

Distinct lines can intersect in at most one point means the lines can intersect in a point or may not intersect.

(3) One and only one : There is one and only one line passing through two distinct points. It means there is one line passing through two distinct points and not more than one or not less than one line is there i.e. unique line passes through two distinct points.

The equation $x + 7 = 10$ has one and only one solution. It means it has a solution and only one solution, i.e. there is not more than one and not less than one solution.

6.6 Some Special Statements

There are some typical statements also which are used in the study of geometry.

(1) Conditional Statement : The statement of the type 'if *p*, then *q*' is called a conditional statement. It is also known as an implication. Here *p* is called a sufficient condition for *q* and *q* is called a necessary condition for *p*.

If m is even, then $m + 1$ is odd. Here m is even is sufficient for $m + 1$ to be odd. Thus '$m + 1$ is odd' is a necessary condition for m to be even. It can also be called a necessary consequence.

If quadrilateral ABCD is a rhombus, all its sides are congruent. Here ABCD is a rhumbus is a sufficient condition. The consequence all its sides are congruent can occur if ABCD is a square also. Thus there may be several sufficient conditions possible for a consequence.

(2) Biconditional Statement : The statement of the type '*p* if and only if *q*' is called a biconditional statement. It is also known as a two way implication.

In fact each two way implication '*p* if and only if *q*' is a conjuction of two conditional statements 'if *p*, then *q*' and 'if *q*, then *p*'.

$x = 3$ if and only if $x + 5 = 8$. This is biconditional statement. It is a conjuction of two conditional statements. 'If $x = 3$, then $x + 5 = 8$' and 'if $x + 5 = 8$, then $x = 3$'.

(3) The Converse of a Statement : The statement obtained by interchanging the sufficient and necessary condition in an implication is called the converse of the given statement

'If *p*, then *q*' is conditional statement. Therefore converse of this statement is 'if *q*, then *p*'.

'If $x = 3$, then $x^2 = 9$' is a conditional statement. Therefore converse of this statement is 'if $x^2 = 9$, then $x = 3$'.

In general a conditional statement and its converse both may be true or any one of them may be true or both may be false.

6.7 Main Parts of Structure of Modern Geometry

Main parts of structure of modern geometry are :

(1) Defined terms (2) Undefined terms (3) Postulates (4) Theorems

The knowledge of any subject is transmitted or spread with the help of the language. Each sentence of any language contains terms. Every term has exact clear meaning. If this meaning is understood, clearly the subject can be studied.

There are two types of terms (1) Defined term (2) Undefined term.

Postulate and Axioms : A self evident statement which is accepted to be true without requiring any proof is called a postulate and commonly called an axiom.

Theorem : A theorem is a conditional statement. A theorem has mainly three parts : (1) Hypothesis (2) Conclusion (3) Proof.

Proof in geometry is divided into two types : (1) Direct proof (2) Indirect proof.

Direct proof : In direct proof, we deduce a statement from the data by means of logical arguments. From one statement we deduce another statement and through such a chain of statements, we derive the statement to be proved by means of logical arguments.

Indirect proof : Some times to prove the statement we have to choose one of the many possible alternatives. In such a situation we investigate each case. If a case leads to the falsehood of the data, then the case itself must be false. Thus eliminating the possibility of all the alternatives other than the statement to prove, we conclude its truth. This method is known as the method of exhausting alternatives.

Another method of indirect proof is "Reductio ad Absurdum". There are two cases for the statement to be proved. Either it is true or false. In this method, first we suppose to the contrary that the statement to be proved is false. Then by means of logical arguments based upon the postulates, definition and previous theorems we deduce the falsehood of the data. We argue that supposition is wrong. So we conclude that the statement to prove is true.

In the proof, data is a sufficient condition and to prove is a necessary condition.

Let us see an example of a proof by direct method.

Example 3 : Prove if n is even, n^2 is even.

Solution : Proof : Suppose $n = 2k$, where $k \in$ N.

(**Note :** All even numbers are obtained by taking $k = 1, 2, 3,...$ etc. Since an even number is a multiple of 2, $n = 2k$).

$\therefore \quad n^2 = 4k^2$

$= 2(2k^2)$

$\therefore \quad n^2$ is even.

If we want an example of indirect proof by exhausting alternatives, consider following.

To prove any real number $a > 0$, we have to prove that $a < 0$, $a = 0$ is not possible. Then it will be proved that $a > 0$.

Let us see a proof by the method of Reductio ad Absurdum.

Example 4 : Prove that two acute angles cannot be supplementary angles.

Solution : Data : $\angle A$ and $\angle B$ are acute angles, so that

$m\angle A < 90$, $m\angle B < 90$

To prove : $\angle A$, $\angle B$ are not supplementary angles.

Proof : Let $\angle A$, $\angle B$ be supplementary angles, if possible.

$\therefore \quad m\angle A + m\angle B = 180$

But, $m\angle A < 90$, $m\angle B < 90$

$\therefore \quad m\angle A + m\angle B < 180$

We get a contradiction.

$\therefore$ $\angle A$ and $\angle B$ cannot be supplementary angles.

Example 5 : Write data and statement to prove for following :

(1) If $y = 3$, then $y^3 = 27$

(2) If $m = n$, then $5m = 5n$

Solution :

(1) Data : $y = 3$

To prove : $y^3 = 27$

(2) Data : $m = n$

To prove : $5m = 5n$

Example 6 : Write the converse of each of the following statement.

(1) If two angles are right angles, then they are congruent.

(2) If there is a good rain, there will be good crops.

Solution :

(1) **Converse :** If two angles are congruent, then they are right angles.

(2) **Converse :** If there are good crops, then there is a good rain.

Example 7 : Explain the meaning of the following statement :

(1) Every line segment has one and only one mid point.

(2) Jignesh is at most 27 years old.

(3) Every set, except an empty set, has at least two subsets.

Solution : Meaning :

(1) Every line segment has one mid-point and not more or less than one mid-point.

(2) The age of Jignesh is 27 years or less but not more than 27 years.

(3) Every set, except the empty set, has two or more than two subsets but not less than two substs.

EXERCISE 6

1. Explain : Direct proof.

2. Prove : If x is an even number, then $x + 1$ is odd.

3. Explain the following statements :

(1) Jayendra can eat at most five cups of icecream.

(2) Every youth should contribute at least 10 hours per month for social services.

(3) $m + 7 = 10$ has one and only one solution.

(4) A line can intersect a circle in at most two points.

4. Write the Data and To Prove for the following statements :

(1) If $X \subset Y$ and $Y \subset X$, then $X = Y$.

(2) The sum of measures of all the three angles of triangle is 180.

(3) If B is not an empty set, then it has at least two subsets.

(4) If today is Sunday, then there is a holiday in the school.

5. Answer the following questions in short :

(1) What is an implication ?

(2) 'If $x + 5 = 7$, then $x = 2$'. Identify necessary and sufficient condition.

(3) Write the parts of a theorem.

(4) Write the types of proofs in geometry.

(5) Write two types of indirect proof.

(6) Name the main parts of the structure of modern geometry.

*

Summary

1. In this chapter we have studied about development of geometry.

2. Study of works of Euclid.

3. Meaning of special phrases like (1) At least (2) At most (3) One and only one.

4. Meaning of typical statements like (1) Conditional statement (2) Biconditional statement (3) Converse of a statement.

5. Main parts of structure of modern geometry like (1) Defined terms (2) Undefined terms (3) Postulate (4) Theorem and types of proofs of a theorem.

●

CHAPTER 7

SOME PRIMARY CONCEPTS IN GEOMETRY : 1

7.1 Introduction

There are four basic concepts in geometry. They are 'point', 'line', 'plane' and 'space'. All these terms are undefined terms. We shall understand these terms in the context of sets. **In the study of geometry, space is taken as universal set.**

Point : It is an undefined term. A point is represented by a dot. A fine dot made by a sharp pencil on a paper represents picture of a point very closely. A point has no length, width or thickness. In general points are denoted by capital letters X, Y, Z, M etc. In figure 7.1, X represents a point.

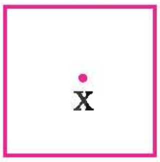
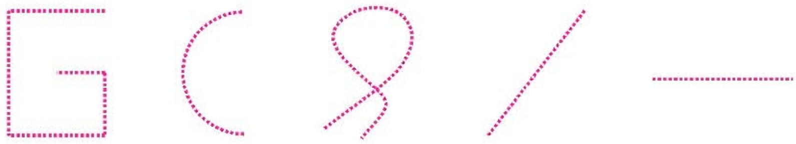


Figure 7.1



(a) (b) (c) (d) (e)

Figure 7.2

Line : Look at the above figure 7.2. They are all sets of points. Do they all represent a 'line' ? No. Figure 7.2 (a), (b), (c) do not represent a line, where as figure 7.2 (d), (e) resemble very closely our imagination of a line.

So a line is a set of points which can be compared to a stretched thread or edge of a ruler extended indefinitely in both directions.

"In geometry, line is a set of points which extends endlessly in both the directions."

Line is denoted by small letters *l*, *m*, *n*, etc. A line has infinitely many points.

Space : Space is the 'largest' set of points. It is the universal set in geometry. Line, plane and other point sets are subsets of space.

7.2 Line as a Set of Points

We know that a line has infinitely many points, but how many minimum number of points are required to determine a line ? This we can understand from the following postulates of line.

Postulate 1 : Every line has at least two distinct points.

We deduce from this postulate that a line has two or more points but not less than two points.

So, (1) A line is not an empty set.

(2) A line is not a singleton.

(3) A line has two or more points.

Postulate 2 : For any two distinct points X and Y, there is one and only one line that contains both the points i.e. passes through them. We deduce from this postulate that there exists exactly one line passing through both X and Y.

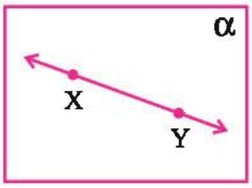


Figure 7.3

Thus two distinct points determine a line.

There is one and only one line passing through two distinct points. (figure 7.3)

7.3 Relation Between Point and Line

We have seen that line is a point set. So for relation between a line m and a point A, there are two alternatives, (i) point A is on line m i.e. $A \in m$ or (ii) point A is not on the line m that is $A \notin m$. Suppose A and B are two distinct points on a line m and point C is not on the line m. This is represented by figure 7.4(a) as follows :

We know that line m is also same as line AB, symbolically it is represented as $\overleftrightarrow{AB}$.

Figure 7.4(a)

If points P, Q, R are on line n, then

$n = \overleftrightarrow{PQ} = \overleftrightarrow{QR} = \overleftrightarrow{RP}$ (figure 7.4(b))

So, we can represent a line by selecting any two points on it.

Figure 7.4(b)

7.4 Collinear Points and Non-Collinear Points

In the figure 7.5, the points P_1, P_2, and P_3 are lying on the same straight line l. Such points which lie on the same line are said to be collinear points. There does not exist a line passing through points P_1, P_4 and P_5. So they are called non-collinear points. So, three or more points are said to be collinear, if there is a single straight line passing through them.

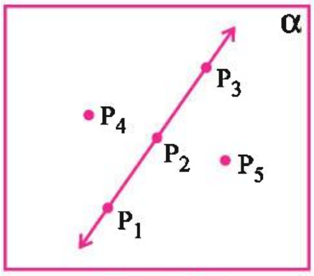


Figure 7.5

According to postulates 1 and 2, 'a unique line passes through two distinct points.' Hence two distinct points are always collinear.

Collinear Points : If three or more distinct points lie on a straight line, they are said to be collinear points or all points lying on a straight line are said to be collinear points.

Non-collinear Points : Points which can not lie on the same straight line are called non-collinear points.

or **"If there does not exist a line containing given points then we say that these points are non-collinear."**

Example 1 : Answer the following questions. Justify your answer by drawing proper figures.

(1) How many lines can be determined by three distinct points ?

(2) How many lines can be determined by four distinct points ?

Solution :

(1) For three distinct points there are two possibilities : (i) Given points A, B, C are collinear. As shown in figure 7.6 they determine only one line.

Figure 7.6

(ii) A, B, C are non-collinear points. They determine three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ as shown in figure 7.7.

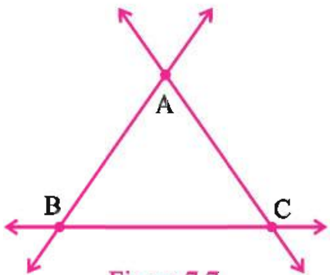


Figure 7.7

(2) There are three possibilities for four distinct points.

(i) They are all collinear. Four points P, Q, R and S in figure 7.8 are collinear. They determine only one line.

Figure 7.8

(ii) Three of them are collinear but not all are collinear. As shown in figure 7.9 P, Q, R are collinear but P, Q, R and S are non-collinear. These four points determine four lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$, $\overleftrightarrow{PS}$ and $\overleftrightarrow{QS}$.

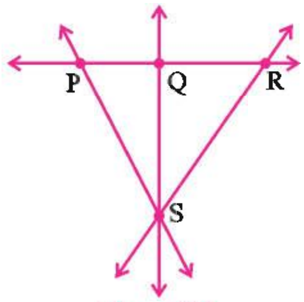


Figure 7.9

(iii) No three out of four points P, Q, R, S are collinear (see figure 7.10). In this case, these four points will determine six lines. $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{PS}$, $\overleftrightarrow{QR}$, $\overleftrightarrow{QS}$ and $\overleftrightarrow{RS}$.

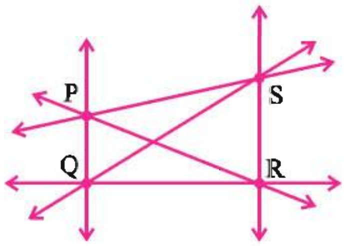


Figure 7.10

Example 2 : (1) How many distinct lines are there in the figure 7.11 ? Mention their names without involving points on them.

(2) What are other names of lines l_1 and l_2 ?

(3) Give other names of $\overleftrightarrow{EG}$.

(4) Is $\overleftrightarrow{CD} = \overleftrightarrow{AB}$?

(5) Is $\overleftrightarrow{ED} = \overleftrightarrow{DE}$?

(6) List all collinear triplets of points.

(7) Give four sets of non-collinear points.

(8) List all the lines containing D.

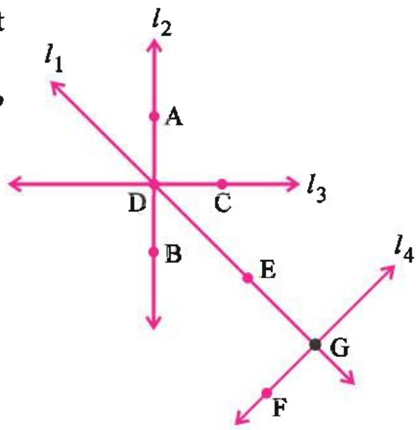


Figure 7.11

Solution :

(1) There are in all four lines l_1, l_2, l_3, l_4.

(2) Other names of line l_1 are $\overleftrightarrow{DE}$, $\overleftrightarrow{DG}$ and $\overleftrightarrow{EG}$ and those of line l_2 are $\overleftrightarrow{AD}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BD}$.

(3) Other names of $\overleftrightarrow{EG}$ are $\overleftrightarrow{DE}$, $\overleftrightarrow{DG}$ and l_1.

(4) No

(5) Yes

(6) (i) A, D, B (ii) D, E, G

(7) (i) A, C, E (ii) C, E, F (iii) A, E, F (iv) B, C, F

(8) Lines l_1, l_2, l_3 contain point D.

EXERCISE 7.1

1. Answer the following :

(1) Write the postulates of line.

(2) Define collinear and non-collinear points.

(3) State the number of lines containing two distinct points.

(4) If P, Q and R are distinct non-collinear points, then what is the relation between P and $\overleftrightarrow{QR}$?

(5) At most how many lines can four distinct points determine ? At least how many lines can four distinct points determine ?

2. Represent the following situations using a figure :

(1) $\overleftrightarrow{XY} = \overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$

(2) A, B and C are collinear, l is a line, $A \notin l$, $B \in l$, $C \notin l$.

(3) P, Q, R and P, S, T are collinear triplets but P, Q, S and P, R, T are not collinear.

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7.5 Intersection of Two Lines

Observe the following figures :

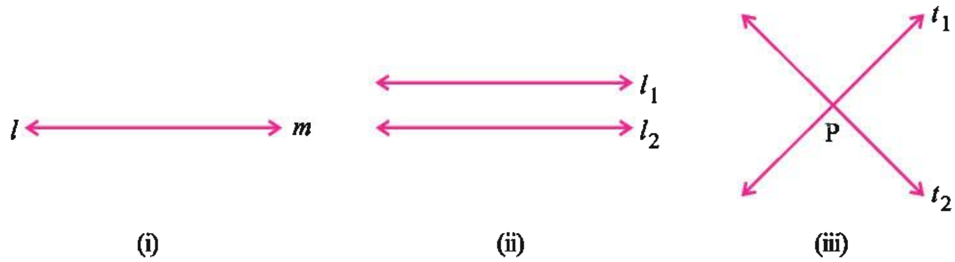

Figure 7.12

In figure 7.12(i) lines l and m are same, so $l = m$.

In figure 7.12(ii) lines l_1 and l_2 do not meet each other (clearly $l_1 \neq l_2$)

In figure 7.12(iii) lines t_1 and t_2 meet each other in P.

Now, we restate each of the above statements in terms of sets.

$l = m$, so l and m have same elements. So their intersection is the line l (or m) itself. $l \cap m = l = m$.

l_1 and l_2 do not intersect, i.e. they do not have any common element. Hence their intersection is the empty set. $l_1 \cap l_2 = \emptyset$.

Lines t_1 and t_2 have a common element (point P). Hence their intersection is contains P. i.e. $P \in t_1 \cap t_2$.

If the lines l and m intersect, then at least one point is common to l and m. Hence we assume that the point P is there in the intersection set of the lines l and m, that is $P \in l \cap m$.

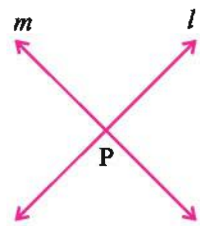

Figure 7.12(a)

Even if you imagine lines t_1, t_2 extended indefinitely in both directions, they will not meet again. So we may conclude that if two distinct lines intersect in one point, then they do not intersect in any other point. We shall accept this fact as a theorem without proof.

Theorem 7.1 : If two distinct lines intersect in one point, then they do not intersect in any other point.

Example 3 : Draw the figure representing the following situations :

For distinct lines m_1, m_2, m_3, m_4 : $m_2 \cap m_3 = \emptyset$, $m_2 \cap m_4 = \{X\}$,

$m_1 \cap m_3 = \{Y\}$, $m_1 \cap m_4 = \{Z\}$, $\overleftrightarrow{XW} = m_2$, $W \in m_1$.

Solution : $m_2 \cap m_3 = \emptyset$, so draw lines m_2 and m_3 in such a way that they do not intersect even if they are extended indefinitely.

$m_2 \cap m_4 = \{X\}$. Select point X on m_2 and draw a line m_4 passing through X. This line must be different from m_2.

$m_1 \cap m_3 = \{Y\}$. Select point Y on m_3 and draw a line m_1 passing through Y which is different from m_3.

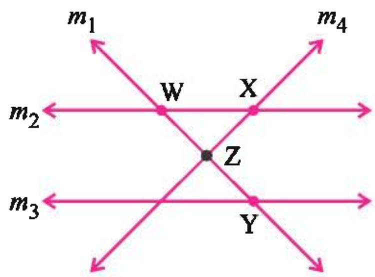

Figure 7.13

$m_1 \cap m_4 = \{Z\}$. So m_1 and m_4 intersect at Z and m_1 passes through Y also.

The other name of m_2 is $\overleftrightarrow{XW}$ as X, $W \in m_2$.

$W \in m_1$ means $m_1 = \overleftrightarrow{WY} = \overleftrightarrow{YZ}$.

EXERCISE 7.2

1. Draw figures for the following situations for lines m and n. :

   (1) $m \cap n = m$ (2) $m \cap n = \{A\}$ (3) $m \cap n = \emptyset$

2. Draw figures representing following situations :

   (1) $l_1 \cap l_2 = \{X\}$, $l_2 \cap l_3 = \emptyset$, $Y \notin l_2$, $Y \in l_1 \cap l_3$

   (2) X, Y, Z and X, A, B are triplets of collinear points but X, Y, A and X, Z, B are non-collinear points.

3. Fill in the blanks with reference to the figure 7.14 :

(1) $l \cap m$ = ...

(2) $p \cap q$ = ...

(3) $q \cap r$ = ...

(4) $p \cap r$ = ...

(5) $l \cap r$ = ...

(6) $m \cap r$ = ...

(7) $l \cap p$ = ...

(8) $m \cap q$ = ...

(9) $l \cap q$ = ...

(10) $m \cap p$ = ...

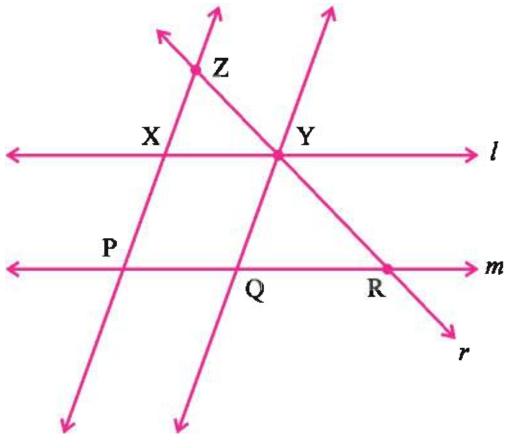

Figure 7.14

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7.6 The Concept of Distance

In ancient times distance was measured using finger, palm and arms. Later, in order to have uniformity, distance was measured with the help of the Kings (Ruler) foot. But every time the ruler changed the measurement would change. Then later units like inch, foot and yard came into existence. Presently, centimetre, metre and kilometre are extensively used units of length. In practice, when we measure distance using a scale, we take two points, plot them on the image of a ruler and then we get the distance which is associated with a definite non-negative real number. This is the postulate of distance.

Postulate 3 : Distance Postulate : With each pair of points, there corresponds one and only one non-negative real number called the distance between these points.

Distance between P and Q is denoted by PQ or d(P, Q).

Properties : For any two points P and Q, (1) PQ $\geq$ 0 i.e. the distance between two points is a non-negative real number.

(2) PQ = QP for any two points P and Q.

(3) PQ = 0 if and only if P = Q.

(4) For points P, Q, R, PQ + QR $\geq$ PR.

How do we measure the length of a given line-segment ?

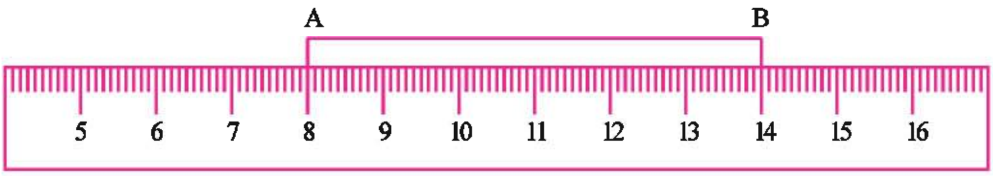


Figure 7.15

As shown in the figure 7.15, to measure the distance between A and B, we first place the scale close to the points A and B and read the two numbers on the scale corresponding to the points A and B. We then take the positive difference of the two numbers on the scale. Now we shall state ruler postulate.

Postulate 4 : Ruler Postulate :

(1) Corresponding to each point on a line, there is one and only one real number.

(2) Corresponding to each real number, there is one and only one point on the line.

(3) There is a one-to-one correspondence between points on a line and real numbers such that for every pair of distinct points on the line, the positive difference of the corresponding real numbers is the distance between them. We get a one to one correspondence between the set of points on line and real numbers such that if real numbers *a* and *b* corresponds to points A and B on line respectively, then the distance between A and B is the non-negative difference of *a* and *b*.

We know that for non-negative difference of two numbers the term modulus is used.

Thus $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Let a and b respectively correspond to A and B.

AB = | number corresponding to A − number corresponding to B |

$= |a - b|$

We know that,

$|a - b| = a - b$ if $a > b$

$= b - a$ if $a < b$

$= 0$ if $a = b$

Example 4 : Find the modulus of the following :

(1) 6 (2) 0 (3) −9 (4) 4 (5) 3.

Solution :

(1) $|6| = 6$ as $6 > 0$ (2) $|0| = 0$ (3) $|-9| = -(-9) = 9$ as $-9 < 0$

(4) $|4| = 4$ as $4 > 0$ (5) $|3| = 3$ as $3 > 0$

Example 5 : On a line l, M corresponds to −5 and N corresponds to −4. Find MN.

m = the number corresponding to M = −5

and n = the number corresponding to N = −4

$MN = |m - n|$

$MN = |m - n| = |(-5) - (-4)| = |-5 + 4| = |-1| = 1$

Given a point O on line l and a positive real number x, there are exactly two points X_1 and X_2 on l on either side of O such that,

$OX_1 = OX_2 = x$

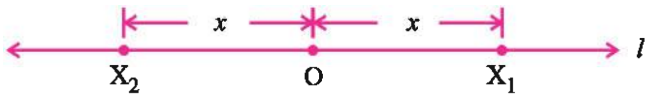


Figure 7.16

Example 6 : Point P on line l corresponds to 6.3. Find the real numbers corresponding to the point which are 3 units away from P on line l.

Solution 1 : Suppose Q is the required point. Then Q may be on the left or the right of P as shown in figure 7.17.

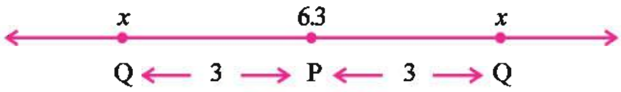


Figure 7.17

According to postulate 4, PQ = $| p - x |$, where $p = 6.3$ and PQ = 3. Here p and x are numbers corresponding to P and Q respectively.

$\therefore$ PQ = $| 6.3 - x |$

Since PQ = 3, $| 6.3 - x | = 3$

But $| \pm 3 | = 3$

$\therefore$ $6.3 - x = 3$ or $6.3 - x = -3$

$\therefore$ $6.3 - 3 = x$ or $6.3 + 3 = x$

$\therefore$ $x = 3.3$ or $x = 9.3$

Solution 2 : PQ = $| 6.3 - x |$

But PQ = 3

$\therefore$ $| 6.3 - x | = 3$

Now if $x < 6.3$, then $6.3 - x > 0$

$\therefore$ $6.3 - x = 3$

$\therefore$ $x = 6.3 - 3 = 3.3$

If $x > 6.3$, then $6.3 - x < 0$

$\therefore$ $| 6.3 - x | = -(6.3 - x) = 3$

$\therefore$ $x = 6.3 + 3 = 9.3$

$x = 3.3$ or 9.3

Thus the real number corresponding to Q is 3.3 or 9.3.

Note : In general, if $| a | = 5$, we take $a = 5$ or $a = -5$ and proceed.

Usually when we measure the distance between two distinct points A and B on a line l, we set a scale in such a way that the point A corresponds to 0 and the point B corresponds to a positive real number. In the following postulate we assume that such an arrangement of scale is possible which is similar to the representation on a number line.

Postulate 4A : Based on ruler postulate, if O and B are two distinct points on a line *l*, then the correspondence can be chosen in such a way that O corresponds to 0 and B corresponds to a positive number.

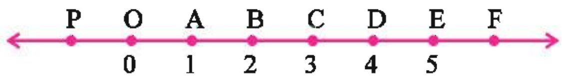


Figure 7.18

With reference to the above number line, let us assume that the point A corresponds to 1 and point B corresponds to 2, etc.

We are familiar with the following facts :

(1) The point O corresponding to 0 is called the origin or the initial point. Origin is usually denoted by letter O.

(2) The direction to the right of the origin i.e. from O towards B is called the positive direction and the direction to the left of the origin i.e. from O towards P is called the negative direction.

(3) Since O corresponds to 0, A corresponds to 1, we can conclude that OA = AB = 1. OA or AB is called the **unit distance**.

7.7 Betweenness

What do we mean when we say that a point B is between A and C ? Which of the figures 7.19(i) to (v) gives us the idea of betweenness ? :

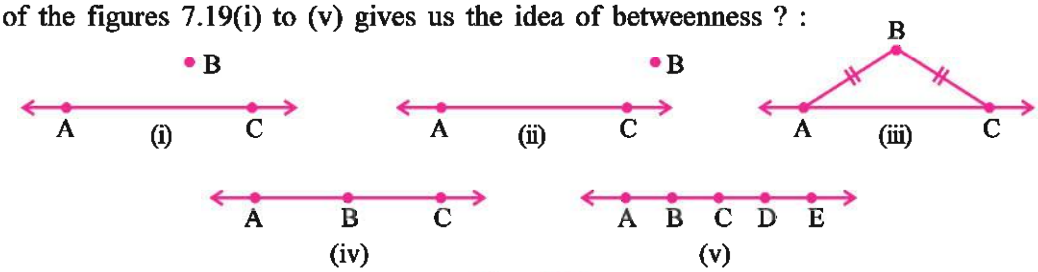


Figure 7.19

In the figure 7.19(i) and 7.19(ii) the points A, B and C are not collinear. So, the question of 'betweenness' does not arise.

In the figure 7.19(iii), the point B is equidistant from A and C but we cannot say that B is between A and C as they are not collinear points.

In figure 7.19(iv), points A, B and C are collinear and in figure 7.19(v), the points A, B, D, E, C are all collinear. So the question of betweenness aries.

Now we can conclude that the term **'betweenness'** can be used if we have at least three distinct collinear points.

For any three distinct collinear points A, B and C, if B lies between A and C, we write it symbolically as A–B–C or C–B–A (read as C dash B dash A).

Now if A, B and C are distinct collinear points, then one and only one of the following will be true. A–B–C or B–A–C or A–C–B.

How will we know which point is between the other two ? It is possible if we know the numbers corresponding to the points. From these numbers we can determine which number is between the other two and then the point corresponding to it will be between the other two points.

i.e. if a, b, c are the numbers corresponding to points A, B and C respectively then if $a > b > c$ or $a < b < c$, then we can conclude that A–B–C or C–B–A.

Conversely if A–B–C, then either $a < b < c$ or $a > b > c$.

Conditions of Betweenness : If p, q, r are real numbers corresponding to the points P, Q, R respectively, and $p < q < r$ or $p > q > r$ then Q is between P and R. We can write it as P–Q–R or R–Q–P.

P, Q and R are distinct collinear points and $p < q < r$ or $p > q > r$.

If P–Q–R then PQ + QR = PR

Can we say P–Q–R if PQ + QR = PR ? No; P, Q, R should be distinct.

Example 7 : Given P–Q–R. Suppose the numbers –3.7 and 7.8 correspond to P and R respectively. If PQ = 5.6, then find QR.

Solution :

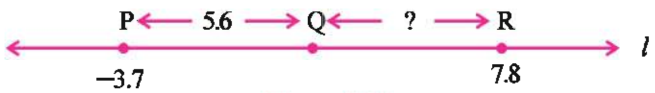


Figure 7.20

Here, PR = | Number corresponding to P – Number corresponding to R |

$= | -3.7 - 7.8 | = | -11.5 | = 11.5$

Because of P–Q–R, we have

PR = PQ + QR. Also, PQ = 5.6

$\therefore$ 11.5 = 5.6 + QR

$\therefore$ QR = 11.5 – 5.6 = 5.9

$\therefore$ QR = 5.9

Example 8 : As shown in figure 7.21, a straight road goes from Kavita's home to a garden via school. Kavita goes from her home to the garden and comes back to school. For this, she has to walk 717 metres. If the distance between Kavita's home and school is 237 metres, find the distance from Kavita's home to garden and from the school to the garden.

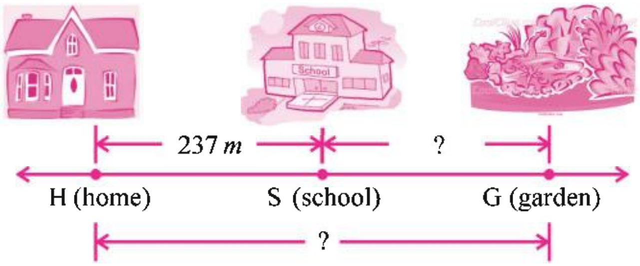


Figure 7.21

Solution :

If H, S, G represent home, school and garden respectively, then H–S–G

$\therefore$ HS + SG = HG (i)

To go to garden and then school, Kavita has to walk from H to G and then G to S.

Thus she covers the distance

HG + GS = 717

$\therefore$ HS + SG + GS = 717

$\therefore$ 237 + 2GS = 717 **(GS = SG)**

$\therefore$ 2SG = 717 − 237 = 480

$\therefore$ SG = $\frac{480}{2}$ = 240 *m*

Also, from (i), HG = HS + SG = 237 + 240 = 477

Thus, the distance between the school and garden is 240 metres and that between home and garden is 477 metres.

EXERCISE 7.3

1. Answer the following :

(1) State the distance postulate.

(2) Explain the symbol X–Y–Z.

(3) State the ruler postulate.

(4) Mention the conditions of 'betweenness'.

2. X, Y and Z are the points on a line *m* and the numbers corresponding to them are 6, −3 and −1 respectively. Find XY, YZ and ZX.

3. P, Q and R are distinct collinear points. P corresponds to 7, Q corresponds to −3 and R corresponds to 3. Which point lies between the other two ?

4. On a line *l*, point A corresponds to −3, B ∈ *l* and AB = 5. Find the number corresponding to B.

5. If A–B–C, BC = 3 and AC = 9, then find AB.

6. Find the missing values in the following table (If there are two values of X and Y take the larger value of them) :

No.	Number Corresponding to X	Number Corresponding to Y	Distance XY
1.	–2	....	5
2.	....	4	7
3.	–6.5	2.5	....
4.	....	–6	8
5.	$3\frac{1}{2}$	....	5
6.	8	–5	....

7. Find the possible values of a :

(1) $|-a| = 5$ (2) $|a - 4| = 5$ (3) $|7 - a| = 10$

(4) $|9 - a| = 11$ (5) $\left|a - \left(-\frac{3}{2}\right)\right| = 4.5$

*

7.8 Line-segment

We have learnt about line-segments in earlier standards. Now we shall learn it in terms of sets.

Like a line, a line-segment is also a set of points.

Figure 7.22

The above figure shows the line-segment PQ as a part of a line. (shown in fig. 7.22)

Line-segment : It is the subset of $\overleftrightarrow{PQ}$ consisting of points lying between P and Q and including P and Q.

- Line-segment is a set of points.
- We denote line-segment PQ as $\overline{PQ}$.
- Line-segment is a subset of a line.
- P and Q are the end-points of $\overline{PQ}$.
- $\overline{PQ}$ includes all the points between P and Q.
- **Length of a line-segment :** If the numbers a, b correspond to points A and B respectively we define the length of $\overline{AB}$ as $|a - b|$ and denote it using symbol AB.

i.e. $AB = | a - b | = \begin{cases} a - b & \text{if } a > b \\ b - a & \text{if } a < b \end{cases}$

where a and b are the numbers corresponding to A and B respectively.

The length of $\overline{AB}$ is represented by AB.

Congruent line segments : We have also learnt that a line-segment AB is represented as $\overline{AB}$.

- Every line-segment has a length.
- If the length of $\overline{MN}$ is x we write MN = x
- **If two line-segments $\overline{XY}$ and $\overline{PQ}$ have equal lengths, then they are said to be congruent. We represent congruent segments $\overline{XY}$ and $\overline{PQ}$ symbolically as $\overline{XY} \cong \overline{PQ}$.**
- $\overline{PQ}$ is a set of points whereas PQ is the length of $\overline{PQ}$. PQ is a non-negative number. Hence we cannot write $\overline{PQ}$ = PQ.
- **Line-segment as a union of sets.**

  $\overline{PQ} = \{P, Q\} \cup \{X \mid P\text{–}X\text{–}Q\}$

  i.e. $\overline{PQ}$ is the union of following two sets :

  (1) Set consisting of the end-points P and Q.

  (2) Set of all the points lying between the end-points P and Q.

Properties of congruent line-segments :

(1) $\overline{XY} \cong \overline{XY}$ **(Reflexivity)**

(2) If $\overline{XY} \cong \overline{PQ}$, then $\overline{PQ} \cong \overline{XY}$ **(Symmetry)**

(3) If $\overline{XY} \cong \overline{PQ}$ and $\overline{PQ} \cong \overline{RS}$, then $\overline{XY} \cong \overline{RS}$ **(Transitivity)**

Example 9 : Answer the following by drawing an appropriate figure :

(1) How many line-segments do four distinct points give rise to ?

(2) How many line-segments do five distinct points give rise to ?

Solution : (1) For example, points A, B, C, D will determine six line-segments $\overline{AB}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$, $\overline{AC}$, $\overline{AD}$. (Figure 7.23)

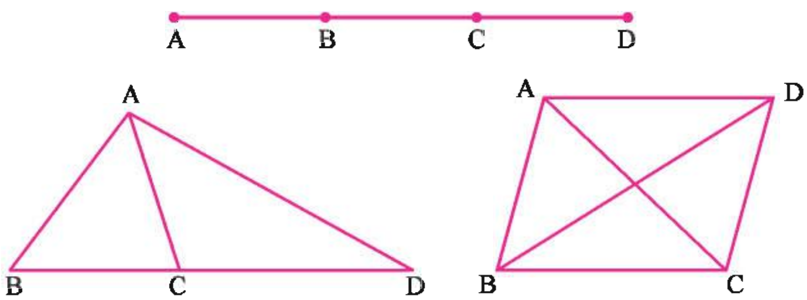


Figure 7.23

(2) Points A, B, C, D, E will determine ten line-segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{BC}$, $\overline{BD}$, $\overline{BE}$, $\overline{CD}$, $\overline{CE}$, $\overline{DE}$. (Figure 7.24)

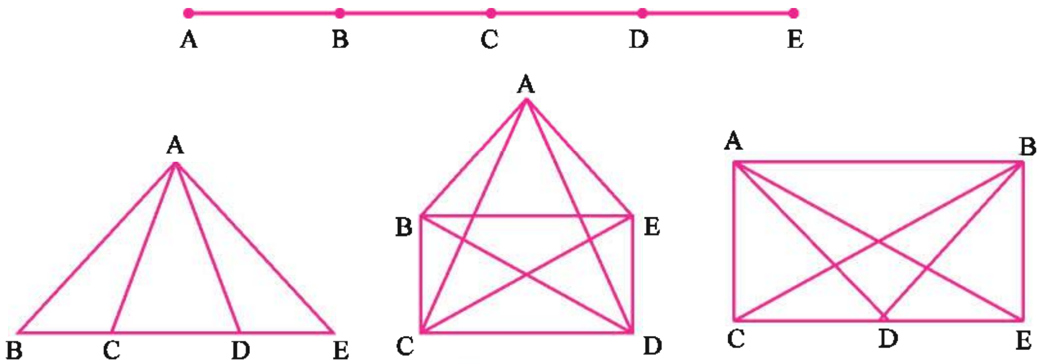


Figure 7.24

Intersection of two line-segments : The intersection of two line-segments would be either (a) a point or (b) a line-segment or (c) empty set.

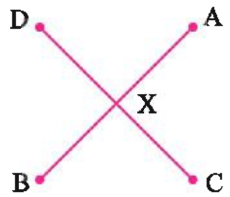


Figure 7.25

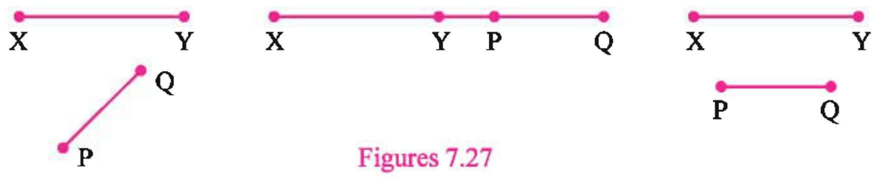
(a) If the intersection of two line-segments is a point then we represent it in set notation as $\overline{AB} \cap \overline{CD} = \{X\}$. (Figure 7.25)

(b) If the intersection of two line-segments is a line-segment we represent it in set notation as : (Figure 7.26)

Figure 7.26

$\overline{PS} \cap \overline{RQ} = \overline{RS}$	$\overline{PS} \cap \overline{PQ} = \overline{PS}$
$\overline{PS} \cap \overline{RS} = \overline{RS}$	$\overline{PR} \cap \overline{PQ} = \overline{PR}$
$\overline{PQ} \cap \overline{RS} = \overline{RS}$	$\overline{RQ} \cap \overline{PQ} = \overline{RQ}$
$\overline{PQ} \cap \overline{SQ} = \overline{SQ}$	$\overline{RQ} \cap \overline{RS} = \overline{RS}$

(c) If the intersection of two line-segments is the empty set, then we can write it in set notation as $\overline{XY} \cap \overline{PQ} = \emptyset$. (Figure 7.27)

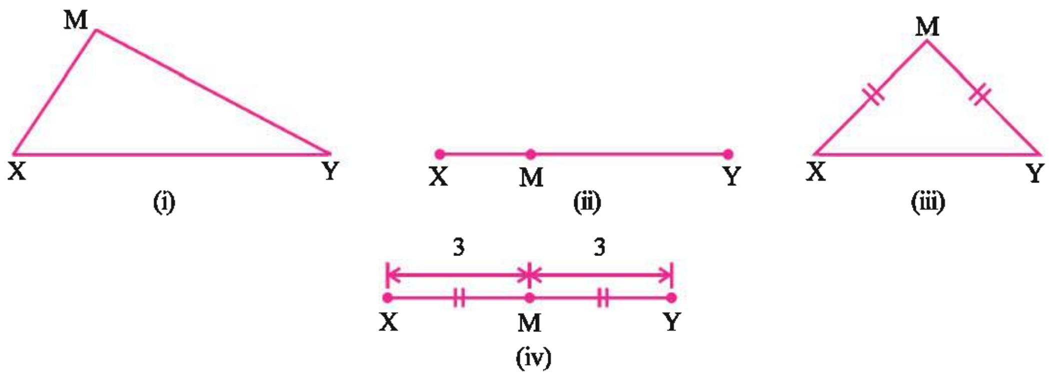


Figures 7.27

All the above figures represent the condition that $\overline{XY} \cap \overline{PQ} = \emptyset$.

Midpoint of a line-segment : A point M is said to be a midpoint of $\overline{XY}$ if (1) M lies between X and Y i.e. X–M–Y.

(2) M is equidistant from X and Y. i.e. XM = MY.



Figures 7.28

In figure 7.28(i) neither X–M–Y nor XM = MY. So M is not a mid-point of $\overline{XY}$.

In figure 7.28(ii), X–M–Y but XM ≠ MY. So M is not a mid-point of $\overline{XY}$.

In figure 7.28(iii), XM = MY but condition X–M–Y is not fulfilled, so M is not a midpoint of $\overline{XY}$.

In figure 7.28(iv), X–M–Y and XM = MY, so we can say that M is a mid-point of $\overline{XY}$. It is clear from the above figure 7.28(iv) that for each line-segment $\overline{XY}$ there exists a unique point M such that X–M–Y and XM = MY. i.e. each line-segment has one and only one mid-point. So we will now say M is the mid-point of $\overline{XY}$ and no other point is a mid-point of $\overline{XY}$. We accept it as a theorem.

Theorem 7.2 : Every line-segment has one and only one mid-point.

- Every line-segment (say $\overline{XY}$) has a mid-point (say M).
- If $\overline{XY}$ has mid-point M, then it cannot have any mid-point other than M.

Number corresponding to the Mid-point : If X, Y correspond to 3 and 9 respectively, then we can find the number corresponding to the mid-point M by two methods. Let m be the number corresponding to the midpoint M.

1st method : We know from the definition of mid-point that X–M–Y.

Figures 7.29

∴ $3 < m < 9$

XM = $| x - m | = | 3 - m | = m - 3$ **(3 < m)**

MY = $| m - 9 | = | 9 - m | = 9 - m$ **(m < 9)**

XM = MY **(definition of mid-point)**

∴ $m - 3 = 9 - m$

∴ $m + m = 9 + 3$

∴ $2m = 12$

∴ $m = 6$

2nd method : If x and y are the numbers corresponding to the points X and Y respectively, then

Number corresponding to M

$$= \frac{\text{Number corresponding to X + Number corresponding to Y}}{2}$$

$$= \frac{x+y}{2} \quad \text{(Prove it !)}$$

$$= \frac{3+9}{2}$$

$$= \frac{12}{2} = 6$$

Example 9 : Let M be the mid-point of $\overline{AB}$. If A and B correspond to $-\frac{5}{2}$ and 8 respectively, then find AM and the number corresponding to M.

Solution :

We know that AB = | number corresponding to A − number corresponding to B |

$$= \left|-\frac{5}{2} - 8\right| = \left|\frac{-5-16}{2}\right| = \left|\frac{-21}{2}\right| = \frac{21}{2} = 10.5$$

Now, $AM = \frac{1}{2}AB = \frac{1}{2}\left(\frac{21}{2}\right) = \frac{21}{4} = 5\frac{1}{4} = 5.25$

Now, number corresponding to M

$$= \frac{\text{Number corresponding to A + Number corresponding to B}}{2}$$

$$= \frac{-\frac{5}{2}+8}{2} = \frac{-5+16}{2 \times 2} = \frac{11}{4} = 2\frac{3}{4} = 2.75$$

Thus, AM = 5.25 and the number corresponding to M is 2.75.

Example 10 : The mid-points of $\overline{PQ}$ and $\overline{PM}$ are M and N respectively. (i) If PN = 6.4, then find NQ. (ii) If PQ = 18.6, find PN.

Figures 7.30

Solution : (i) Here N is the mid-point of $\overline{PM}$ and PN = 6.4

So, PM = 2PN = 2 × 6.4 = 12.8

Now, M is the mid-point of $\overline{PQ}$.

∴ PQ = 2PM = 2 × 12.8 = 25.6

Now, since P–N–Q,

PN + NQ = PQ

∴ 6.4 + NQ = 25.6

∴ NQ = 25.6 − 6.4 = 19.2

Thus, NQ = 19.2

(ii) Now, PQ = 18.6 and M is the mid-point of $\overline{PQ}$.

∴ $PM = \frac{1}{2}PQ = \frac{1}{2} \times 18.6 = 9.3$

Also, since N is the mid-point of $\overline{PM}$,

$PN = \frac{1}{2}PM = \frac{1}{2} \times 9.3 = 4.65$

So, PN = 4.65

Example 11 : The mid-point of $\overline{RS}$ is T. If R corresponds to −7 and T corresponds to $\frac{3}{2}$, then find the number corresponding to S.

Solution : Let a real number x correspond to S. Now the number corresponding to the mid-point T of $\overline{RS}$ is given by

$$\frac{\text{Number corresponding to R + Number corresponding to S}}{2}$$

∴ $\frac{3}{2} = \frac{-7+x}{2}$

∴ $3 = -7 + x$

∴ $x = 3 + 7 = 10$

Thus, the number corresponding to S is 10.

EXERCISE 7.4

1. State the conditions for T to be the mid-point of $\overline{SU}$.

2. Look at the figure 7.31 and answer the following questions :

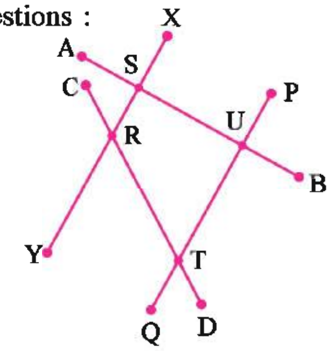
(1) If XY = PQ, then what can you say about $\overline{XY}$ and $\overline{PQ}$?

(2) Find $\overline{AB} \cap \overline{SB}$

(3) Find $\overline{AB} \cap \overline{CD}$

(4) Find $\overline{XY} \cap \overline{PQ}$

(5) Find $\overline{CD} \cap \overline{XY}$



Figures 7.31

3. If A corresponds to 4 and B corresponds to −3, find the number corresponding to the mid-point of $\overline{AB}$.

4. Find number corresponding to the mid-point of $\overline{PQ}$ in each of the following cases, where the number corresponding to P and Q are given as :

(1) -5.5 and 7.5 (2) $2\frac{1}{2}$ and -4.5

(3) -8 and -1 (4) $\sqrt{2}+1$ and $\sqrt{2}-1$

5. Can we conclude that A–B–C, if $\overline{AB} \cup \overline{BC} = \overline{AC}$? Explain by a figure.

*

7.9 Ray

In daily life we use the term ray for light rays, sun rays, X-rays etc. In all these cases there is a single point or source from where the rays are emitted in different directions. Mathematically we may say that a ray has one end-point (initial point).

Ray is a set of points.

Ray : The set of points A and all the points on the side of A towards B on the line $\overleftrightarrow{AB}$ is called ray AB. It is symbolically written as $\overrightarrow{AB}$.

- Point A is called the initial point of $\overrightarrow{AB}$.
- $\overrightarrow{AB}$ extends indefinitely towards B.
- B is not an end-point of $\overrightarrow{AB}$.
- $\overline{AB} \subset \overrightarrow{AB} \subset \overleftrightarrow{AB}$

Figures 7.32

Ray as a union of two sets :

We can write ray XY as $\overrightarrow{XY} = \overline{XY} \cup \{P \mid X\text{–}Y\text{–}P,\ P \in \overleftrightarrow{XY}\}$

- $\overrightarrow{XY}$ is the union of two sets

  1st set : Set of points of $\overline{XY}$.

  2nd set : Set of points P on $\overleftrightarrow{XY}$ which satisfy X–Y–P.

Types of pairs of rays : Observe the pair of rays shown in each part of the figure 7.33 :

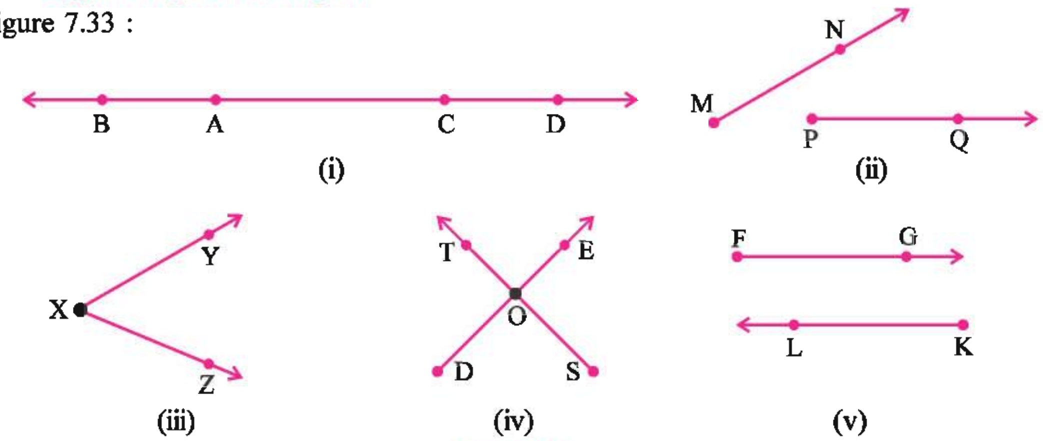


Figure 7.33

The pair of rays in figure 7.33(ii) to (v) use different point sets. These rays are called **distinct rays**. If distinct rays do not intersect each other, then they are called **disjoint rays**. For example, the rays in figure 7.33(ii) and (v) do not intersect each other, i.e. their intersection is null set. These rays are disjoint rays. The rays in figure (iii) and (iv) are distinct but not disjoint, because they intersect in a point.

Rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in figure 7.33(i) are disjoint and distinct rays, but $\overrightarrow{AB}$ and $\overrightarrow{AC}$ or $\overrightarrow{AD}$ and $\overrightarrow{CB}$ are distinct but not disjoint. Why ? (Think !) Similarly, $\overrightarrow{AC}$ and $\overrightarrow{CD}$ are distinct rays, (because all the points on both rays are not same) but not disjoint.

What can you say about $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in figure 7.33(i) ? Those points of line l are in $\overrightarrow{AC}$ are also in $\overrightarrow{AD}$ i.e. the point set of both rays are same. Hence $\overrightarrow{AC} = \overrightarrow{AD}$ and they are **equal rays**. Similarly observe that $\overrightarrow{BA} = \overrightarrow{BC} = \overrightarrow{BD}$.

Now see $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in figure 7.33(i). $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are in same line. Initial point of both rays is A and they are in opposite direction. These rays are called **opposite rays.**

Opposite Rays : Two distinct rays in the same line and having the same initial point are called rays opposite to each other or opposite rays.

Point plotting on a ray : If $\overrightarrow{AB}$ and positive number x are given, then there exists a point P on $\overrightarrow{AB}$ such that AP = x.

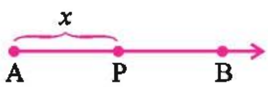


Figure 7.34

Bisector of a line-segment :

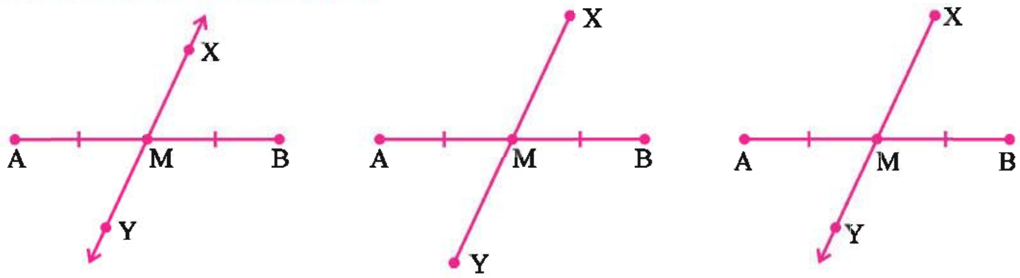


Figure 7.35

Bisector of a line-segment : A line, line-segment or a ray passing through the mid-point of a line-segment is called a bisector of the segment.

It is clear from the definition that,

- We can find the bisector of only a line-segment and not of a ray or a line.
- A bisector can be a line, line-segment or a ray.
- The bisector of $\overline{AB}$ intersects at its mid-point.

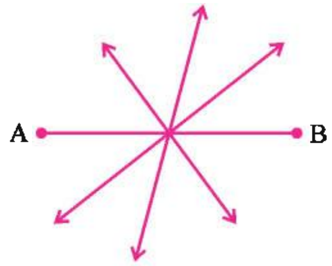


Figure 7.36

- A line-segment has unique mid-point, but bisector of a line-segment is not unique. Every line-segment has infinitely many bisectors.

EXERCISE 7

1. Answer the following questions :

(1) How many end-points does $\overrightarrow{AB}$ have ?

(2) Express $\overrightarrow{XY}$ and $\overrightarrow{AB}$ as a union of two sets.

(3) Name the opposite rays formed when X–Y–Z.

(4) Define the bisector of a line-segment.

2. Draw a figure for each of the following :

(1) $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ are opposite rays.

(2) $\overrightarrow{PQ} \cap \overrightarrow{PR} = \{P\}$ and $\overrightarrow{PR} \cap \overrightarrow{PM} = \{P\}$

(3) $A \in \overrightarrow{BC}$ and $\overrightarrow{BC} \cap \overrightarrow{DA}$ is a singleton set.

3. Answer the following questions with reference to the figure 7.37 :

(1) What is $\overline{BE} \cap \overline{EG}$?

(2) What is $\overline{SG} \cap \overline{FR}$?

(3) What is $\overrightarrow{DF} \cap \overrightarrow{EF}$?

(4) Name the opposite ray of $\overrightarrow{DE}$.

(5) Name the points in the figure lying on line m.

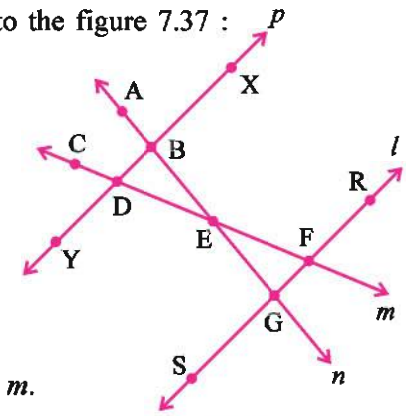


Figure 7.37

4. Write the data and to prove for each of the following statements :

(1) If P–Q–R–S and PR = QS, then PQ = RS.

(2) If P–Q–R and PQ = QR, then Q is the mid-point of $\overline{PR}$.

5. Answer the following :

(1) If X corresponds to -5 and Y corresponds to 7, find the number corresponding to M, the mid-point of $\overline{XY}$.

(2) If A, B and C are points on a line l corresponding to the numbers 7, -2 and 3 respectively, then find AB, BC and AC.

(3) M is the mid-point of $\overline{AB}$. If AB = 10 and M corresponds 7, find the numbers corresponding to A and B.

(4) If X and Y correspond to -4 and 6 respectively, then find the number corresponding to the mid-point of $\overline{XY}$.

(5) M is the mid-point of $\overline{XY}$. If X corresponds -5 and XY = 8, find the number corresponding to Y and M.

6. X, Y, Z are distinct collinear points corresponding to the numbers 2.5, $-\sqrt{2}$ and $\frac{1}{2}$ respectively. Determine which of these three lies between the other two. Represent your answer symbolically.

7. Represent each of the following by a figure :

(1) $P \notin \overleftrightarrow{AB}$ but $Q \in \overleftrightarrow{PB}$.

(2) $\overleftrightarrow{AB} = \overleftrightarrow{PQ}$ but $\overleftrightarrow{AB} \neq \overleftrightarrow{PR}$ and $S \in \overleftrightarrow{QR}$ and R–Q–S.

(3) A–B–C, B–D–E, A–P–E and P–D–Q.

8. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If P–Q–R, ... is the ray opposite to $\overrightarrow{QR}$. ☐

(a) $\overrightarrow{PQ}$ (b) $\overrightarrow{QP}$ (c) $\overrightarrow{RQ}$ (d) $\overrightarrow{RP}$

(2) If PQ = 9 and RS = 9, we can write ...... ☐

(a) $\overline{PQ} \cong \overline{RS}$ (b) $\overline{PQ} = \overline{RS}$ (c) $\overrightarrow{PQ} = \overline{RS}$ (d) $PQ \cong RS$

(3) ...... represents ray XY. ☐

(a) $\overline{XY}$ (b) $\overrightarrow{YX}$ (c) $\overrightarrow{XY}$ (d) $\overleftrightarrow{XY}$

(4) If $\overrightarrow{AB} = \overrightarrow{AC}$, then ...... is not possible. ☐

(a) A–B–C (b) A–C–B (c) B–A–C (d) $\overrightarrow{AB} \cap \overrightarrow{AC} = \overrightarrow{AB}$

(5) If P–Q–R, then point ...... on $\overrightarrow{PQ}$ cannot lie between any two other points of $\overrightarrow{PQ}$. ☐

(a) R (b) P (c) Q (d) all

(6) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays then $\overrightarrow{AB} \cap \overrightarrow{AC}$ = ...

(a) {A} (b) $\overline{AC}$ (c) $\overline{AB}$ (d) $\emptyset$

(7) If X–Y–Z then $\overrightarrow{XZ}$ = ...

(a) $\overleftrightarrow{YZ}$ (b) $\overleftrightarrow{ZX}$ (c) $\overrightarrow{XY}$ (d) $\overrightarrow{YX}$

(8) If X–Y–Z then $\overrightarrow{YZ} \cap \overrightarrow{ZY}$ = ...

(a) $\overleftrightarrow{YZ}$ (b) $\overrightarrow{YZ}$ (c) $\overline{YZ}$ (d) $\overline{XY}$

(9) Every line has at least ...... distinct points.

(a) 1 (b) 2 (c) 3 (d) 4

*

Summary

In this chapter we have learnt :

1. Point, line, line-segment in the context of sets.
2. Relation between a point and a line.
3. Collinear points and non-collinear points.
4. Intersection of two lines.
5. Concept of distance
6. Concept of betweenness
7. Line segment, its length and congruent line segment.
8. Mid-point of a line-segment
9. Ray and types of pairs of rays.
10. Bisector of a line-segment

•

CHAPTER 8

SOME PRIMARY CONCEPTS IN GEOMETRY : 2

8.1 Introduction

Like point and line, plane is also an **undefined term**. As we have learnt in the previous chapter, line is a set of points. Exactly in a similar way plane and space are also sets of points. In this chapter **universal set** will be space.

The top of a desk, the ceiling of a room, a page of a book are the examples of a limited portion (bounded part) of a plane. Like a line, a plane can also be represented by a figure. We usually represent a plane by a rectangle or a parallelogram and we denote a plane by the symbols α, β, γ or π, or sometimes by the alphabets X, Y, Z, P, Q etc. (See figure 8.1)

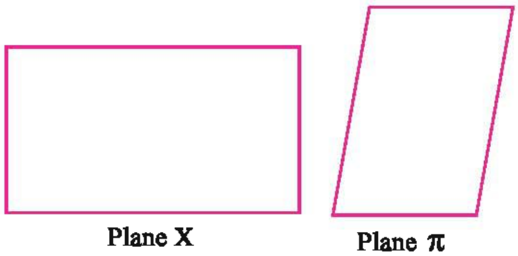


Figure 8.1

Just like a line, a plane is also a set of points but it is quite different from a line in many respects. One of the characteristics of a line is its **'straightness'**, whereas that of plane is its **'flatness'**. A plane extends (expands) in all the directions. Thus a plane does not have any boundary. In figure 8.1, a bounded part of a plane is drawn. Note that a line and a plane are subsets of space.

8.2 Postulates of a Plane

Postulate 1 : Each plane contains at least three non-collinear points.

This postulate says that a plane must have at least three non-collinear points. Further given any three non-collinear points, there is exactly one plane containing them.

Postulate 2 : Three non-collinear points determine one and only one plane.

8.3 Coplanar and Non-coplanar Points and Lines

A plane consists of some points of space. All these points are called **coplanar points**. If there does not exist a plane containing given points, then they are called **non-coplanar points**.

Coplanar points : If there exists a plane containing all of the given points, the points are said to be coplanar.

Non-coplanar points : If there does not exist a plane containing all of the given points, we say they are non-coplanar.

Look at the points A and B in plane α in figure 8.2. Draw line $\overleftrightarrow{AB}$. All the points of $\overleftrightarrow{AB}$ are points of plane α. Why ? Thus, $\overleftrightarrow{AB} \subset$ plane α. Based on this observation, we have the following postulate.

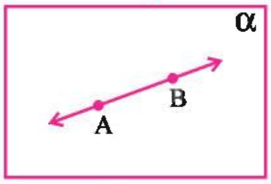


Figure 8.2

Postulate 3 : A line passing through two distinct points of a plane is a subset of the plane.

Each plane contains at least three non-collinear points. If a plane α contains non-collinear points A, B and C, then $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{CA}$ are subsets of the plane α. Such lines are called coplanar lines.

Coplanar lines : If there exists a plane containing all of the given lines, we say the lines are coplanar.

Skew lines : The lines which are not coplanar are called skew lines.

or

Non-coplanar lines : If there does not exist a plane containing all of the given lines, then such lines are called non-coplanar or skew lines.

8.4 Partition of a Plane and Half Plane

Look at the following figure. In each plane there is one line and the line partitions the given plane into three subsets.

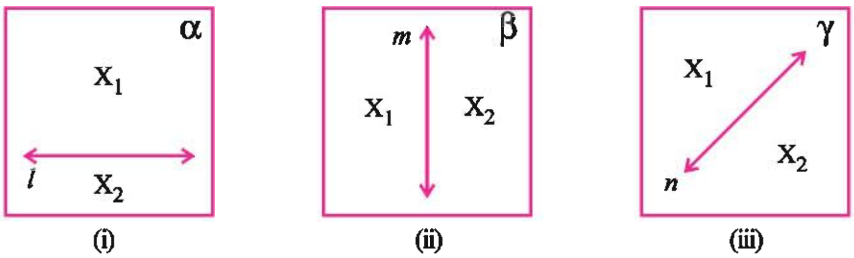


Figure 8.3

In each of the above figures, a line partitions the corresponding plane into three mutually disjoint subsets of points of the plane.

The subsets of the plane on each side of the line are called half planes. If we take union of the line with any of these half planes, the resulting set is called a closed half plane.

Here $X_1 \cup l$ and $X_2 \cup l$ are closed half planes.

For example in the figure 8.3(i) these parts are as follows :

(1) The line itself (line l)

(2) Part X_1 of the plane on one side of the line.

(3) Part X_2 of the plane on the other side of the line.

All these three sets are disjoint sets from each other because if any point is in one set then it cannot be in any of the remaining two sets.

$$l \cap X_1 = \emptyset,\ X_1 \cap X_2 = \emptyset \text{ and } l \cap X_2 = \emptyset .$$

Also, the union of these three sets is equal to the whole plane.

i.e. $X_1 \cup l \cup X_2 = \alpha$.

Now observe the figure 8.4. Line l is in plane α. Point X is on line l. Points P and Q are on different sides of line l. whereas the points Q and R are on the same side of the line l. Draw $\overline{PQ}$ and $\overline{QR}$. Observe that $\overline{PQ}$ intersects the line l but $\overline{QR}$ does not intersect l. Also $\overline{PR}$ intersects the line l, because points P and R are on different sides of the line l.

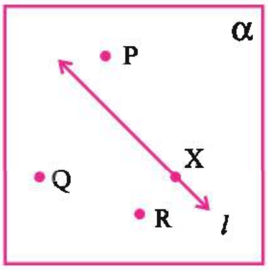


Figure 8.4

We note the following on the basis of the above discussion :

(1) A point on a line in a plane is never in any of the half planes made by the line l.

(2) If two points are in different half planes made by a line, then the line-segment joining them always intersects the line.

(3) If two points are in the same half plane made by a line, then the line-segment joining them never intersects the line.

(4) If points P, Q are in different half planes made by a line l and if points Q and R are in the same half plane made by the line l, the points P and R are in the different half planes made by line l.

8.5 Conditions to Determine a Plane Uniquely

We have already learnt that three non-collinear points determine a plane uniquely.

In figure 8.5 the plane α contains $\overleftrightarrow{XY}$ and point Z outside it. We have learnt earlier that a line has at least two distinct points X and Y. Also Z is a point such that $Z \notin \overleftrightarrow{XY}$. So we can conclude that X, Y, Z are three

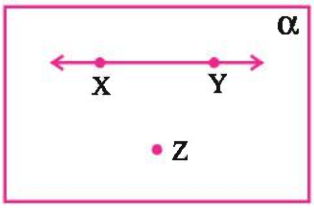


Figure 8.5

non-collinear points. Hence they determine a unique plane. So we can conclude following theorem. We will accept this theorem without proof.

Theorem 8.1 : A line and a point not on it determine a plane uniquely.

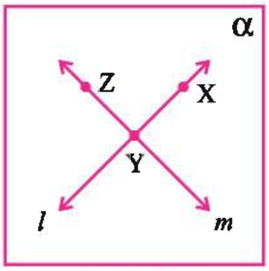


Figure 8.6

With reference to the figure 8.6 can we say that the plane α which contains line *l* and *m* such that $l \cap m = \{Y\}$ is the unique plane containing lines *l* and *m* ?

We know that a line contains at least two distinct points. So we can say that the line *l* contains at least the point Y [point of intersection] and any other point say X. Similarly line *m* contains point Y and any other point, say Z. Obviously since *l* and *m* are intersecting lines, X, Y and Z are non-collinear points. So we can conclude that X, Y and Z determine a plane uniquely or we can conclude the following theorem. We will accept this theorem without proof.

Theorem 8.2 : Two distinct intersecting lines determine a plane uniquely.

So we can conclude that the following three conditions determine a plane uniquely : **(1) Three non-collinear points (2) A line and a point not lying on it (3) Two distinct intersecting lines.**

8.6 Intersection of Two Planes

Till now we have studied various aspects of one plane only. Now we shall consider two or more planes.

Since a plane is a set of points, we can consider various set operations like union and intersection of planes.

There could be two possibilities for relations between any two planes.

(1) Parallel or non-intersecting planes

(2) intersecting planes.

(1) Parallel planes : When two planes are such that there intersection is the empty set, then they are said to be parallel planes.

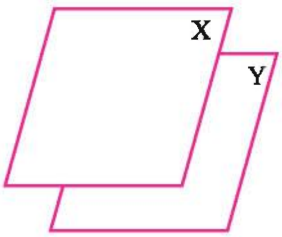


Figure 8.7

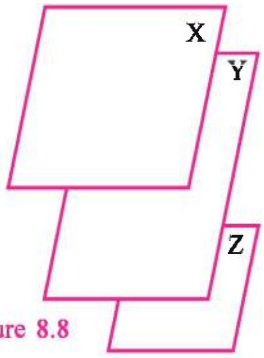


Figure 8.8

In figure 8.7, $X \cap Y = \emptyset$.

In figure 8.8, $X \cap Y = Y \cap Z = Z \cap X = X \cap Y \cap Z = \emptyset$.

(2) Intersecting planes : The intersection of two distinct intersecting planes is a line. (figure 8.9). Thus two distinct planes may or may not intersect. If they intersect, then the intersection is a line. We take this as a postulate.

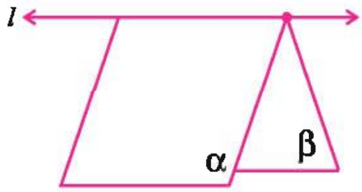


Figure 8.9

Postulate 4 : Two distinct intersecting planes intersect in a line.

In figure 8.9, planes α and β intersect in a line l. Symbolically $\alpha \cap \beta = l$.

8.7 Intersection of Three Distinct Planes

The intersection of three distinct planes could be (1) a line or (2) a point. The figures will explain the situation.

The figure 8.10 explains the intersection of three planes as a line. Symbolically we may write $\alpha \cap \beta \cap \gamma = l$.

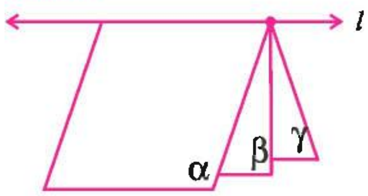


Figure 8.10

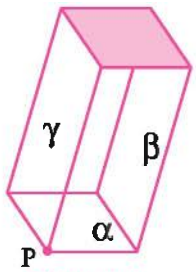


Figure 8.11

Figure 8.11 explains the intersection of three planes as a point. Symbolically we may write $\alpha \cap \beta \cap \gamma = \{P\}$.

Thus the intersection of three distinct planes is null set or a singleton set or a line.

EXERCISE 8.1

1. State the conditions that determine a plane uniquely.
2. Look at the figure 8.12 and answer the following questions :
   (1) Which points belong to plane β ?
   (2) Write symbolically the relation between the line l and plane β.
   (3) Mention the points lying in the each of the half planes, determined by line l.

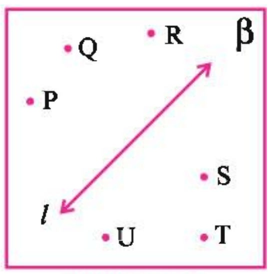


Figure 8.12

3. Discuss intersection of two or three planes.
4. Define : (1) Coplanar and non-coplanar points.
   (2) Coplanar and non-coplanar lines.
5. Look at the figure 8.13 and answer the following :
   (1) Name the three planes.
   (2) Write the points shown in plane γ.
   (3) Are A, B, C, D coplanar ?
   (4) Are P, Q, D, C coplanar ?
   (5) What do you say about the points A, B, S, T, collectively ?
   (6) Of which plane is $\overleftrightarrow{XY}$ a subset ?

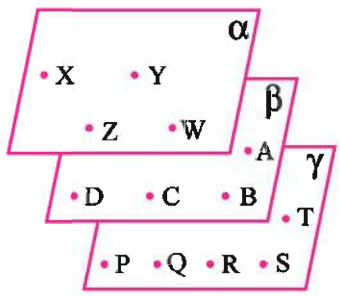


Figure 8.13

6. Draw the figures representing the following situations :

(1) Line $l \cap$ plane $X = \{A\}$

(2) Distinct planes X, Y and Z are such that $X \cap Y \cap Z = \emptyset$

*

8.8 Angle

We have already studied how to identity different angles in a given shape. Observe the 'angles' in the figures 8.14 :

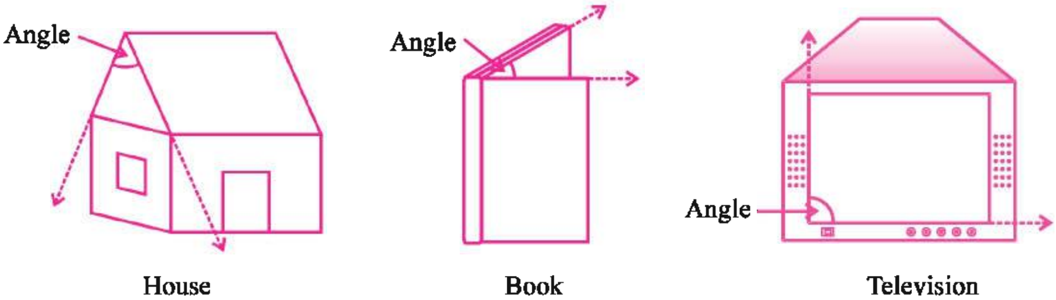


Figure 8.14

From the figures 8.14 we can conclude that where there is an 'angle', there are two rays with a common initial point forming an angle. Both these rays are not in the same line.

We shall now define an angle in the language of set theory.

Angle : The union of two distinct rays having the same initial point and not lying in the same line is called an angle.

In the figure 8.15 distinct rays $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ have the same initial point Y. They are also not in the same line.

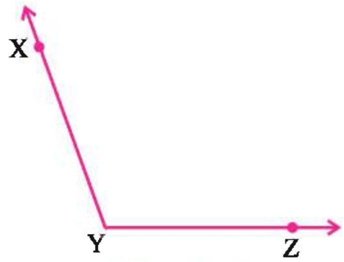


Figure 8.15

Thus, we can say that the union of $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ is an angle and it is denoted as $\angle XYZ$ or $\angle ZYX$.

Thus, $\overrightarrow{YX} \cup \overrightarrow{YZ} = \angle XYZ$ or $\angle ZYX$.

Thus, an $\angle XYZ$ is the union of two rays $\overrightarrow{YX}$ and $\overrightarrow{YZ}$. Here point Y is called the **vertex** of the $\angle XYZ$. The rays $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ are called the **arms** of the $\angle XYZ$. If there is no confusion i.e. if Y is also not a vertex of another angle we call it $\angle Y$.

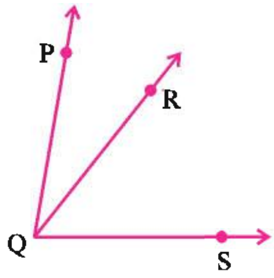


Figure 8.16

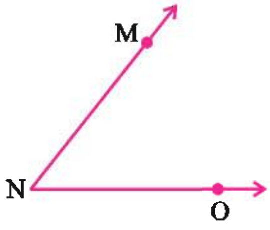


Figure 8.17

In the figure 8.16, Q is the vertex of ∠PQR, ∠RQS and ∠PQS whereas in figure 8.17, N is the vertex of only one angle ∠MNO. So in such cases where a point is the vertex of only one angle, we can write the angle ∠N only for ∠MNO. But we cannot write ∠Q for ∠PQR as ∠Q may represent ∠PQS or ∠RQS.

Example 1 : Mention the vertex and arms of ∠ABC. Also draw the figure and express it as a union of two rays.

Solution : Point B is the vertex of ∠ABC.

$\overrightarrow{BA}$ and $\overrightarrow{BC}$ are the arms of the angle.

$\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$ (fig. 8.18)

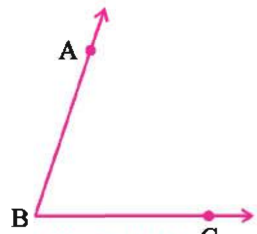


Figure 8.18

Example 2 : For which of the figures 8.19 can we write ∠B for ∠ABC ? Give reasons.

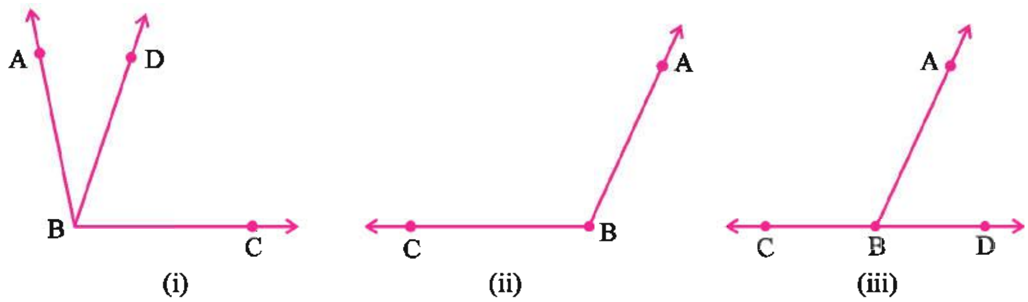


Figure 8.19

Solution : With reference to figure (i) and (iii) we cannot write ∠B for ∠ABC as point B is the vertex of more than one angle.

But in figure (ii) we can see that vertex B is the vertex of ∠ABC only. So in this case we can write ∠B for ∠ABC.

8.9 Interior of an Angle

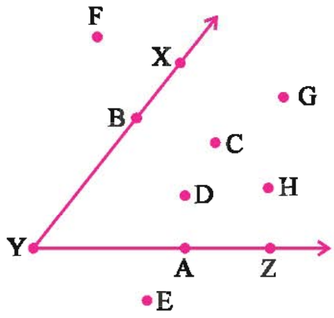


Figure 8.20

Observe the figure 8.20.

Where do the points C, D, G and H lie ? Where are the points X, B, A and Z ? Where do the points E and F lie ?

The points C, D, G and H lie in the **'interior'** of ∠XYZ. The points X, B, A and Z are on the ∠XYZ and the points F and E are in the **'exterior'** of ∠XYZ.

So we can say that the interior points of ∠XYZ lie in the half plane made by $\overleftrightarrow{YZ}$ containing the point X and also in the half plane made by $\overleftrightarrow{XY}$ containing point Z.

So, the **interior of ∠XYZ is the intersection of the half plane of $\overleftrightarrow{YZ}$ containing X and the half plane of $\overleftrightarrow{XY}$ containing Z.**

The interior of ∠XYZ = (The half plane of $\overleftrightarrow{YZ}$ containing X) ∩ (the half plane of $\overleftrightarrow{XY}$ containing Z).

8.10 Partition of the Plane by an Angle

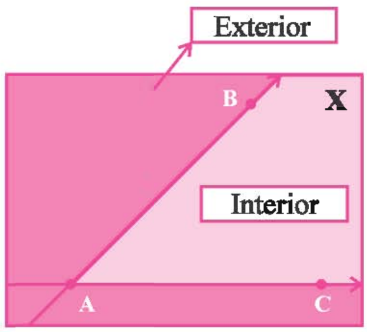


Figure 8.21

Like a line, an angle also divides the plane into three mutually disjoint sets. The angle ∠BAC divides the plane into the following three subsets of plane X.

(1) ∠BAC

(2) Interior of ∠BAC

(3) Exterior of ∠BAC

8.11 An Important Result

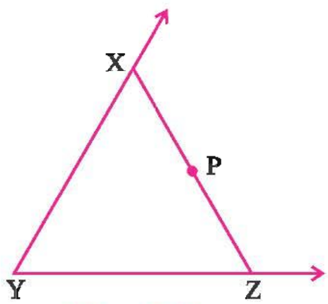


Figure 8.22

Observe the figure 8.22. The end-points of $\overline{XZ}$ are on ∠XYZ. What can you say about the point P which is on $\overline{XZ}$?

Obviously, it is in the interior of ∠XYZ.

So we can say that for any ∠XYZ all the points lying on $\overline{XZ}$ are in the interior of ∠XYZ.

i.e. for ∠XYZ, if X–P–Z, then P is in the interior of ∠XYZ.

Cross Bar Theorem : Look at the figure 8.23. Points X, Y and Z are in the interior of ∠PQR. Now draw $\overline{PR}$, $\overrightarrow{QX}$, $\overrightarrow{QY}$, $\overrightarrow{QZ}$. **We will observe that $\overrightarrow{QX}$, $\overrightarrow{QY}$, $\overrightarrow{QZ}$ will intersect $\overline{PR}$.** This concept we shall state as a theorem called **the cross bar theorm**. We will not prove it.

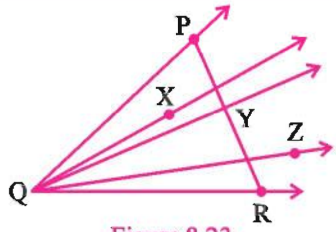


Figure 8.23

Theorem 8.3 (Cross Bar theorem) : If point D is in the interior of an angle ∠BAC, then $\overrightarrow{AD}$ intersects $\overline{BC}$.

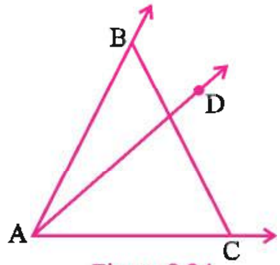


Figure 8.24

Example 3 : Look at the figure 8.25 and answer the following questions :

(1) Name an angle shown in plane α.

(2) List the points which are in the interior, of ∠XYZ and the points which are in the exterior of ∠XYZ as shown.

(3) Which rays intersect $\overline{XZ}$?

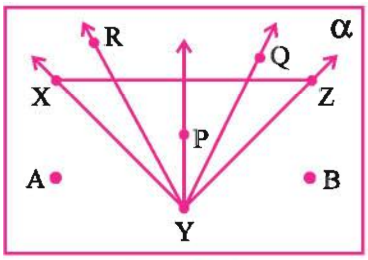


Figure 8.25

Solution :

(1) One of the angles shown in the plane α is ∠XYZ.

(2) Points P, Q and R are in the interior of ∠XYZ. Points A and B are in the exterior of ∠XYZ.

(3) $\overrightarrow{YP}$, $\overrightarrow{YQ}$ and $\overrightarrow{YR}$ intersect $\overline{XZ}$ according to cross bar theorem.

EXERCISE 8.2

1. Look at the figure 8.26 and answer the following questions :

(1) Name the arms of ∠ABC.

(2) Where are the points D, J and G ?

(3) Name the vertex of ∠ABC.

(4) State the partition of the plane α by ∠ABC.

(5) According to Cross Bar theorem name all the possible rays which intersect $\overline{AC}$.

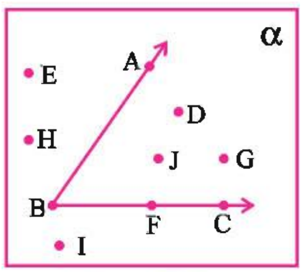


Figure 8.26

2. Define : (1) Coplanar lines (2) Skew lines
(3) Coplanar points (4) A closed half plane.

3. P $\notin \overleftrightarrow{QR}$ and P–M–R. In the interior of which angle does the point M lie ? If P-N-Q, then in the interior of which angle does N lie ?

4. Write the cross bar theorem. Explain it with a figure.

5. $\angle$ABC is given and points X and Y are such that X–A–C–Y. With respect to $\angle$ABC, where do points X and Y lie ?

*

8.12 Measure of an Angle

Just like a line-segment, an angle can also be measured. We know how to draw and measure an angle using a protractor. In doing so we have been using some postulates of angles unknowingly. Now we shall state these postulates.

Postulate 5 : (Postulate for the measure of an angle) Corresponding to every angle, there is one and only one real number between 0 and 180, called the measure of the angle.

It is clear from the postulate of the measure of an angle that -

(1) an angle has exactly one measure

(2) an angle does not have the measure 0 or 180 or less than 0 or greater than 180.

We use '*m*' to denote the measure of an angle. e.g., the measure of $\angle$BAC is written as $m\angle$BAC. Clearly, $\angle$BAC is a set where as $m\angle$BAC is a number. Thus **$\angle$BAC = $m\angle$BAC is meaningless.**

Postulate 6 : (Postulate of Unique Ray) : If $0 < x < 180$ and $\overrightarrow{AB}$ is given, we can get a unique ray $\overrightarrow{AC}$ originating from A in each closed half plane made by $\overleftrightarrow{AB}$ of the plane X such that $m\angle CAB = x$

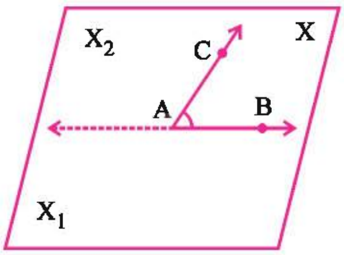


Figure 8.27

Postulate 7 : (Postulate about sum of the measures of angles.) If a point W is in the interior of $\angle$XYZ, then

$$m\angle XYW + m\angle WYZ = m\angle XYZ.$$

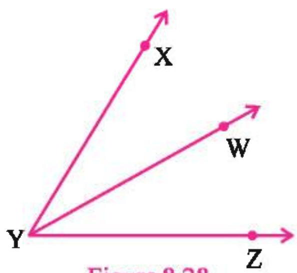


Figure 8.28

Example 4 : Point C is in the interior of ∠BOA. The measures of ∠BOC and ∠COA are in the ratio 3:2. If $m\angle BOA = 60$, then find the measure of each angle.

Solution : Let $m\angle BOC = 3x$ and so $m\angle COA = 2x$.

By postulate 7 $m\angle BOC + m\angle COA = m\angle BOA$

$\therefore \quad 3x + 2x = 60$ (Postulate 6)

$\therefore \quad 5x = 60$

$\therefore \quad x = \frac{60}{5} = 12$

$m\angle BOC = 3x = 3 \times 12 = 36,$

$m\angle COA = 2x = 2 \times 12 = 24$

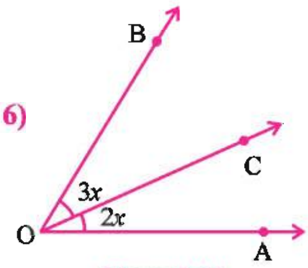


Figure 8.29

8.13 Types of angles based on the measure

(1) Right Angle : An angle having the measure 90 is called a right angle.

(2) Acute Angle : An angle having the measure less than 90 is called an acute angle.

(3) Obtuse Angle : An angle having the measure more than 90 is called an obtuse angle.

8.14 Types of pairs of angles based on their measures

(1) Complementary angles : Two angles are said to be complementary to each other if the sum of their measures is 90.

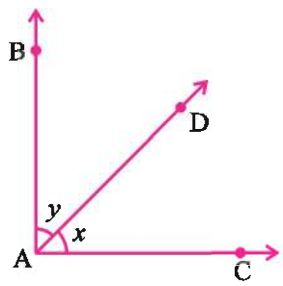


Figure 8.30

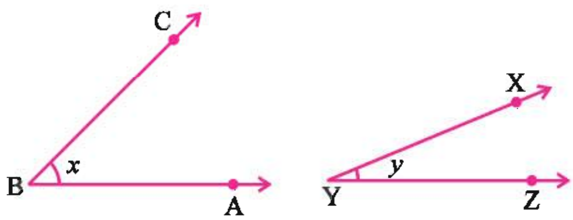


Figure 8.31

In figure 8.30 x and y denote measures of angles ∠BAD and ∠DAC $x + y = 90$. So ∠BAD and ∠CAD are complementary angles of each other. Similarly in figure 8.31, ∠ABC and ∠XYZ are complementary angles of each other if $x + y = 90$.

(2) Supplementary angles : Two angles are said to be supplementary to each other if the sum of their measures is 180.

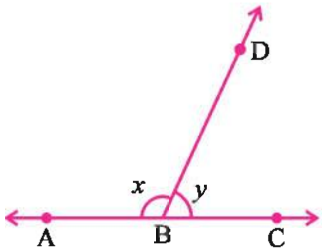


Figure 8.32

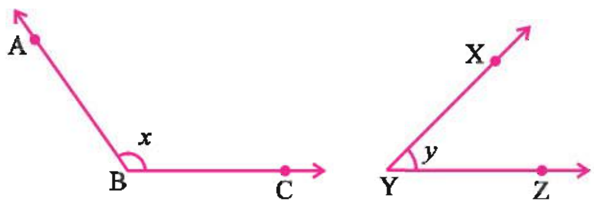


Figure 8.33

x and y denote measures of ∠ABD and ∠DBC in figure 8.32 and $x + y = 180$. So ∠ABD and ∠CBD are supplementary angles of each other. Similarly in figure 8.33, ∠ABC and ∠XYZ are supplementary angles of each other if $x + y = 180$.

(3) Congruent angles : If two angles have same measure, they are said to be congruent angles and ≅ is the symbol used to show that two angles are congruent.

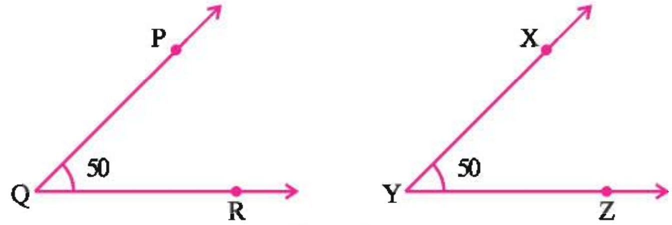


Figure 8.34

With reference to the figure 8.34, ∠PQR and ∠XYZ are said to be congruent angles as their measures are equal.

∴ $m\angle PQR = m\angle XYZ = 50$

∴ ∠PQR ≅ ∠XYZ

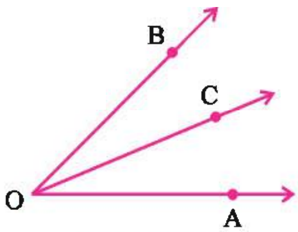


Figure 8.35 (i)

8.15 Types of Pairs of Angles Based on Their Arms

(1) Adjacent angles : Two angles are said to be adjacent angles if

(i) They have the same vertex.

(ii) They have a common arm.

(iii) Uncommon arms are on either side of the common arm i.e. in different half planes of the line containing the common arm.

∴ In figure 8.35(i) ∠AOC and ∠BOC are adjacent angles and O is their common vertex and $\overrightarrow{OC}$ is their common arm. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are on the either side of $\overrightarrow{OC}$.

(2) Linear pairs of angles : Two adjacent angles are said to form a linear pair, if their uncommon arms are opposite rays.

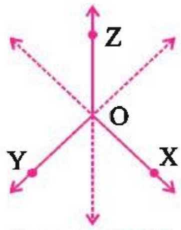


Figure 8.35 (ii)

In figure 8.35 (ii) ∠XOY and ∠YOZ; ∠YOZ and ∠ZOX and ∠ZOX and ∠XOY are adjacent angles.

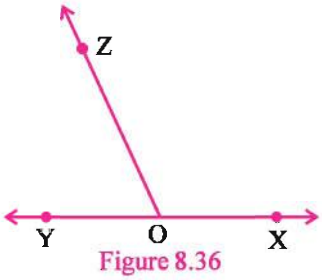


Figure 8.36

In the figure 8.36, ∠XOZ and ∠YOZ are adjacent angles and the uncommon arms $\overrightarrow{OY}$ and $\overrightarrow{OX}$ are opposite rays. So ∠XOZ and YOZ form supplementary angles.

Note : All linear pair of angles are adjacent angles but converse is not always true.

(3) Vertically opposite angles : Two angles are said to form a pair of vertically opposite angles, if their arms form two pairs of opposite rays.

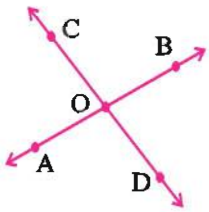


Figure 8.37

In the figure 8.37, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in O. Thus ∠AOC and ∠BOD, ∠COB and ∠AOD are formed.

∠AOC and ∠BOD form a pair of vertically opposite angles. Similarly ∠COB and ∠AOD also from a pair of vertically opposite angles.

8.16 Bisector of an Angle

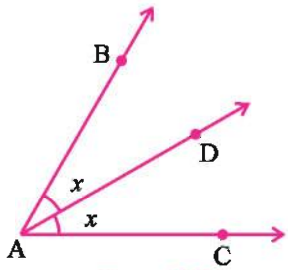


Figure 8.38

In the figure 8.38, point D is inside ∠BAC such that congruent angles ∠BAD and ∠DAC are formed i.e. $m\angle BAD = m\angle DAC$. $\overrightarrow{AD}$ is the bisector of ∠BAC.

Bisector of an Angle : If D is in the interior of ∠BAC in such a way that $m\angle BAD = m\angle DAC$, then $\overrightarrow{AD}$ is called a bisector of ∠BAC.

Example 5 : The measure of an angle is equal to five times the measure of its complementary angle. Determine its measure.

Solution : Let the measure of the given angle be x. Then, its complementary angle has measure $(90 - x)$.

The measure of given angle = 5 × measure of complementary angle of the given angle

∴ $x = 5(90 - x)$

$x = 450 - 5x$

$6x = 450$

$x = 75$

∴ Measure of the given angle is 75.

Example 6 : $\overrightarrow{QA}$ and $\overrightarrow{QB}$ are the bisectors of ∠PQR and ∠PQA respectively. If $m\angle AQB = 17$, then find $m\angle PQR$.

Solution : $\overrightarrow{QB}$ is the bisector of ∠PQA.

∴ $m\angle PQB = m\angle AQB = 17$

∴ $m\angle PQB = 17$

Also, $m\angle PQB + m\angle AQB = m\angle PQA$

(Postulate about sum of measures of angles)

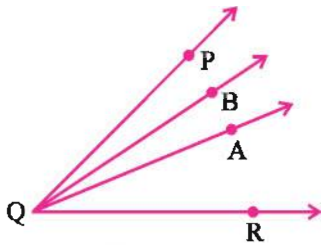


Figure 8.39

∴ $m\angle PQA = 17 + 17 = 34$

Now $\overrightarrow{QA}$ bisects ∠PQR.

∴ $m\angle AQR = m\angle PQA = 34$

Also $m\angle PQA + m\angle AQR = m\angle PQR$ **(Postulate about sum of measures of angles)**

∴ $m\angle PQR = 34 + 34 = 68$

8.17 Theorems About Angles

We have defined the bisector of an angle. But does every angle have a bisector or more than one bisectors ? A line-segment has exactly one mid-point. If an angle has a bisector, will it be unique ? Let us think.

Draw ∠CAB having measure 60. Take point D in the interior of ∠BAC, such that $m\angle DAB = 30$. Now using the postulate of sum of measures of two angles, we see that $m\angle CAD = 30$. Here $\overrightarrow{AD}$ is a bisector of ∠CAB. Can we get yet another bisector of ∠CAB ? No, according to unique ray postulate we can't have another angle bisector. We have the following theorem which we accept without proof.

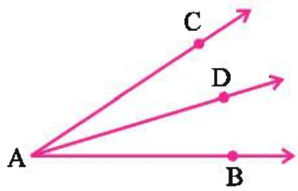


Figure 8.40

Theorem 8.4 : Every angle has one and only one bisector.

Look at the figure 8.41. ∠RPS and ∠SPQ are congruent angles and also supplementary to each other.

∴ $m\angle RPS + m\angle QPS = 180$

also $\angle RPS \cong \angle QPS$

∴ $m\angle RPS = m\angle QPS$

∴ $2m\angle RPS = 180$

∴ $m\angle RPS = 90$ and $m\angle QPS = 90$

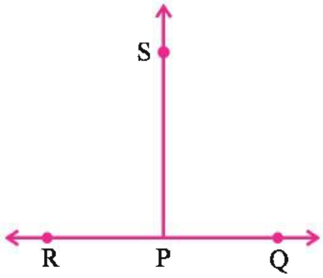


Figure 8.41

Thus if two congruent angles are supplementary, then each angle is a right angle. We will accept this theorem without proof.

Theorem 8.5 : If two congruent angles are supplementary, then each of them is a right angle.

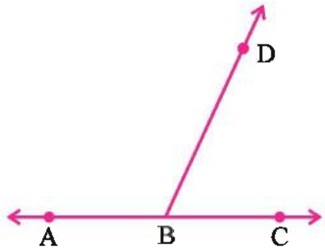


Figure 8.42

Now look at the figure 8.42. ∠ABD and ∠CBD form a linear pair. Find their measures and add. See that the sum is 180. i.e. they are supplementary. We accept this result as a postulate.

Postulate 8 : Angles forming a linear pair are supplementary.

If two lines intersect each other, then four angles are formed at the point of intersection. These angles are vertically opposite angles. We will prove the theorem given below about vertically opposite angles.

Theorem 8.6 : If two lines intersect at a point, vertically opposite angles are congruent.

Data : Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O. So (i) ∠AOD and ∠COB (ii) ∠AOC and ∠DOB are vertically opposite angles.

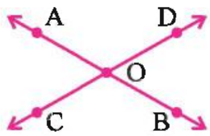


Figure 8.43

To prove that : $\angle AOD \cong \angle COB$

$\angle AOC \cong \angle BOD$

Proof : ∠AOD and ∠BOD are angles of a linear pair.

$m\angle AOD + m\angle BOD = 180$ **(Postulate 8)** (i)

∠BOD and ∠COB are angles of a linear pair.

$m\angle BOD + m\angle COB = 180$ **(Postulate 8)** (ii)

From results (i) and (ii)

∴ $m\angle AOD + m\angle BOD = m\angle BOD + m\angle COB$

∴ $m\angle AOD = m\angle COB$

∴ $\angle AOD \cong \angle COB$

Similarly we can prove, $\angle AOC \cong \angle BOD$.

Example 7 : In figure 8.44, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect each other at point O. If $m\angle AOD : m\angle BOD = 5:7$, find measures of all the angles.

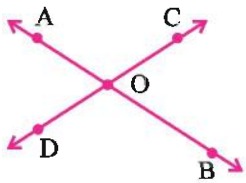


Figure 8.44

Solution : $m\angle AOD + m\angle BOD = 180$ **(Postulate 8)**

But $m\angle AOD : m\angle BOD = 5:7$ **(given)**

Let $m\angle AOD = 5x$, $m\angle BOD = 7x$

$\therefore \quad 5x + 7x = 180$

$\therefore \quad 12x = 180$

$\therefore \quad x = \frac{180}{12} = 15$

$m\angle AOD = 5x = 5 \times 15 = 75$

$m\angle BOD = 7x = 7 \times 15 = 105$

$m\angle BOC = m\angle AOD = 75$ and

$m\angle AOC = m\angle BOD = 105$

Example 8 : From the figure 8.45, prove that sum of all the angles around the point O by $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ and $\overrightarrow{OE}$ is equal to 360.

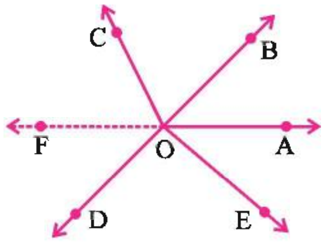


Figure 8.45

Solution : At common vertex O angles are formed by rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ and $\overrightarrow{OE}$ as in figure 8.45.

We need to prove that

$m\angle AOB + m\angle BOC + m\angle COD + m\angle DOE + m\angle EOA = 360$

Let $\overrightarrow{OF}$ be the ray opposite to $\overrightarrow{OA}$.

Since $\overrightarrow{OB}$ intersects $\overleftrightarrow{FA}$ at O, $m\angle AOB + m\angle BOF = 180$ **(Linear pair)**

$\therefore \quad m\angle AOB + m\angle BOC + m\angle COF = 180$ **($m\angle BOF = m\angle BOC + m\angle COF$) (i)**

Again $\overrightarrow{OD}$ intersects $\overleftrightarrow{FA}$ at O.

$\therefore \quad m\angle FOD + m\angle DOA = 180$

$\therefore \quad m\angle FOD + m\angle DOE + m\angle EOA = 180$ **(ii)**

$m\angle AOB + m\angle BOC + m\angle COF + m\angle FOD + m\angle DOE + m\angle EOA$
$= 180 + 180 = 360$

$\therefore \quad m\angle AOB + m\angle BOC + m\angle COD + m\angle DOE + m\angle EOA = 360$

(Since F is inside $\angle COD$, $m\angle COF + m\angle FOD = m\angle COD$)

EXERCISE 8.3

1. In the figure 8.46, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays.

$3x$ and $2y + 5$ are measures of $\angle BOC$ and $\angle COA$ respectively.

(1) If $x = 25$, what is the value of y ?

(2) If $y = 65$, what is the value of x ?

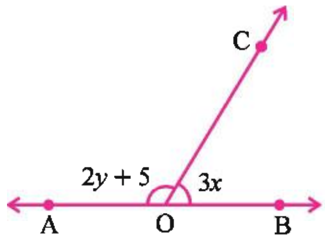


Figure 8.46

2. With reference to figure 8.47, write all pairs of adjacent angles and all the linear pairs.

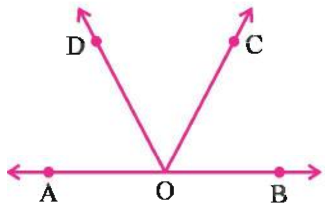


Figure 8.47

3. In the figure 8.48, $\overleftrightarrow{XY}$ and $\overleftrightarrow{MN}$ intersect in O.

$m\angle MOP = a$, $m\angle MOX = b$, $m\angle NOX = c$ If $m\angle POY = 90$ and $a : b = 2 : 3$, find c.

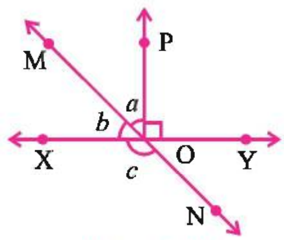


Figure 8.48

4. In the figure 8.49, x is greater than y by one third of a right angle. Find the value of x and y. x and y are measures of angles shown.

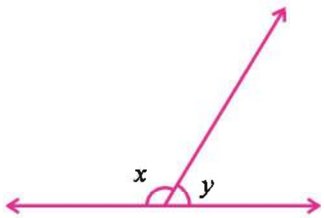


Figure 8.49

5. $\overleftrightarrow{OC}$ is perpendicular to $\overleftrightarrow{AB}$ in the figure 8.50. S is in the interior of $\angle AOC$. Prove that $m\angle COS = \frac{1}{2}(m\angle BOS - m\angle AOS)$.

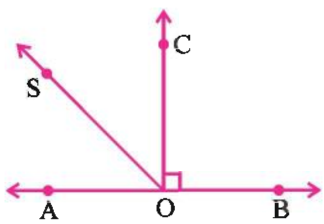


Figure 8.50

6. Answer the following :

(1) If two supplementary angles have equal measures, what is the measure of each angle ?

(2) The measure of two supplementary angles differ by 34. Find the measure of the angles.

(3) Find the measure of the complementary angle of the supplementary angle of the angle having measure 120.

(4) Find the measure of the complementary angle of the angle with following measure :

(i) 42 (ii) 37 (iii) $10 + x$ (iv) 81

(5) Find the measure of the supplementary angle of each of the angle with following measure :

(i) 100 (ii) 89 (iii) $(y - 30)$ (iv) 49

*

8.18 Intersection of Two Lines

Let us review, we studied about intersection of two lines in previous chapter.

Now, there are three possibilities for the intersection of two coplanar lines l and m.

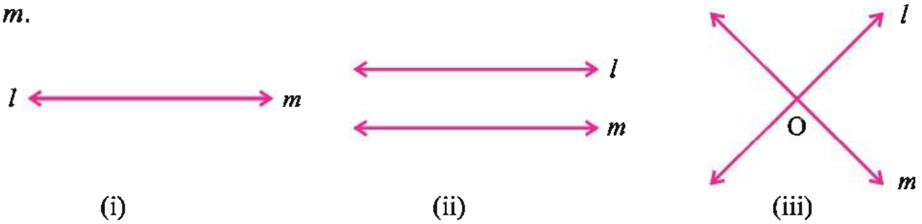


Figure 8.51

In figure 8.51(i) lines l and m are same lines, that is $l = m$. Then all the points of both the lines are the same. So, the intersection of l and m is the line itself. Thus in this case, $l \cap m = l = m$.

In figure 8.51(ii) coplanar lines l and m do not intersect each other. i.e. the lines do not have any point in common. Here the intersection of l and m is the empty set. Thus in this case $l \cap m = \emptyset$. **Such lines are called parallel lines. We denote l is parallel to m by $l \parallel m$. If two distinct coplanar lines do not intersect, they are called parallel lines.**

In figure 8.51(iii) l and m intersect each other in a point, that is l and m have one point in common. Thus, in this case $l \cap m = \{O\}$. **If two distinct lines intersect, then they intersect in exactly one point.**

8.19 Postulate and Theorems for Parallel Lines

Now we will learn more about a theorem and a postulate useful for study of parallel lines. We will accept the theorem.

Theorem 8.7 : Given a line and a point outside it, there exists at least one line passing through the point and parallel to the given line.

Postulate of Parallel Line (Postulate 9) : Given a line and a point outside it, there exists at most one line passing through the point and parallel to the given line.

Combining these two statements into one, we derive following rule :

"Given a line and a point outside it, there exists one and only one line passing through the point and parallel to the given line."

Now we will study proof of two important results. They are useful for study of parallel lines.

Result 1 : Two distinct coplanar lines which are both parallel to another line in the same plane are parallel to each other.

Data : l_1, l_2 and l_3 are three distinct coplanar lines such that $l_1 \parallel l_2$ and $l_3 \parallel l_2$.

To prove : $l_1 \parallel l_3$

Figure 8.52

Proof : For two coplanar lines l_1 and l_3 there can be two possibilities.

(1) $l_1 \parallel l_3$ (2) l_1 is not parallel to l_3.

(1) If l_1 is parallel to l_3, there is nothing to prove.

(2) If l_1 is not parallel to l_3, then l_1 and l_3 are intersecting lines. Let them intersect in P. **(Lines are distinct)**

Now $P \in l_1$ and $P \in l_3$.

Also $P \notin l_2$ as $l_1 \cap l_2 = \emptyset$.

Since $P \notin l_2$, there exists exactly one line through P, parallel to l_2.

But l_1 and l_3 are two lines passing through P and parallel to l_2.

$\therefore$ $l_1 = l_3$ is a contradiction since $l_1 \neq l_3$.

$\therefore$ Alternative (2) is not possible. Hence $l_1 \parallel l_3$.

Result 2 : If l_1, l_2 and l_3 are three distinct coplanar lines such that l_1 intersects l_2 and $l_3 \parallel l_2$, then l_1 intersects l_3 also.

Data : Coplanar distinct lines l_1, l_2 and l_3 are such that l_1 intersects l_2 and $l_3 \parallel l_2$.

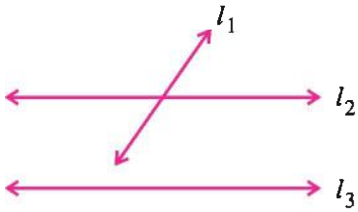


Figure 8.53

To prove : Lines l_1 and l_3 are intersecting lines.

Proof : Let us suppose that lines l_1 and l_3 are non-intersecting coplanar lines.

$\therefore$ l_1 and l_3 are parallel lines. i.e. $l_1 \parallel l_3$

But $l_3 \parallel l_2$

$\therefore$ $l_1 \parallel l_3$ and $l_3 \parallel l_2$.

$\therefore$ $l_1 \parallel l_2$ **(Result 1)**

But this contradicts the data that l_1 and l_2 are intersecting lines.

$\therefore$ Our supposition that l_1 and l_3 are non-intersecting lines [parallel lines] is false.

$\therefore$ l_1 and l_3 are intersecting lines.

8.20 Intersection of Three Lines

Consider three coplanar lines *l*, *m* and *n*. They are shown in the figure 8.54.

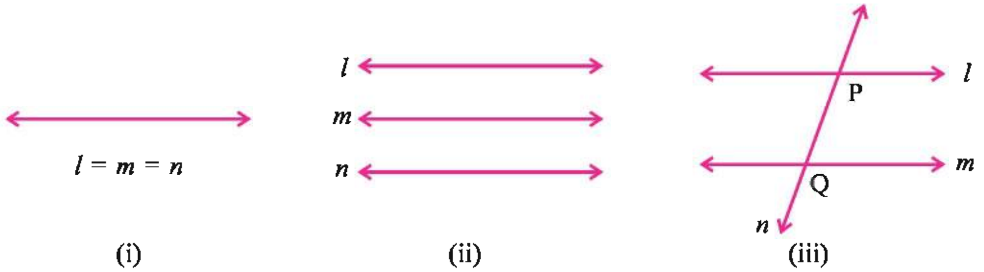


Figure 8.54

There are several possibilities for their intersection.

(1) Line *l*, *m* and *n* are same lines, so all the points of *l*, *m* and *n* are common, that is $l = m = n$. In this case $l \cap m \cap n = l = m = n$.

(2) The three lines are parallel, so that $l \parallel m$ and $m \parallel n$. So $l \cap m \cap n = \emptyset$

(3) Exactly two of the lines *l*, *m* and *n* are parallel. In this case, the third line that is not parallel to the other two, intersects these two lines in two distinct points. In the figure 8.54 (iii) $l \parallel m$ and *n* intersects *l* and *m* both in two distinct points. Here *n* is called a **transversal** of *l* and *m*. $\overline{PQ}$ is called **intercept** made by *l* and *m* on *n*.

If all these three lines are distinct and no two of them are parallel, then we have two possibilities for their intersection.

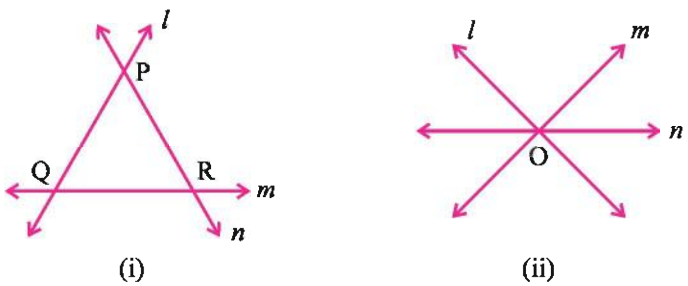


Figure 8.55

(1) Out of all the three lines, three pairs intersect in distinct points.

(2) All the three lines intersect in exactly one point. So, $l \cap m \cap n = \{O\}$.

8.21 Parallel lines and Transversal

Let l and m be two parallel lines and let line t be their transversal.

As in the figure 8.56, a total of eight angles are formed by the coplanar lines l, m and their transversal t. These include

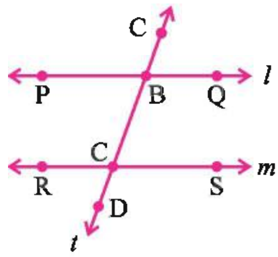


Figure 8.56

- **Four pairs of corresponding angles :**

  (1) ∠ABQ and ∠BCS (2) ∠QBC and ∠SCD

  (3) ∠ABP and ∠BCR (4) ∠PBC and ∠RCD

- **Two pairs of alternate angles :**

  (1) ∠QBC and ∠BCR (2) ∠PBC and ∠BCS

- **Two pairs of interior angles on the same side of transversal :**

  (1) ∠QBC and ∠BCS (2) ∠PBC and ∠BCR

If a transversal intersects any two lines, there may be no relation between the above pair of angles. However if the two lines are parallel, then the following relations are true.

(1) Corresponding angles are congruent

(2) Alternate angles are congruent

(3) Interior angles on the same side of the transversal are supplementary.

Here we shall accept without proof, the first result that corresponding angles are congruent.

Theorem 8.8 : Angles in each pair of corresponding angles formed by a transversal to two parallel lines are congruent.

As shown in figure 8.57, $l \parallel m$ and t is transversal to them. Angles in each pair of corresponding angles formed by t are congruent. Thus,

∠APC ≅ ∠PQE

∠APB ≅ ∠PQD

∠BPQ ≅ ∠DQF and

∠CPQ ≅ ∠EQF

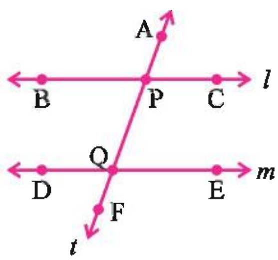


Figure 8.57

The question that arises now is : If angles in a pair of corresponding angles formed by two coplanar lines and their transversal are congruent, then will the two lines be parallel ?

To answer the question take lines l and t intersecting in P as in the figure 8.58.

Take a point Q other than P on t and construct $\angle PQC$ at Q such that $\angle PQC \cong \angle APB$.

If $\overrightarrow{QD}$ is the ray opposite to $\overrightarrow{QC}$, then the line $\overleftrightarrow{CD}$ obtained by $\overrightarrow{QC} \cup \overrightarrow{QD}$ is parallel to l.

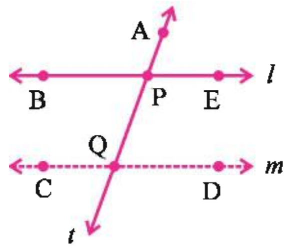


Figure 8.58

Thus, the converse of the theorem 8.8 is also true. We will accept it without proof.

Theorem 8.9 : If the corresponding angles formed by a transversal to two coplanar lines are congruent then the given lines are parallel.

We learn now an example for application of theorem.

Example 9 : If the bisectors of a pair of corresponding angles formed by a transversal with two coplanar lines are parallel, prove that the given coplanar lines are also parallel.

Data : $\overleftrightarrow{EF}$ is transversal for coplanar lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. $\overleftrightarrow{GM}$ and $\overleftrightarrow{HN}$ are the bisectors of the pair of corresponding angles $\angle EGB$ and $\angle GHD$ respectively and $\overleftrightarrow{GM} \parallel \overleftrightarrow{HN}$.

To prove : $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Proof : $\overleftrightarrow{GM} \parallel \overleftrightarrow{HN}$ and $\overleftrightarrow{GH}$ is a transversal which intersects them at G and H respectively.

$\therefore \quad m\angle EGM = m\angle GHN$ **(corresponding angles)**

$2m\angle EGM = 2m\angle GHN$

$m\angle EGB = m\angle GHD$

$\therefore \quad \angle EGB \cong \angle GHD$

But they are a pair of corresponding angles made by transversal $\overleftrightarrow{EF}$ to coplanar lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

$\therefore \quad \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

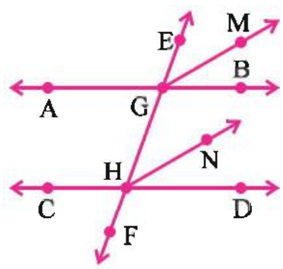


Figure 8.59

EXERCISE 8.4

1. Prove that if a line is perpendicular to one of two given parallel lines, then it is also perpendicular to the other line.

2. In the figure 8.60, if $m \parallel n$ and $m\angle EFB = 65$. Find $m\angle CGF$ and $m\angle DGF$.

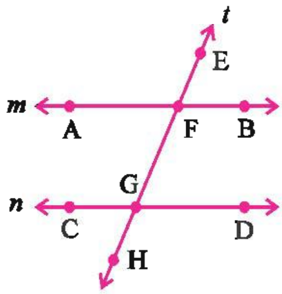


Figure 8.60

3. t is a transversal for lines l and m. $m\angle APB$ is $\frac{4}{3}$ times the measure of a right angle and $m\angle CQD = 120$. Prove that $l \parallel m$.

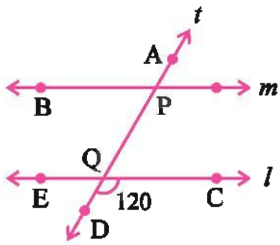


Figure 8.61

4. In the figure 8.62, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, t is their transverasl. If $m\angle FGD = 5x$ and $m\angle EFB = 120 - x$, find $m\angle EFB$ and $m\angle FGD$.

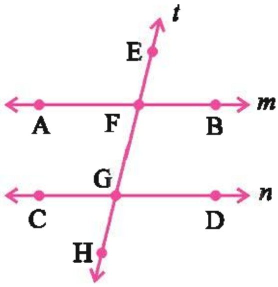


Figure 8.62

5. t is a transversal to lines m and l. If $m\angle APB = m\angle CQD = 85$, prove that $l \parallel m$.

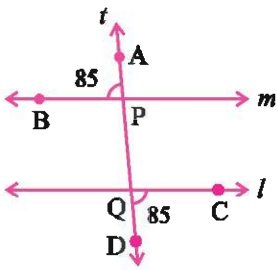


Figure 8.63

*

8.22 Theorems on Parallel Lines

Theorem 8.10 : Angles in each pair of alternate angles formed by a transversal to two parallel lines are congruent.

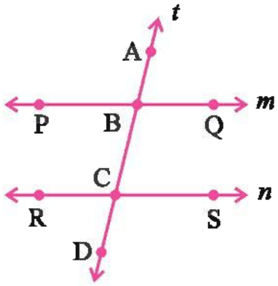


Figure 8.64

Data : Line t is a transversal to the parallel lines m and n. $\angle QBC$ and $\angle BCR$, $\angle PBC$ and $\angle BCS$ are two pairs of alternate angles formed by the transversal.

To prove : (1) $\angle PBC \cong \angle BCS$

(2) $\angle QBC \cong \angle BCR$

Proof : $m \parallel n$ and t is a transversal to m and n. **(Given)**

$\therefore$ $\angle ABQ \cong \angle BCS$ **(corresponding angles)** (i)

and $\angle ABQ \cong \angle PBC$ **(vertically opposite angles)** (ii)

From the result (i) and (ii)

$\angle PBC \cong \angle BCS$

Similarly we can prove that $\angle QBC \cong \angle BCR$

The converse of this theorem is true. We shall accept it without proof.

Theorem 8.11 : If angles in a pair of alternate angles formed by a transversal of two coplanar lines are congruent, then the lines are parallel.

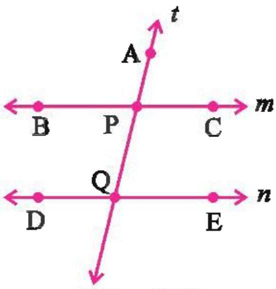


Figure 8.65

Line t is transversal of lines m and n. The alternate angles $\angle BPQ$ and $\angle PQE$ are congruent i.e. $\angle BPQ \cong \angle PQE$.

Then $m \parallel n$.

Example 10 : If the bisectors of alternate angles formed by two coplanar lines and their transversal are congruent, then prove that the lines are parallel.

Solution : Line t is a transversal to two coplanar lines l and m. $\angle AGH$ and $\angle GHD$ are alternate angles formed by t. $\overrightarrow{GM}$ and $\overrightarrow{HN}$ are bisectors of $\angle AGH$ and $\angle GHD$. **(Given)**

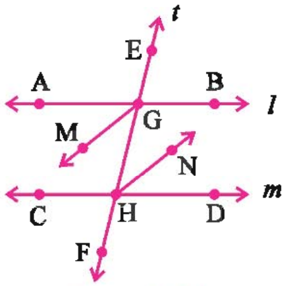


Figure 8.66

$\therefore$ $m\angle AGH = 2m\angle MGH$ and

$m\angle GHD = 2m\angle GHN$

Also $\overleftrightarrow{GM} \parallel \overleftrightarrow{HN}$ and line t is their transversal.

∴ $m\angle MGH = m\angle GHN$ **(alternate angles)**

∴ $2m\angle MGH = 2m\angle GHN$

∴ $m\angle AGH = m\angle GHD$

∴ $\angle AGH \cong \angle GHD$

Thus, the alternate angles ∠GHD and ∠AGH formed by the transversal *t* to lines *l* and *m* are congruent.

∴ $l \parallel m$

8.23 More Theorems on Parallel Lines

Let us study a theorem.

Theorem 8.12 : Interior angles on one side of a transversal to two parallel lines are supplementary.

Data : *t* is a transversal to the parallel lines *l* and *m*. ∠PBC and ∠BCR is a pair of interior angles and ∠QBC and ∠BCS is also pair of interior angle on the same side of the transversal.

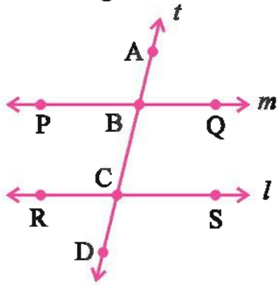


Figure 8.67

To prove : $m\angle PBC + m\angle BCR = 180$

$m\angle QBC + m\angle BCS = 180$

Proof : $l \parallel m$ and *t* is a transversal.

$\angle PBC \cong \angle BCS$ **(alternate angles)**

∴ $m\angle PBC = m\angle BCS$

$m\angle BCR + m\angle BCS = 180$ **(linear pair)**

$m\angle BCR + m\angle PBC = 180$

Similarly we can prove that $m\angle QBC + m\angle BCS = 180$

The converse of this theorem also holds. We shall accept it without proof.

Theorem 8.13 : If the interior angles on the same side of a transversal to two distinct coplanar lines are supplementary, then the lines are parallel.

EXERCISE 8.5

1. Line *t* is a transversal for lines *l* and *m*. Identify the alternate and corresponding angles in figure 8.68.

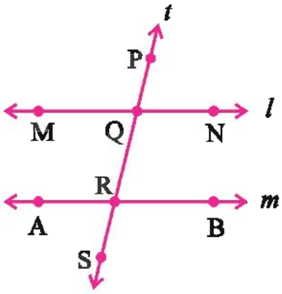


Figure 8.68

2. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$. If $m\angle ABC = 60$, find $m\angle DEF$.

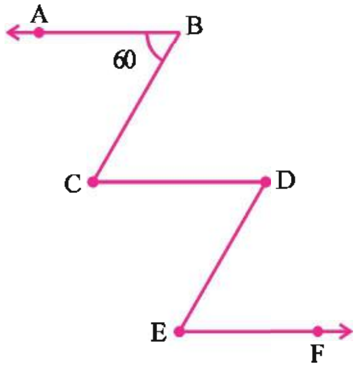


Figure 8.69

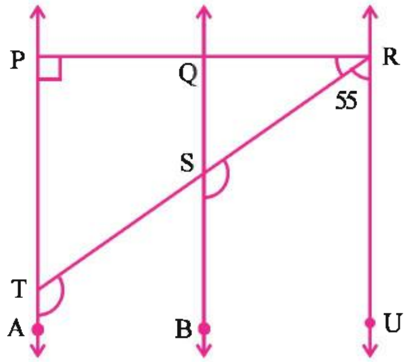
3. 

Figure 8.70

In figure 8.70, $\overleftrightarrow{PT} \parallel \overleftrightarrow{QS}$ and $\overleftrightarrow{QS} \parallel \overleftrightarrow{RU}$. Also $\overleftrightarrow{RP} \perp \overleftrightarrow{PT}$. If $m\angle TRU = 55$, find the value of $m\angle RTA$, $m\angle RSB$, $m\angle PRS$.

4. In figure 8.71, if $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$, $m\angle PXY = 50$ and $m\angle XZS = 127$, find $m\angle YXZ$ and $m\angle XYZ$.

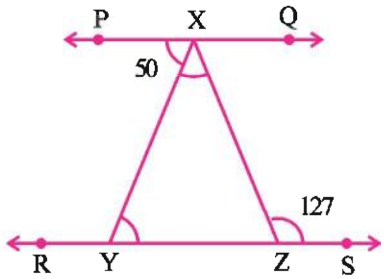


Figure 8.71

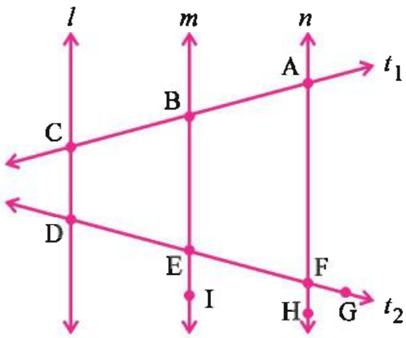
5. 

Figure 8.72

As shown in figure 8.72, t_1 and t_2 are transversals to the lines l, m and n. If $m\angle ABE = m\angle BCD$ and $m\angle FEI = m\angle GFH$, prove that $l \parallel n$.

6. In the figure 8.73, $\overleftrightarrow{MS} \parallel \overleftrightarrow{YZ}$ and $\overleftrightarrow{NR} \parallel \overleftrightarrow{ZL}$. If $m\angle YZX = 48$ and $m\angle RNS = 72$, then determine $m\angle XZL$, $m\angle MNZ$ and $m\angle ZNR$.

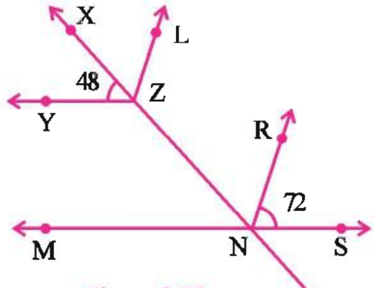


Figure 8.73

7.

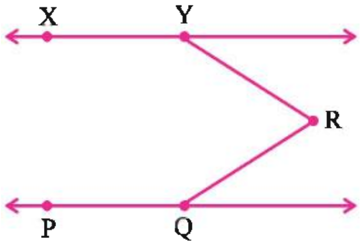


Figure 8.74

In the figure 8.74, $\overleftrightarrow{XY} \parallel \overleftrightarrow{PQ}$ and R is a point as shown in the figure 8.74. Prove that

$m\angle XYR + m\angle YRQ + m\angle PQR = 360$.

EXERCISE 8

1. If $\overleftrightarrow{PQ} \cap \overleftrightarrow{RS} = \{O\}$. $\overrightarrow{OT}$ is a bisector of $m\angle POS$ and $m\angle POT = 75$, then find the measure of all the angles from the figure 8.75.

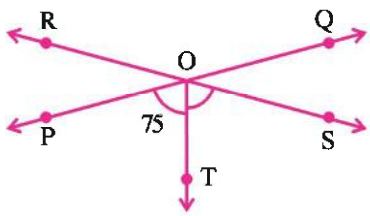


Figure 8.75

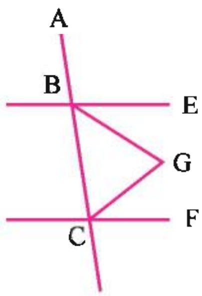


Figure 8.76

2. In the figure 8.76, the bisectors of $\angle CBE$ and $\angle BCF$ intersect at G. Also $\overline{BE} \parallel \overline{CF}$. Prove that $m\angle BGC = 90$.

3. If two parallel lines are intersected by a transversal, prove that the bisectors of the two pairs of interior angles form a rectangle.

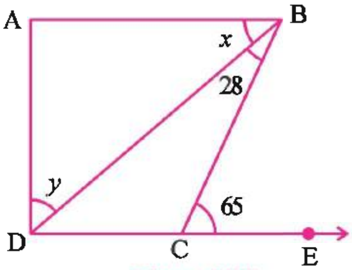


Figure 8.77

4. In the figure 8.77, if $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $m\angle DBC = 28$ and $m\angle BCE = 65$, then find the value of x and y, where $x = m\angle ABD$ and $y = m\angle ADB$.

5. $\angle AOC$ and $\angle BOD$ are vertically opposite angles such that $m\angle AOC = a + 20$, $m\angle BOD = 2a - 50$ and A–O–B. Find $m\angle AOD$.

6. For a linear pair of angles $\angle XOY$ and $\angle YOZ$, $m\angle XOY : m\angle YOZ = 2:3$. Find the measure of each of them.

7. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) An angle is a union of .... ☐
 (a) lines (b) line-segments
 (c) rays (d) a line-segment and a ray

(2) The measure of an angle always lies between .... ☐
 (a) 0 and 90 (b) 90 and 180 (c) 0 and 100 (d) 0 and 180

(3) If $m\angle A = 81$ and $m\angle B$ = ..., then they are complementary angles. ☐
 (a) 99 (b) 19 (c) 81 (d) 9

(4) $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are distinct rays. If ..... then they determine a plane uniquely. ☐
 (a) they are opposite rays (b) they lie in the same line
 (c) they are not opposite rays (d) they are identical rays

(5) If distinct points A and B lie in a plane X, then $X \cap \overleftrightarrow{AB}$ = ... ☐
 (a) {A, B} (b) $\overleftrightarrow{AB}$ (c) plane X (d) $\overline{AB}$

(6) If two lines can not lie in the same plane, they are called ...... lines. ☐
 (a) disjoint (b) skew (c) parallel (d) coplanar

(7) The supplementary angle of the complementary angle of angle having measure 23 has measure ...... ☐
 (a) 67 (b) 90 (c) 113 (d) 23

(8) The complementary angle of an angle having measure $x + 30$ has measure ...... ☐
 (a) $-(x - 60)$ (b) $60 + x$ (c) $x - 60$ (d) $-60 - x$

(9) If one angle of a linear pair is acute, then the other angle is ...... ☐

(a) congruent (b) acute (c) obtuse (d) right angle

(10) If t is a transversal for two parallel lines l and m, interior angles on the same side of the transversal are .... ☐

(a) supplementary (b) linear pair (c) complementary (d) congruent

(11) If two angles forming a linear pair have measure $(6y + 30)$ and $4y$, then y = ... ☐

(a) 30 (b) 15 (c) 60 (d) 90

(12) An angle has measure equal to $\frac{1}{3}$rd measure of its supplementary angle, then the angle has measure .... ☐

(a) 15 (b) 30 (c) 45 (d) 60

*

Summary

1. Postulates of plane
2. Coplaner and Non-coplaner points and lines
3. Partition of a plane and half plane
4. Conditions to determine plane uniquely
5. Intersection of two / three planes, parallel planes
6. Angle
7. Interior of an angle
8. Cross bar theorem
9. Measure of an angle and its postulates
10. Types of angles according to their measures
11. Types of pairs of angles based on their measures
12. Types of pairs of angles based on their arms
13. Bisector of angles
14. Theorems about angles
15. Intersection of two lines
16. Postulates and theorem for parallel lines
17. Intersection of three lines
18. Parallel lines and angles formed by transversal

CHAPTER 9

TRIANGLE

9.1 Introduction

In this chapter, we will learn about a triangle using the terminology of the set theory. We know that a triangle is a closed figure. In a plane the closed figure obtained as a union of three line-segments joining three distinct non-collinear points is a triangle.

9.2 Triangle

Triangle : The union of three line-segments determined by three non-collinear points is called a triangle.

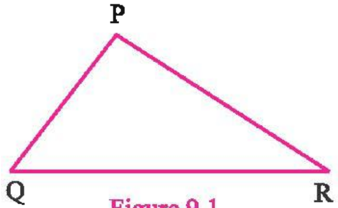


Figure 9.1

Three non-collinear points P, Q, R determine three line-segments $\overline{PQ}$, $\overline{QR}$ and $\overline{RP}$. The union $\overline{PQ} \cup \overline{QR} \cup \overline{RP}$ is a triangle. We shall denote this triangle by ΔPQR.

$\Delta PQR = \overline{PQ} \cup \overline{QR} \cup \overline{RP}$

We know that set operations are commutative and associative.

$\therefore \quad \overline{PQ} \cup \overline{QR} \cup \overline{RP} = \overline{QR} \cup \overline{RP} \cup \overline{PQ} = \overline{RP} \cup \overline{PQ} \cup \overline{QR}$

$\therefore \quad \Delta PQR = \Delta QRP = \Delta RPQ$

Also $\Delta PQR = \Delta PRQ = \Delta RQP = \Delta QPR$

Thus we can name a triangle using the three given points in any order.

We know from the postulates of a plane that three non-collinear points determine exactly one plane. The line segments determined by three points are also in the same plane.

Thus, all the points of a triangle are in one plane.

Let us discuss about some subsets of a triangle and some definitions.

Vertices : In the context of ΔPQR (figure 9.1), the points P, Q, R are called vertices of ΔPQR (each point is a vertex of the triangle.)

Sides : In the context of ΔPQR (figure 9.1), $\overline{PQ}$, $\overline{QR}$ and $\overline{RP}$ are called the sides of ΔPQR.

For the sets A, B and C we know that,

$A \subset (A \cup B \cup C)$, $B \subset (A \cup B \cup C)$ and $C \subset (A \cup B \cup C)$.

Thus from this, it is clear that $\overline{PQ} \subset \Delta PQR = \overline{PQ} \cup \overline{QR} \cup \overline{RP}$. Similarly $\overline{QR} \subset \Delta PQR$ and $\overline{RP} \subset \Delta PQR$.

$\therefore$ **The sides of a triangle are subsets of the triangle.**

Now, three non-collinear points P, Q, R determine one and only one plane α.

Thus, $P \in \alpha$, $Q \in \alpha$, $R \in \alpha$ So, $\overline{PQ} \subset \alpha$, $\overline{QR} \subset \alpha$, $\overline{RP} \subset \alpha$

$\therefore$ $(\overline{PQ} \cup \overline{QR} \cup \overline{RP}) \subset \alpha$

$\therefore$ $\Delta PQR \subset \alpha$

Thus, **a triangle is a plane figure and it is a subset of the plane. Sides of a triangle are subsets of the plane containing the triangle.**

Angles : In the context of ΔPQR (figure 9.1), the angles $\angle PQR$, $\angle QRP$ and $\angle RPQ$ are called the angles of ΔPQR. These angles are also denoted by $\angle Q$, $\angle R$ and $\angle P$ respectively.

Also, $P \in \alpha$, $Q \in \alpha$ and $R \in \alpha$.

$\therefore$ $\overrightarrow{PQ} \subset \alpha$, $\overrightarrow{PR} \subset \alpha$. So $\angle QPR \subset \alpha$

Similarly, $\angle PQR \subset \alpha$ and $\angle PRQ \subset \alpha$.

$\therefore$ **The three angles of a triangle are subsets of the plane containing the triangle.**

We have seen that the sides of a triangle are subsets of that triangle, but what can we say about the angles of a triangle ?

From the figure 9.2, we say that $\angle P = \overrightarrow{PQ} \cup \overrightarrow{PR}$.

Here $Y \in \overrightarrow{PQ}$ but $Y \notin \overline{PQ}$ thus $Y \in \angle QPR$ but $Y \notin \Delta PQR$

$\therefore$ $\angle P \not\subset \Delta PQR$

Similarly $\angle PQR \not\subset \Delta PQR$, $\angle PRQ \not\subset \Delta PQR$.

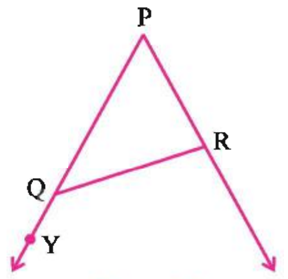


Figure 9.2

$\therefore$ **Angles of a triangle are not subsets of the triangle.**

Each triangle has three sides and three angles. These sides and angles are known as the parts of the triangle. Hence, each triangle has six parts.

We have seen that the sides of a triangle are line-segments. So, if we consider two distinct sides of ΔPQR, say $\overline{QP}$ and $\overline{QR}$, then they have a common end-point Q. So $\overline{PQ} \cap \overline{QR} = \{Q\}$. Similarly, and $\overline{PQ} \cap \overline{PR} = \{P\}$ and $\overline{PR} \cap \overline{QR} = \{R\}$.

Thus, **each pair of distinct sides of a triangle intersect in a vertex of the triangle.**

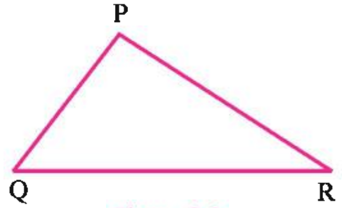


Figure 9.3

For ΔPQR,

$\overline{QP} \subset \overrightarrow{QP}$, $\overline{QR} \subset \overrightarrow{QR}$

$\therefore$ $(\overline{QP} \cup \overline{QR}) \subset (\overrightarrow{QP} \cup \overrightarrow{QR})$

$\therefore$ $(\overline{QP} \cup \overline{QR}) \subset \angle PQR$

$\angle Q$ is called the included angle of sides $\overline{QP}$ and $\overline{QR}$.

Included Angle : While referring to two sides of a triangle, the angle represented in a single letter notation by the point of intersection of the two sides of the triangle is called the included angle of the sides. The included angle of two sides is the angle whose arms contain both the sides as subsets.

Here $\angle Q$ is the included angle of sides $\overline{QP}$ and $\overline{QR}$ in ΔPQR. (fig. 9.3)

Sides $\overline{QP}$ and $\overline{QR}$ are subsets of $\overrightarrow{QP}$ and $\overrightarrow{QR}$ respectively and $\angle PQR$ of ΔPQR is formed by rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$, $\angle PQR$ is called the included angle of sides $\overline{QP}$ and $\overline{QR}$. In short, since $\overline{QP} \cap \overline{QR} = \{Q\}$, $\angle Q$ in single letter notation i.e. $\angle PQR$ is included angle of sides $\overline{QP}$ and $\overline{QR}$. Similarly $\angle P$ is included angles of sides $\overline{PQ}$ and $\overline{PR}$ and $\angle R$ is included angle of side $\overline{RP}$ and $\overline{RQ}$.

Side Opposite to an Angle of a Triangle : The side opposite to an angle of a given triangle is the side other than the sides for which given angle is the included angle. It is the side that is not a subset of the included angle. It is the side joining two vertices other than the vertex of the angle.

In the above ΔPQR (figure 9.3), which is the side opposite to $\angle Q$? Here $\angle Q$ is the included angle of the sides $\overline{PQ}$ and $\overline{QR}$. The side other than $\overline{PQ}$ and $\overline{QR}$ is $\overline{PR}$. This side is the side opposite to $\angle Q$. Similarly $\overline{QR}$ is the side opposite to $\angle P$ and $\overline{PQ}$ is the side opposite to $\angle R$. When an angle is written using only one vertex, the side determined by the other two vertices is the side opposite to the angle.

The Angle Opposite to a Side of a Triangle : The angle opposite to a given side is the included angle of the other two sides of the triangle.

In the ΔPQR, (figure 9.3), $\angle Q$ is the angle opposite to the side $\overline{PR}$; because $\angle Q$ is the included angle of the other two sides $\overline{QP}$ and $\overline{QR}$. Similarly, $\angle R$ is the angle opposite to the side $\overline{PQ}$. Can you state the angle opposite to the side $\overline{QR}$?

Included side : The included side of two angles of a triangle is the side opposite to the remaining angle.

From the figure 9.3, we say that, $\overline{QR}$ is the side opposite to $\angle P$. So, $\overline{QR}$ is the included side of the angles $\angle Q$ and $\angle R$. Similarly, the side opposite to $\angle Q$ is $\overline{PR}$. Thus $\overline{PR}$ is the included side of the angles $\angle P$ and $\angle R$. $\overline{PQ}$ is the included side of the angles $\angle P$ and $\angle Q$. If two angles of a triangle are named by their vertices only, then the segment joining the vertices is the included side of given two angles.

Partition of a Plane by a Triangle :

Triangle is a plane closed figure. We have seen that a triangle is a subset of a plane. The plane containing a triangle is partitioned into three parts by the triangle.

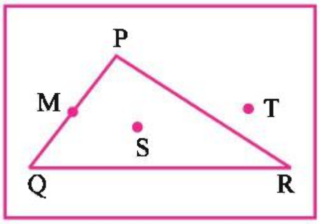


Figure 9.4

In the figure 9.4, point S is in the interior of ΔPQR. T is in the exterior of ΔPQR. The point M is on ΔPQR.

In figure 9.5, the interior of $\angle P$ is coloured and it contains more points than our intuitive understanding of the interior of the triangle.

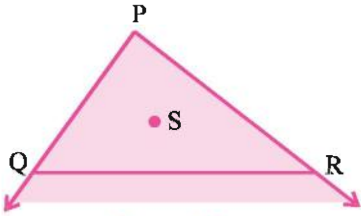


Figure 9.5

The intersection of interiors of $\angle P$, $\angle Q$ and $\angle R$ seems to comply with our understanding of interior of ΔPQR. The region obtained is the interior of the triangle.

Interior of a Triangle :

If S is in the interior of $\angle P$, then S and R are on the same side of $\overleftrightarrow{PQ}$ and S and Q are on the same side of $\overleftrightarrow{PR}$ (fig. 9.5).

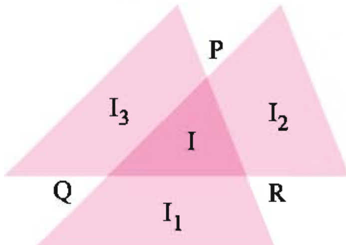


Figure 9.6 (i)

Similarly, if S is in the interior of $\angle Q$, then S and P on the same side of $\overleftrightarrow{QR}$ and S and R are on the same side of $\overleftrightarrow{PQ}$. So let S be in the interior of $\angle R$, then which are the points on the same side of $\overleftrightarrow{PR}$ and $\overleftrightarrow{QR}$? Also what can you say about a point S in the interior of the angles $\angle P$, $\angle Q$ and $\angle R$?

The intersection of the interiors of all the angles of a triangle is called interior of the given triangle.

Thus, from the figure 9.6(i) **if the interiors of $\angle P$, $\angle Q$ and $\angle R$ are I_1, I_2 and I_3 respectively and the interior of ΔPQR is I, then $I = I_1 \cap I_2 \cap I_3$**

So, interior of a triangle is also a subset of the plane containing the triangle.

Exterior Region of a Triangle : The set of all points of the plane containing the given triangle which are neither in the interior nor on the triangle is called the exterior of the triangle.

If the set of all exterior points of a triangle is represented by E and α is the plane containing the triangle, then complement of $I \cup \Delta PQR$ with respect to α is E.

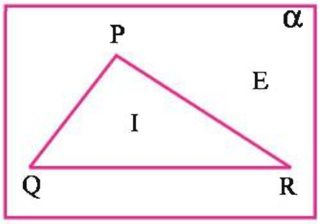


Figure 9.6 (ii)

Now, $I \cap E = E \cap \Delta PQR = I \cap \Delta PQR = \emptyset$

and $\alpha = I \cup E \cup \Delta PQR$

Thus, a triangle partitions the plane containing it into three mutually disjoint sets, namely **(1) the triangle (2) the interior of the triangle and (3) the exterior of the triangle.**

In fig. 9.6(ii), I is the interior of ΔPQR and E is the exterior of ΔPQR. Thus the union of ΔPQR, interior of ΔPQR and exterior of ΔPQR form the plane α.

A point in the exterior of a triangle is called an **exterior point** of the triangle. In figure 9.4 T is an exterior point of ΔPQR. A point in the interior of a triangle is called an **interior point** of the triangle.

An Important Result : $I = I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_1 = I_1 \cap I_2 \cap I_3$

In figure 9.6(i) we denote the interior of $\angle P$, $\angle Q$ and $\angle R$ by I_1, I_2 and I_3 respectively. Let $S \in I_1 \cap I_2$. Then $S \in I_1$ and $S \in I_2$. So S is in the interior of $\angle P$ and also in the interior of $\angle Q$.

So, we have

S and R are on the same side of $\overleftrightarrow{PQ}$. **(i)**

S and Q are on the same side of $\overleftrightarrow{PR}$.

S and R are on the same side of $\overleftrightarrow{PQ}$. **(ii)**

S and P are on the same side of $\overleftrightarrow{QR}$.

Hence, from (i) and (ii), we get,

S and Q are on the same side of $\overleftrightarrow{PR}$.

S and P are on the same side of $\overleftrightarrow{QR}$.

$\therefore$ S is in the interior of $\angle R$.

$\therefore$ $S \in I_3$.

Thus if $S \in I_1$, $S \in I_2$, then $S \in I_3$. i.e. $S \in I_1 \cap I_2 \cap I_3$

$\therefore$ $S \in I$ **$(I = I_1 \cap I_2 \cap I_3)$**

$\therefore \quad (I_1 \cap I_2) \subset (I_1 \cap I_2 \cap I_3)$ **(iii)**

Also, from properties of set operations, we know that,

$(A \cap B \cap C) \subset (A \cap B)$, $(A \cap B \cap C) \subset (B \cap C)$ and $(A \cap B \cap C) \subset (C \cap A)$

Thus, $(I_1 \cap I_2 \cap I_3) \subset (I_1 \cap I_2)$ **(iv)**

So from (iii) and (iv), it is clear that

$$I_1 \cap I_2 = I_1 \cap I_2 \cap I_3$$

Similarly, if we consider $S \in I_2 \cap I_3$, then $I_2 \cap I_3 = I_1 \cap I_2 \cap I_3$

and if $S \in I_3 \cap I_1$, then $I_3 \cap I_1 = I_1 \cap I_2 \cap I_3$

$\therefore \quad I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_1 = I_1 \cap I_2 \cap I_3 = I$

Thus, **if a point is in the interior of any two angles of a triangle, it is an interior point of the triangle.**

Exterior Angle of a Triangle :

Suppose ΔPQR is given. A point S is on the opposite ray of $\overrightarrow{RQ}$ such that Q–R–S. $\angle PRS$ and $\angle PRQ$ form a linear pair. $\angle PRS$ is called **an exterior angle** of ΔPQR.

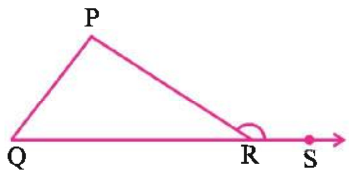


Figure 9.7

Exterior Angle of a Triangle : An angle forming a linear pair with any angle of a triangle is called an exterior angle of the triangle.

In figure 9.7, $\overrightarrow{RQ}$ and $\overrightarrow{RS}$ are opposite rays. $\angle PRS$ and $\angle PRQ$ form a linear pair. So, $\angle PRS$ is an exterior angle of ΔPQR. Similarly in figure 9.8 $\overrightarrow{RP}$ and $\overrightarrow{RT}$ are opposite rays. $\angle QRT$ and $\angle PRQ$ form a linear pair. So, $\angle QRT$ is an exterior angle of ΔPQR. These two exterior angles are vertically opposite angles. Hence they are congruent. Similarly from figure 9.9, we get two congruent exterior angles $\angle OPR$ and $\angle WPQ$ of ΔPQR at the vertex P, two congruent exterior angles $\angle UQR$ and $\angle PQY$ at the vertex Q of ΔPQR.

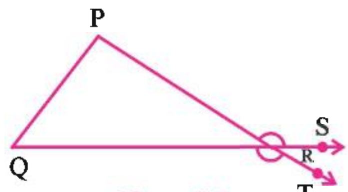


Figure 9.8

Thus, at each vertex of a triangle, there are two exterior angles of the triangle. So, a triangle has six exterior angles.

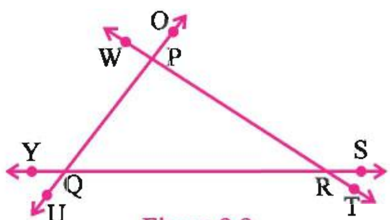


Figure 9.9

Interior Opposite Angles :

In figure 9.7, ∠PRS is an exterior angle of ΔPQR. ∠PRS and ∠PRQ form a linear pair. The two angles ∠P and ∠Q of ΔPQR are called the interior opposite angles of ∠PRS.

Interior Opposite Angles : The angles other than the angle, with which an exterior angle forms a linear pair, are called interior opposite angles corresponding to that exterior angle.

In the context of the exterior angle ∠PRS at the vertex R, ∠P and ∠Q are the interior oppsite angles. Similarly, ∠P and ∠R are interior opposite angles related to exterior angle at the vertex Q and ∠Q and ∠R are interior opposite angles related to the exterior angle at the vertex P.

9.3 Some Properties of a Triangle

If we measure the exterior angle ∠PRS of ΔPQR and its interior opposite angles ∠P and ∠Q, then we see that the measure of the exterior angle is larger than the measure of each of its interior opposite angles.

We accept this theorem without proof.

Theorem 9.1 : Measure of any exterior angle of a triangle is larger than the measure of each of its interior opposite angles.

In fact, measure of an exterior angle of a triangle equals the sum of the measures of its interior opposite angles.

We accept this theorem without proof.

Theorem 9.2 : Measure of any exterior angle of a triangle is equal to the sum of the measures of its two interior opposite angles.

Let us understand the above theorem by following example :

Example 1 : ∠PRS is an exterior angle of ΔPQR. $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RY}$. $m\angle PRY = 50$. Y is in the interior of ∠PRS. $m\angle YRS = 70$. Find $m\angle PRS$ and also find the measure of each of its interior opposite angles.

Solution : ∠PRS is an exterior angle of ΔPQR. Thus, ∠P and ∠Q are interior opposite angles of ∠PRS.

Here $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RY}$ and $\overleftrightarrow{PR}$ is the transversal. ∠P and ∠PRY are alternate angles.

$\therefore \quad m\angle P = m\angle PRY$

But $m\angle PRY = 50$. Thus, $m\angle P = 50$

Now, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RY}$ and $\overleftrightarrow{QR}$ is their transversal.

∠Q and ∠YRS are corresponding angles.

$\therefore \quad m\angle Q = m\angle YRS$. But $m\angle YRS = 70$.

$\therefore \quad m\angle Q = 70$

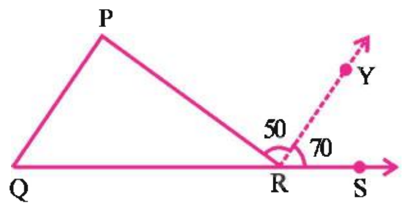


Figure 9.10

Now, Y is in the interior of ∠PRS.

∴ $m\angle PRS = m\angle PRY + m\angle YRS$

$= 50 + 70$

$= 120$

∴ $m\angle PRS = 120$, $m\angle P = 50$, $m\angle Q = 70$

Now, if we construct a triangle and measure its angles, we find that sum of measures all the three angles is 180. We shall prove it as follows :

Theorem 9.3 : The sum of the measures of all the three angles of a triangle is 180.

Data : ΔABC is given.

To prove : $m\angle A + m\angle B + m\angle C = 180$

Proof : Let D be a point on the ray opposite to $\overrightarrow{CB}$. Exterior ∠ACD is formed making a linear pair with ∠ACB.

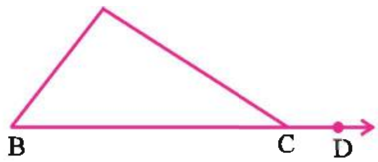


Figure 9.11

∴ $m\angle ACB + m\angle ACD = 180$ **(i)**

∠A and ∠B are the interior opposite angles of the exterior angle ∠ACD of ΔABC.

∴ $m\angle ACD = m\angle A + m\angle B$ **(ii)**

∴ By (i) and (ii), $m\angle ACB + m\angle A + m\angle B = 180$

∴ $m\angle C + m\angle A + m\angle B = 180$ or $m\angle A + m\angle B + m\angle C = 180$

Alternate proof can also be given for this theorem as below :

$A \notin \overleftrightarrow{BC}$.

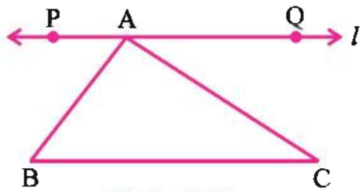


Figure 9.12

∴ There is a line l parallel to $\overleftrightarrow{BC}$ passing through the point A. Select two points P and Q on line l other than A such that P–A–Q.

Now $\overleftrightarrow{AB}$ is a transversal of the parallel lines l and $\overleftrightarrow{BC}$.

Hence ∠PAB and ∠ABC are the alternate angles.

∴ $m\angle PAB = m\angle ABC$ (i)

Similarly $\overleftrightarrow{AC}$ is also a transversal of the parallel lines l and $\overleftrightarrow{BC}$.

Thus, ∠QAC and ∠ACB are the alternate angles.

∴ $m\angle QAC = m\angle ACB$ (ii)

Now, $m\angle PAB + m\angle BAC + m\angle QAC = 180$

$\therefore$ $m\angle ABC + m\angle BAC + m\angle ACB = 180$ (by (i) and (ii))

$\therefore$ $m\angle B + m\angle A + m\angle C = 180$ or $m\angle A + m\angle B + m\angle C = 180$

An Important Result :

If one angle of a triangle is a right angle, then the other two angles are acute angles. (Thus no triangle can have two right angles.)

Date : $\angle ACB$ of ΔABC is a right angle.

To prove : $\angle A$ and $\angle B$ are acute angles.

Proof : Let D be on the opposite ray of $\overrightarrow{CB}$.

Now $\angle ACD$ be an exterior angle of the triangle which forms a linear pair with the right angle $\angle ACB$.

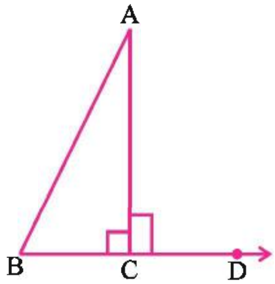


Figure 9.13

$\therefore$ $\angle ACD$ is a right angle.

$\therefore$ $m\angle ACD = 90$ (i)

Now $\angle A$ and $\angle B$ are the interior opposite angles of $\angle ACD$.

$\therefore$ $m\angle A < m\angle ACD$ and $m\angle B < m\angle ACD$

$\therefore$ $m\angle A < 90$ and $m\angle B < 90$ (from (i))

$\therefore$ $\angle A$ and $\angle B$ are acute angles.

Example 2 : For ΔABC, if $m\angle A = 40$, $m\angle B = 60$, then find $m\angle C$.

Solution : For ΔABC, $m\angle A + m\angle B + m\angle C = 180$.

$\therefore$ $40 + 60 + m\angle C = 180$

$\therefore$ $100 + m\angle C = 180$

$\therefore$ $m\angle C = 180 - 100 = 80$

Example 3 : For ΔABC, if $m\angle A = 2m\angle B$ and $m\angle B = 3m\angle C$, then find the measures of all the angles of ΔABC.

Solution : For ΔABC, $m\angle A + m\angle B + m\angle C = 180$.

Here, $m\angle B = 3m\angle C$ (i)

$\therefore$ $2m\angle B = 6m\angle C$

$\therefore$ $m\angle A = 6m\angle C$ (ii)

We know that since, $m\angle A + m\angle B + m\angle C = 180$

$6m\angle C + 3m\angle C + m\angle C = 180$

$\therefore \quad 10m\angle C = 180$

$\therefore \quad m\angle C = 18$

Now $m\angle A = 6m\angle C$ gives $m\angle A = 6(18) = 108$ and $m\angle B = 3(18) = 54$

Example 4 : If the measures of the angles of a triangle are in proportion 4 : 5 : 6, then find the measure of all the angles of the triangle.

Solution : Let A, B, C be the angles of ΔABC.

Now $m\angle A : m\angle B : m\angle C = 4 : 5 : 6$

$\therefore \quad \frac{m\angle A}{m\angle B} = \frac{4}{5}$ giving us $m\angle A = \frac{4}{5}\, m\angle B$ (i)

and $\frac{m\angle B}{m\angle C} = \frac{5}{6}$ giving us $m\angle B = \frac{5}{6}\, m\angle C$ (ii)

$\therefore \quad$ by (i), $m\angle A = \frac{4}{5} \times \frac{5}{6}\, m\angle C = \frac{4}{6}\, m\angle C$

For ΔABC, $m\angle A + m\angle B + m\angle C = 180$.

$\frac{4}{6}\, m\angle C + \frac{5}{6}\, m\angle C + m\angle C = 180$

$\therefore \quad \frac{15}{6}\, m\angle C = 180$

$\therefore \quad m\angle C = \frac{180 \times 6}{15} = 12 \times 6 = 72$

$m\angle B = \frac{5}{6} \times 72 = 5 \times 12 = 60$

$m\angle A = \frac{4}{5} \times 60 = 4 \times 12 = 48$

We can solve this example by another method also.

Solution : Suppose the angles of ΔABC have measures $4x$, $5x$, $6x$.

Thus, $4x + 5x + 6x = 180$

$\therefore \quad 15x = 180$

$\therefore \quad x = \frac{180}{15} = 12$

$\therefore \quad 4x = 4 \times 12 = 48,\ 5x = 5 \times 12 = 60,\ 6x = 6 \times 12 = 72$

Hence the measures of all the angles of the triangle are 48, 60, 72.

Example 5 : In ΔABC, sum of measures two angles is 60 and difference between them is 20, then find the measures of all the angles of ΔABC.

Solution : For ΔABC,

suppose $m\angle A + m\angle B = 60$ **(i)**

and $m\angle A - m\angle B = 20$ **(ii)**

Now we know that $m\angle A + m\angle B + m\angle C = 180$

$\therefore$ $60 + m\angle C = 180$ **(using (i))**

$\therefore$ $m\angle C = 120$

$\therefore$ by adding both sides of (i) and (ii) $2m\angle A = 80$

$\therefore$ $m\angle A = 40$

and by (i) $40 + m\angle B = 60$

$\therefore$ $m\angle B = 60 - 40 = 20$

$\therefore$ $m\angle B = 20$

Example 6 : In ΔABC, if $m\angle A = 30$, $m\angle C = 50$ and bisector of $\angle B$ meets $\overline{AC}$ at D, then find $m\angle ADB$ and $m\angle CDB$.

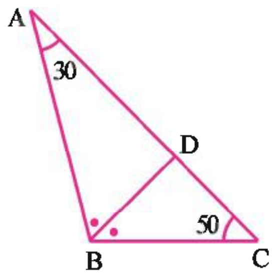


Figure 9.14

Solution : For ΔABC,

$m\angle A + m\angle B + m\angle C = 180$

$\therefore$ $30 + m\angle B + 50 = 180$

$\therefore$ $m\angle B = 100$ **(i)**

$\overrightarrow{BD}$ is the bisector of $\angle B$.

Hence $m\angle ABD = m\angle CBD = \frac{1}{2}m\angle B = \frac{1}{2}(100) = 50$ **(by (i))**

Now, for the ΔADB,

$m\angle BAD + m\angle ABD + m\angle ADB = 180$

$\therefore$ $30 + 50 + m\angle ADB = 180$

$\therefore$ $m\angle ADB = 180 - 80 = 100$

Now, $\angle ADB$ makes a linear pair with $\angle CDB$.

Thus, $m\angle ADB + m\angle CDB = 180$

$\therefore$ $m\angle CDB + 100 = 180$

$\therefore$ $m\angle CDB = 180 - 100 = 80$

$\therefore$ $m\angle ADB = 100$ and $m\angle CDB = 80$

Example 7 : Prove that the sum of the measure of all the exterior angles of a triangle is 720.

Solution : Here, we have six exterior angles namely $\angle ABE$, $\angle ACD$, $\angle BAF$, $\angle BCG$, $\angle CAH$, $\angle CBI$.

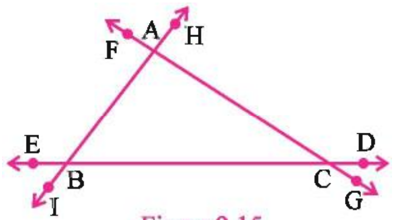


Figure 9.15

According to theorem,

$m\angle ACD = m\angle A + m\angle B$

$m\angle ABE = m\angle A + m\angle C$

$m\angle BAF = m\angle B + m\angle C$

$m\angle BCG = m\angle A + m\angle B$

$m\angle CAH = m\angle B + m\angle C$

$m\angle CBI = m\angle A + m\angle C$

Now, taking the sum of all the exterior angles of ΔABC, we get

$m\angle ACD + m\angle ABE + m\angle BAF + m\angle BCG + m\angle CAH + m\angle CBI$

$= 4(m\angle A + m\angle B + m\angle C)$

$= 4(180)$

$= 720$

Example 8 : The measure of an exterior angle $\angle ACD$ of ΔABC is 105 and $m\angle B = 35$. Then find the measures of the remaining angles of ΔABC.

Solution : Here $\angle ACD$ is an exterior angle of ΔABC and $m\angle ACD = 105$.

It forms a linear pair with $\angle ACB$

Thus, $m\angle ACB + m\angle ACD = 180$

$\therefore \quad m\angle ACB + 105 = 180$

$\therefore \quad m\angle ACB = 180 - 105 = 75$

Now, $m\angle A + m\angle B + m\angle ACB = 180$

$\therefore \quad m\angle A + 35 + 75 = 180$

$\therefore \quad m\angle A + 110 = 180$

$\therefore \quad m\angle A = 70$ and $m\angle C = 75$

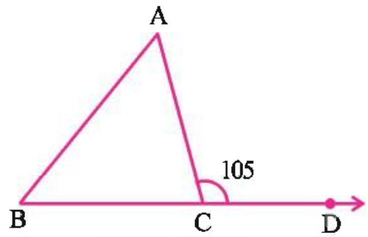


Figure 9.16

We can solve this example by an alternate method which is as follow :

$m\angle ACD = m\angle A + m\angle B$

$\therefore \quad 105 = m\angle A + 35$

$\therefore \quad m\angle A = 70$

$m\angle C = 180 - m\angle A - m\angle B = 180 - 70 - 35 = 75$

$\therefore \quad m\angle A = 70$ and $m\angle C = 75$

EXERCISE 9.1

1. If the exterior $\angle ACD$ of ΔABC has the measure 120 and one of two internal opposite angles has the measure 40, then find the measures of the remaining angles of ΔABC.

2. In ΔABC, $\overline{BE} \perp \overline{AC}$ (figure 9.17) $D \in \overline{BC}$ such that B–D–C. If $m\angle EBC = 40$, $m\angle DAC = 30$, then find $m\angle BCE$ and $m\angle ADC$.

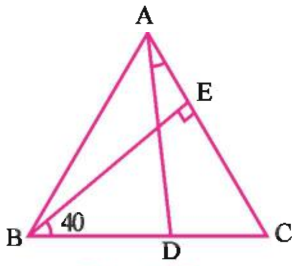


Figure 9.17

3. If the measures of the angles of a triangle are in proportion 2 : 3 : 5, then find the measures of all the angles of ΔABC.

4. Compute the value of x in each of the following figures : x is the measure of the angle shown.

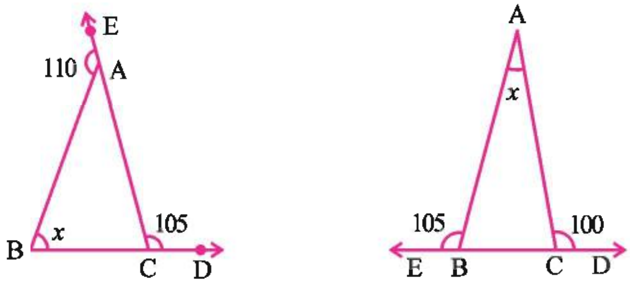


Figure 9.18

5. In ΔABC, if $m\angle A - m\angle B = 70$ and $m\angle B - m\angle C = 40$, then find the measures of all the angles of ΔABC.

6. In ΔABC, if $m\angle A = \frac{m\angle B}{2} = \frac{m\angle C}{3}$, then find the measures of all the angles of ΔABC.

7. If $\angle ACD$ is an exterior angle of ΔABC and the angle bisector of $\angle A$ intersects $\overline{BC}$ at M, then prove that $m\angle ABC + m\angle ACD = 2m\angle AMC$.

8. For ΔABC, if $\angle ABE$ and $\angle CAD$ are exterior angles and their measures are 100 and 125 respectively, then find $m\angle ACB$.

*

Correspondence :

We know about one-one correspondence between two sets. If two finite sets have the same number of elements, then it is said that there exists a one-one correspondence between them.

Consider the sets {A, B, C} and {P, Q, R} of vertices of ΔABC and ΔPQR respectively. All the six possible one-one correspondences between them are

$A \leftrightarrow P$	$A \leftrightarrow P$	$A \leftrightarrow Q$	$A \leftrightarrow Q$	$A \leftrightarrow R$	$A \leftrightarrow R$
$B \leftrightarrow Q$	$B \leftrightarrow R$	$B \leftrightarrow P$	$B \leftrightarrow R$	$B \leftrightarrow P$	$B \leftrightarrow Q$
$C \leftrightarrow R$	$C \leftrightarrow Q$	$C \leftrightarrow R$	$C \leftrightarrow P$	$C \leftrightarrow Q$	$C \leftrightarrow P$

Each of the above correspondence gives rise to a correspondence between the parts of ΔABC and ΔPQR. e.g. for correspondence $ABC \leftrightarrow PQR$, $\angle A \leftrightarrow \angle P$, $\angle B \leftrightarrow \angle Q$, $\angle C \leftrightarrow \angle R$ shows the correspondence between the angles and the $\overline{AB} \leftrightarrow \overline{PQ}$, $\overline{BC} \leftrightarrow \overline{QR}$, $\overline{CA} \leftrightarrow \overline{PR}$ is the correspondence between the sides.

Above correspondence between the vertices of the triangles can be called a correspondence between triangles. Any pair of angles formed by a correspondence of triangles is called the pair of corresponding angles and any pair of sides formed by a correspondence of triangles is called the pair of corresponding sides.

Given correspondence $ABC \leftrightarrow PQR$ for ΔABC and ΔPQR, $\angle B$ and $\angle Q$ are corresponding angles and $\overline{BC}$ and $\overline{QR}$ are corresponding sides. We can also say that $\angle B$ corresponds to $\angle Q$ and $\overline{BC}$ corresponds to $\overline{QR}$.

Corresponding Angles and Corresponding Sides :

Let a correspondence between two triangles (or between a triangle and itself) be given. Then this correspondence gives rise to a correspondence between the sides and the angles of these two triangles. The sides or angles of the triangles so associated are called corresponding sides or corresponding angles.

It is clear that for a given correspondence between two triangles, to each angle of one triangle there corresponds one angle of the other triangle and to each side of one triangle, there corresponds one side of the other triangle.

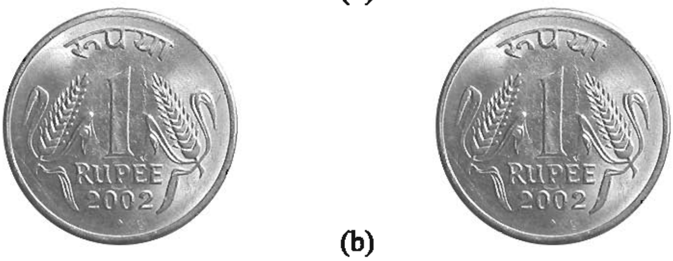
It is not necessary for a correspondence that the two triangles be distinct. We can also have a correspondence between a triangle and itself. For example, $ABC \leftrightarrow CBA$ is a correspondence between the vertices of ΔABC. In this case, $\angle A$ and $\angle C$ are corresponding angles and $\overline{BC}$ and $\overline{BA}$ are corresponding sides.

Note : Now onwards we will call the correspondence between vertices of two triangles as the correspondence between two triangles.

9.4 Congruence of Triangles

In day-to-day life, we often observe that two copies of our photographs, post-cards, ATM cards issued by the same bank are identical. Similarly, two coins of one rupee, two bangles of the same size etc. are same in size and shape. The figures whose shapes and sizes both are same are considered as **congruent figures**. These figures cover (super-impose) each other completely. We can understand this concept by figures.

(a)



(b)

Figure 9.19

In figure 9.19(a) two ATM cards are shown which superimpose each other completely. In figure 9.19(b) two coins of one rupee are shown and they also superimpose each other completely. These figures are congruent. (congruent means equal in all respects) The relation of being congruent is called a congruence.

Congruence : If there is a one-one correspondence between the vertices of two triangles (or a triangle with itself) such that the three sides and three angles of a triangle are congruent to the corresponding sides and corresponding angles of the other triangle, then such a correspondence is called a congruence between the two triangles and such triangles are called congruent triangles.

Activity :

Take a piece of a paper and a card-board. Draw a ΔABC the lengths of the sides of which are as shown in the figure 9.20 on the card-board.

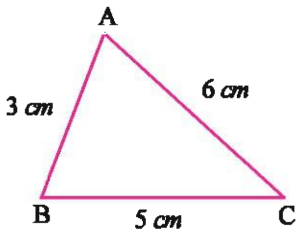


Figure 9.20

Now draw triangles ΔPQR, ΔDEF, ΔXYZ on a paper. The lengths of the sides of all these triangles are shown in the figure 9.21.

Now cut off each triangle from the paper and try to superimpose it on the triangle which is drawn on the card-board.

We observe that if we rotate the triangles ΔDEF and ΔPQR and try to superimpose on ΔABC, then it is clear that these two triangles exactly cover ΔABC.

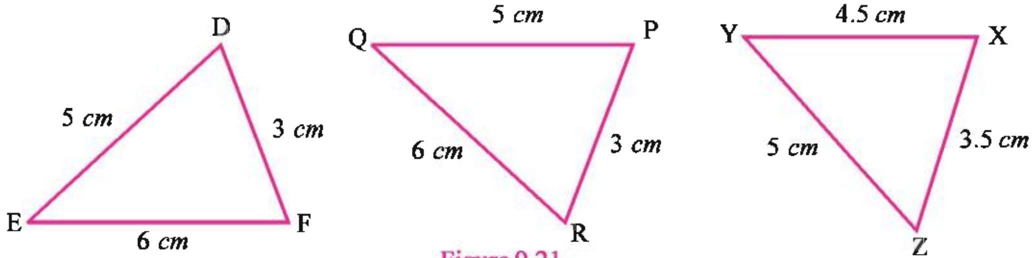


Figure 9.21

So ΔDEF and ΔPQR are congruent to ΔABC. But ΔXYZ can not be exactly superimposed on ΔABC. Hence it is not congruent to ΔABC.

Let us think further about the congruence of two triangles. We have to talk about the congruence of their parts, angles and sides.

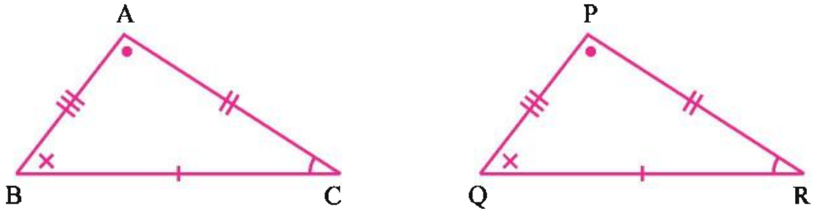


Figure 9.22

For ΔABC and ΔPQR, if ABC ↔ PQR is a correspondence and if (i) $\overline{BC} \cong \overline{QR}$, $\overline{CA} \cong \overline{RP}$, $\overline{AB} \cong \overline{PQ}$ (ii) $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$, then we say correspondence ABC ↔ PQR is a congruence between the vertices of ΔABC and ΔPQR. ΔABC and ΔPQR, are called congruent triangles and we write this symbolically as ΔABC ≅ ΔPQR.

If two triangles are congruent under another correspondence, then for another correspondence giving a congruence of these triangles, the conditions would be different. For the congruence ABC ↔ QPR, the resulting conditions for congruence are (i) $\overline{AB} \cong \overline{QP}$, $\overline{BC} \cong \overline{PR}$, $\overline{AC} \cong \overline{QR}$, (ii) $\angle A \cong \angle Q$, $\angle B \cong \angle P$, $\angle C \cong \angle R$.

If two triangles are congruent, then out of six possible correspondences between them at least one should be a congruence.

For ΔABC and ΔPQR, if $\angle A \cong \angle R$, $\angle B \cong \angle Q$, $\angle C \cong \angle P$, $\overline{AB} \cong \overline{RQ}$, $\overline{BC} \cong \overline{PQ}$, $\overline{AC} \cong \overline{RP}$, then the correspondence ABC ↔ RQP is a congruence.

EXERCISE 9.2

1. If the correspondence DEF ↔ PQR is a congruence, then mention the congruent sides and angles of ΔDEF and ΔPQR.

2. In the figure 9.23, the measures of the sides and angles of each triangle are mentioned. State which correspondence between two triangles is a congruence.

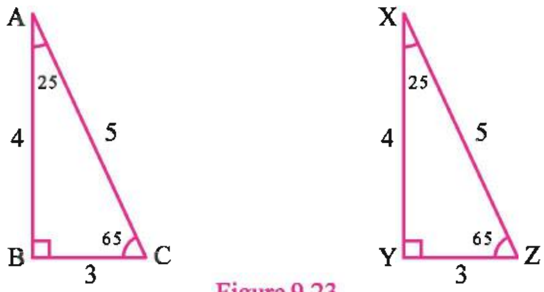


Figure 9.23

3. In the figure 9.24, two congruent triangles are given (corresponding congruent parts are marked using the same signs). State which correspondence between them is a congruence.

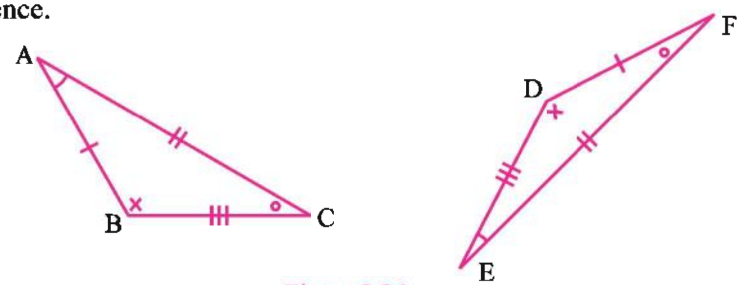


Figure 9.24

4. In ΔDEF and ΔXYZ if $\overline{DE} \cong \overline{XY}$, $\angle E \cong \angle Y$, $\overline{EF} \cong \overline{YZ}$, then which correspondence between them could be a congruence ?

*

9.5 Criteria for Congruence of Triangles

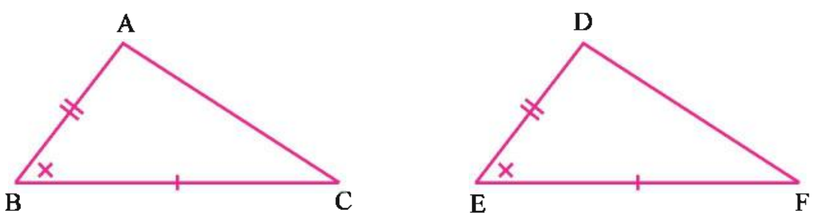


Figure 9.25

In figure 9.25, $\angle B$ is the included angle of the sides $\overline{AB}$ and $\overline{BC}$ in ΔABC and $\angle E$ is the included angle of the sides $\overline{DE}$ and $\overline{EF}$ in ΔDEF.

If we construct these triangles such that $\angle B \cong \angle E$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, then we see that the correspondence $ABC \leftrightarrow DEF$ is a congruence. So if two sides and included angle of one triangle are congruent to corresponding sides and corresponding included angle of the other triangle, the correspondence is a congruence. This criterion is known as SAS postulate.

1. SAS Postulate : (Side-Angle-Side) :

SAS Postulate : If a correspondence between two triangles (or a triangle with itself) is such that two sides and included angle of one triangle are congruent to corresponding two sides and the corresponding included angle of the other triangle, then this correspondence is a congruence and the triangles are congruent. This criterion is known as SAS criterion for congruence of two triangles.

Remember that if two sides and one angle (not included between two sides) of a triangle are congruent to two sides and one angle of the other triangle, then the triangles need not to be congruent by SAS criterion.

Types of the Triangle : (According to the length of sides) :

We have learnt about types of triangles. Let us recall them and study further.

Equilateral Triangle : If all the three sides of a triangle are congruent, then the triangle is called an equilateral triangle.

Isosceles Triangle : A triangle having two sides congruent is called an isosceles triangle.

Scalene Triangle : If no two sides of a triangle are congruent, then the triangle is called a scalene triangle.

Types of triangle (According to the measures of the angle)

Acute angle triangle : If all angles of a triangle are acute, we say it is an acute angle triangle.

Right angle triangle : A triangle is called a right angled triangle or a right triangle, if one of its angle is a right angle.

Obtuse angle triangle : A triangle with one obtuse angle is called an obtuse angle triangle.

Note that every equilateral triangle is an isosceles triangle, because according to the definition of an isosceles triangle, two of its sides should be congruent. Any two sides of an equilateral triangle are always congruent.

In ΔABC, $\overline{AB}$ and $\overline{AC}$ have same length. Now if we measure the angles opposite to $\overline{AB}$ and $\overline{AC}$ (i.e. $\angle C$ and $\angle B$), then we find that $m\angle B = m\angle C$ (i.e. $\angle B \cong \angle C$). **Thus in a triangle if the measures of two sides are equal, then the measures of their opposite angles are also equal.** We shall give a proof of this theorem.

Theorem 9.4 : If two sides of a triangle are congruent, then angles opposite to them are also congruent.

Figure 9.26

(**Note :** ΔABC and ΔACB are the same triangles but the replica is shown for clarity.)

Data : In ΔABC, $\overline{AB} \cong \overline{AC}$

To prove : $\angle B \cong \angle C$

Proof : For the correspondence ABC $\leftrightarrow$ ACB of ΔABC and ΔACB (the same triangle)

$\overline{AB} \cong \overline{AC}$ **(data)**

$\angle A \cong \angle A$ **(Reflexivity)**

$\overline{AC} \cong \overline{AB}$ **(data)**

$\therefore$ The correspondence ABC $\leftrightarrow$ ACB is a congruence. **(SAS postulate)**

$\therefore$ $\angle B \cong \angle C$

Let us apply the above theorem to the following examples.

Example 9 : In ΔPQR, if $\overline{PQ} \cong \overline{PR}$, $m\angle Q = 50$ and Q–D–R, find $m\angle PRD$.

Solution : In ΔPQR, $\overline{PQ} \cong \overline{PR}$

$\therefore$ $\angle Q \cong \angle R$

$\therefore$ $m\angle Q = m\angle R$

Now, $m\angle Q = 50$

$\therefore$ $m\angle R = 50$

$\therefore$ $m\angle PRQ = 50$

Since Q–D–R, $\overrightarrow{RD} = \overrightarrow{RQ}$

So, $m\angle PRQ = m\angle PRD$ **($\angle PRQ \cong \angle PRD$)**

$\therefore$ $m\angle PRD = 50$

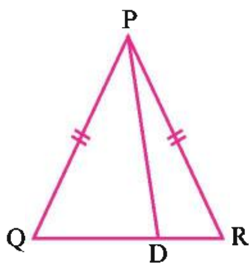


Figure 9.27

Example 10 : In ΔABC, $\overrightarrow{BE}$ is the bisector of $\angle ABC$, $\overline{AB} \cong \overline{AC}$ and $m\angle ABE = 40$, find $m\angle C$.

Solution : Here $m\angle ABC = 2m\angle ABE$

$= 2(40)$

$= 80$

Now, $\overline{AB} \cong \overline{AC}$

$\therefore \quad m\angle C = m\angle ABC$

$\therefore \quad m\angle C = 80$

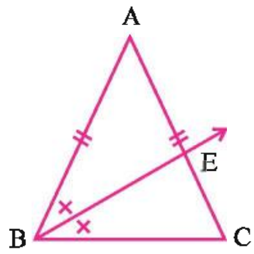


Figure 9.28

Note : If a ray originating from the vertex of a triangle bisects the angle at the vertex, then the ray is called the **bisector of the angle** of the triangle.

Corollary 1 : An equilateral triangle is also an equiangular triangle and measure of their congruent angles is 60. (Equiangular means all angles congruent.)

Data : ΔPQR is an equilateral triangle.

To prove : ΔPQR is an equiangular triangle.

Proof : Here, in ΔPQR, $\overline{PQ} \cong \overline{PR}$ **(data)**

$\therefore \quad \angle R \cong \angle Q$ **(i)**

Now, $\overline{PQ} \cong \overline{QR}$ **(data)**

$\therefore \quad \angle R \cong \angle P$ **(ii)**

$\therefore \quad \angle R \cong \angle Q$ and $\angle R \cong \angle P$

$\therefore \quad \Delta PQR$ is an equiangular triangle.

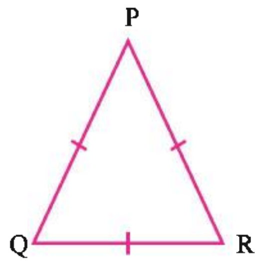


Figure 9.29

Hence, $m\angle P = m\angle Q = m\angle R$ **(iii) (by (i) and (ii))**

Now, we know that,

$m\angle P + m\angle Q + m\angle R = 180$

$\therefore \quad m\angle P + m\angle P + m\angle P = 180$ **(by (iii))**

$\therefore \quad 3m\angle P = 180$

$\therefore \quad m\angle P = 60$

Thus, $m\angle Q = m\angle R = 60$ **(by (iii))**

$\therefore \quad \Delta PQR$ is an equiangular triangle and measure of each of their congruent angles is 60.

EXERCISE 9.3

1. In ΔPQR, if M and N are the mid-points of two congruent sides $\overline{PQ}$ and $\overline{PR}$ respectively, then prove that QN = RM.

2. In an isosceles ΔXYZ, $\overline{XY} \cong \overline{XZ}$. If M and N are the points on $\overline{YZ}$ such that YN = MZ, then prove that XM = XN.

3. Prove that the triangle obtained by joining the mid-points of the sides of an isosceles triangles is also an isosceles triangle.

4. From the figures 9.30, find which correspondence between the triangles is a congruence :

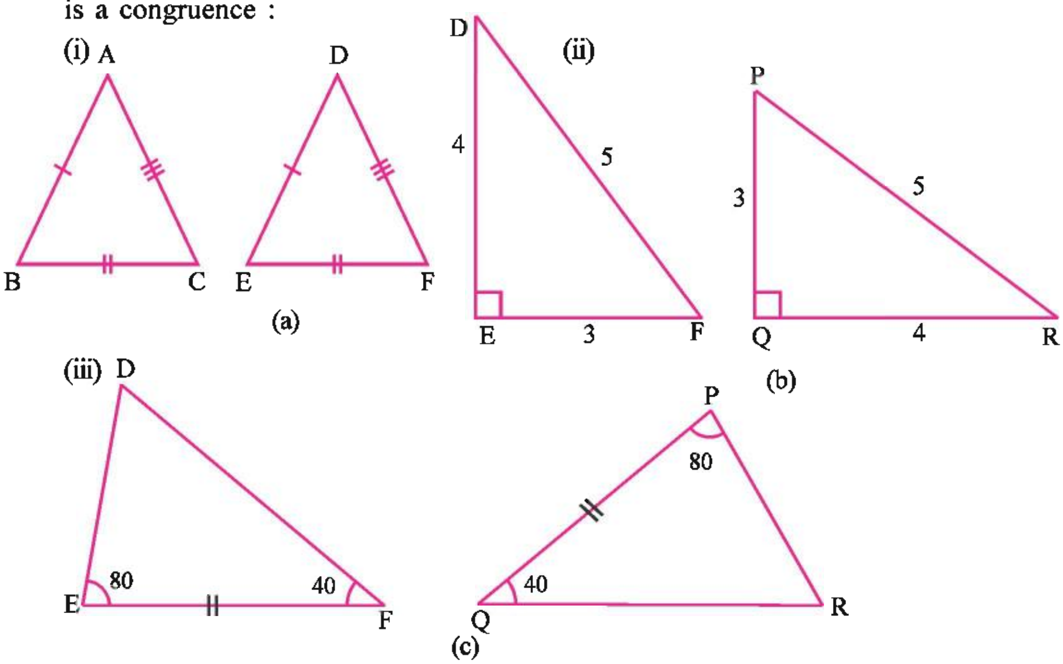


Figure 9.30

5. In ΔABC, if $\overrightarrow{AD}$ is the bisector of $\angle A$ intersecting $\overline{BC}$ at D and $\overline{AB} \cong \overline{AC}$, prove that D is the mid-point of $\overline{BC}$.

6. For ΔABC, if $m\angle A = x$, $m\angle B = 3x$, $m\angle C = y$ and $3y - 5x = 30$, then identify the type of the triangle.

7. If a line m is the perpendicular bisector of $\overline{AB}$ and $P \in m$, then prove that P is equidistant from A and B.

8. In ΔABC, if $\overline{AB} \cong \overline{AC}$ and $m\angle BAC = 50$, then find the measures of the remaining angles of ΔABC.

9. In ΔABC, if $\overline{AB} \cong \overline{AC}$, $\overrightarrow{BD}$ is the angle bisector of $\angle B$ such that $m\angle ABD = 40$, then find the measures of all the angles of ΔABC.

10. In an isosceles triangle, if the third angle has measure greater by 60 than the measure of its congruent angles, then find the measures of all the angles of the triangle.

*

2. ASA Theorem :

For SAS postulate, two sides and the included angle between the sides of a triangle are congruent to the corresponding sides and included angle of the other triangle. Now we shall learn about the side included between two angles. If two angles and included side of one triangle are congruent to corresponding parts of the other triangle in a correspondence, then these triangles are congruent. This criterion is known as ASA (Angle-Side-Angle) theorem.

To prove this theorem, we will use following property of real numbers. Any two real numbers a and b obey following rule called law of trichotomy. For any two real numbers a and b, $a > b$ or $a = b$ or $a < b$. In other words if $a \neq b$, then $a > b$ or $a < b$.

Theorem 9.5 : (ASA theorem) : A correspondence between two triangles (or a triangle with itself) is given. If two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle under the given correspondence, then this correspondence is a congruence.

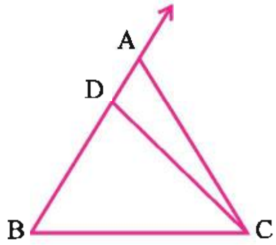


Figure 9.31

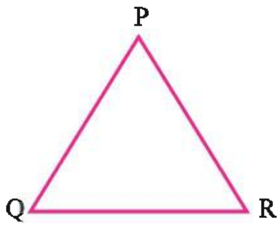


Figure 9.32

Data : In correspondence ABC $\leftrightarrow$ PQR of the vertices of two triangles ΔABC and ΔPQR, $\angle B \cong \angle Q$, $\angle C \cong \angle R$ and $\overline{BC} \cong \overline{QR}$.

To prove : The correspondence ABC $\leftrightarrow$ PQR is a congruence.

Proof : In ΔABC and ΔPQR, if AB = PQ then by applying SAS postulate, we can assert that given correspondence is a congruence.

If AB $\neq$ PQ, then according to the law of trichotomy AB > PQ or AB < PQ.

Suppose AB > PQ

According to the point plotting theorem, we get a point D on $\overrightarrow{BA}$ such that $\overline{BD} \cong \overline{PQ}$.

Now for the correspondence PQR $\leftrightarrow$ DBC,

$\overline{PQ} \cong \overline{DB}$ **(by point-plotting theorem)**

$\overline{QR} \cong \overline{BC}$ **(given)**

$\angle Q \cong \angle B$ **(given)**

∴ correspondence PQR ↔ DBC is a congruence. **(SAS postulate)**

∴ $\angle PRQ \cong \angle DBC$

but $\angle PRQ \cong \angle ACB$ **(data)**

∴ $\angle ACB \cong \angle DCB$

Also, $D \in \overrightarrow{BA}$

D and A are on the same side of $\overleftrightarrow{BC}$.

∴ According to the postulate of unique ray, $\overrightarrow{CD}$ and $\overrightarrow{CA}$ are same.

∴ D and A both are on $\overleftrightarrow{CA}$.

Also D and A both are on $\overrightarrow{BA}$ and $\overleftrightarrow{CA} \neq \overleftrightarrow{BA}$

∴ D = A

∴ $\overline{AB} \cong \overline{PQ}$

∴ correspondence ABC ↔ PQR is a congruence **(SAS postulate)**

The case AB < PQ can be dealt with similarly.

We know that if two sides of a triangle are congruent, then their opposite angles are also congruent. Now we accept the following theorem.

Theorem 9.6 : If two angles of a triangle are congruent, then the sides opposite to them are congruent.

Equiangular triangle : If all the three angles of a triangle are congruent, then the triangle is called an equiangular triangle.

Corollary 2 : An equiangular triangle is an equilateral triangle.

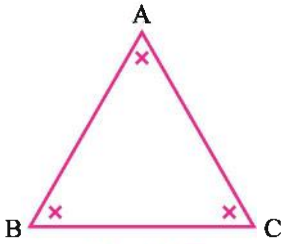


Figure 9.33

Data : ΔABC is an equiangular triangle.

To prove : ΔABC is an equilaterial triangle.

Proof : For ΔABC, $\angle A \cong \angle B$ and $\angle B \cong \angle C$

Now, since $\angle A \cong \angle B$ **(given)**

∴ $\overline{BC} \cong \overline{AC}$ **(i)**

Now, $\angle B \cong \angle C$ **(given)**

∴ $\overline{AC} \cong \overline{AB}$ **(ii)**

$\overline{BC} \cong \overline{AC}$ and $\overline{AC} \cong \overline{AB}$ **(by (i) and (ii))**

∴ BC = AC and AC = AB

∴ ΔABC is an equilateral triangle.

3. AAS condition :

In any correspondence between two triangles (or a triangle with itself) if two angles and non-included side of any triangle are congruent to the corresponding angles and corresponding non-included side of the other triangle then both triangles are congruent. This is known as AAS (Angle-Angle-Side) condition. We do not prove it. (Try it !)

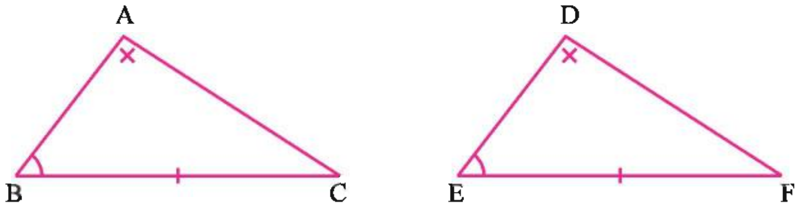


Figure 9.34

Here, $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$.

Here $\overline{BC}$ is non-included side for $\angle A$ and $\angle B$ and similarly corresponding $\overline{EF}$ is the non-included side for $\angle D$ and $\angle E$. Thus ABC $\leftrightarrow$ DEF is a congruence.

$\therefore \quad \Delta ABC \cong \Delta DEF$

Let us understand the above condition by following example.

Example 11 : The measures of some parts of two triangles are given in the figure 9.35. Then prove that $\Delta ABC \cong \Delta DEF$.

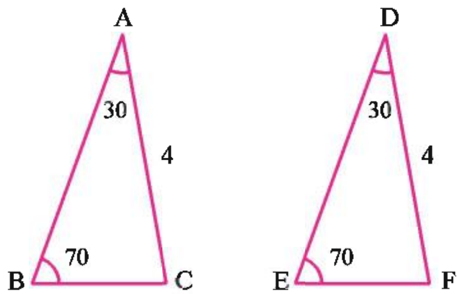


Figure 9.35

Solution : Here $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\overline{AC} \cong \overline{DF}$.

Now, we know that $m\angle A + m\angle B + m\angle C = 180$

$\therefore \quad 30 + 70 + m\angle C = 180$

$\therefore \quad m\angle C = 180 - 100$

$\therefore \quad m\angle C = 80$

Similarly we can prove that $m\angle F = 80$

$\therefore \quad m\angle C = m\angle F \qquad$ Thus, $\angle C \cong \angle F$

Hence, $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$ and $\angle C \cong \angle F$

So by ASA theorem, we say that, ABC $\leftrightarrow$ DEF is a congruence.

$\therefore$ $\Delta ABC \cong \Delta DEF$

The above condition can be proved as shown below :

If any two angles and a non-included side of one triangle are congruent to the corresponding angles and corresponding non-included, side of the other triangle, then both the triangles are congruent.

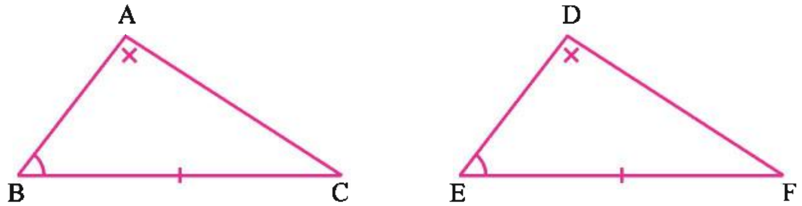


Figure 9.36

Hint : $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$ are given

Thus $m\angle A = m\angle D$ and $m\angle B = m\angle E$ (i)

Now, $m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F = 180$ (ii)

$\therefore$ by (i), $m\angle A + m\angle B = m\angle D + m\angle E$

$\therefore$ by (ii), $180 - m\angle C = 180 - m\angle F$

$\therefore$ $m\angle C = m\angle F$

$\therefore$ $\angle C \cong \angle F$

Now proceed to apply ASA condition.

SSA condition does not imply congruence.

EXERCISE 9.4

1. In ΔABC, if $\overline{AB} \cong \overline{AC}$ and the angle bisector of $\angle A$ intersects $\overline{BC}$ at D, then prove that $\overline{AD}$ the perpendicular bisector of $\overline{BC}$.
2. In ΔABC, if $\overline{AB} \cong \overline{AC}$, $\angle ACD$ and $\angle CAP$ are exterior angles of ΔABC such that B–A–P and B–C–D and $m\angle ACD = 110$, then find measures of all the angles of ΔABC and also find $m\angle CAP$.
3. For ΔABC, if $D \in \overline{BC}$ such that $AD = BD = CD$, then prove that $m\angle A = 90$.
4. In ΔPQR, bisector of $\angle P$ is perpendicular to $\overline{QR}$. Then prove that ΔPQR is an isosceles triangle.
5. In ΔABC, if $\overline{AB} \cong \overline{AC}$ and D is the mid-point of $\overline{BC}$, then prove that $m\angle ADB = m\angle ADC = 90$.
6. If P is in the interior of ΔABC and $PA = PB = PC$, then prove that $m\angle A = m\angle ABP + m\angle ACP$.

7. In the figure 9.37, if $\overline{AF} \cong \overline{AE}$ and $\overline{AB} \cong \overline{AC}$, then prove that $\overline{BE} \cong \overline{CF}$.

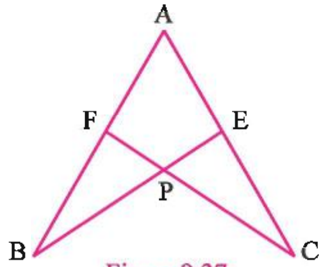


Figure 9.37

*

4. SSS Theorem :

Now we will learn one more criterion for the congruence.

Theorem 9.7 : If there is a correspondence between the vertices of two triangles such that three sides of one triangle are congruent to the corresponding three sides of the other triangle, then the correspondence is a congruence.

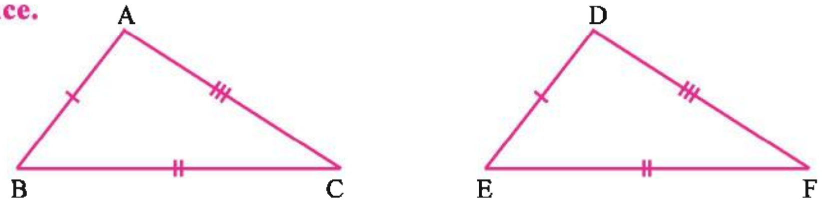


Figure 9.38

In ΔABC and ΔDEF, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$, then $ABC \leftrightarrow DEF$ is a congruence. So we say that both triangles are congruent. We will accept the SSS theorem without proof.

Let us try to understand the above theorem by the following example.

Example 12 : In ΔABC, $\overline{AB} \cong \overline{AC}$. A point D is in the interior of ΔABC such that $\angle DBC \cong \angle DCB$. Then prove that $\overrightarrow{AD}$ bisects $\angle BAC$.

Solution :

Data : $\overline{AB} \cong \overline{AC}$ and a point D is in the interior of ΔABC such that $\angle DBC \cong \angle DCB$.

To prove : $\overrightarrow{AD}$ is the bisector of $\angle A$.

Proof : $\angle DBC \cong \angle DCB$ **(data)**

$\therefore \quad \overline{BD} \cong \overline{CD}$

For the correspondence, $ADB \leftrightarrow ADC$,

$\overline{AD} \cong \overline{AD}$, $\overline{BD} \cong \overline{DC}$ and $\overline{AB} \cong \overline{AC}$

Thus, by SSS theorem, $ADB \leftrightarrow ADC$ is a congruence.

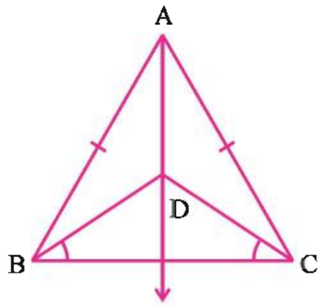


Figure 9.39

∴ $\angle BAD \cong \angle CAD$

Now D is in the interior of $\angle A$.

∴ $\overrightarrow{AD}$ bisects $\angle BAC$.

5. R.H.S. Theorem (Right Angle - Hypotenuse - Side theorem)

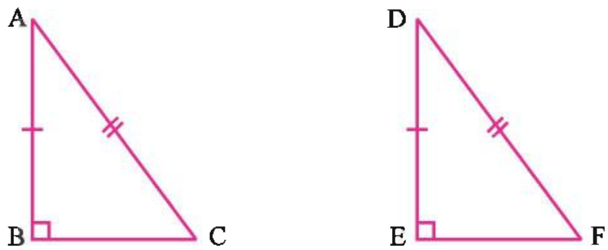


Figure 9.40

In ΔABC and ΔDEF, $m\angle B = m\angle E = 90$ and $\overline{AC}$ and $\overline{DF}$ are the hypotenuses. If they are congruent and any one side of ΔABC is congruent with the corresponding side of ΔDEF, then the correspondence is a congruence. It is known as RHS theorem stated as follows :

Theorem 9.8 (R.H.S. theorem) : In a correspondence between two right angle triangles, if one side and the hypotenuse of one triangle are congruent to the corresponding side and hypotenuse of the other triangle, then the correspondence is a congruence and the triangles are congruent.

We will accept the theorem without proof.

Let us try to understand this theorem by the following example.

Example 13 : In ΔABC, if $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are congruent altitudes, then prove that ΔABC is an equilateral triangle.

Solution : Here $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are the altitudes on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

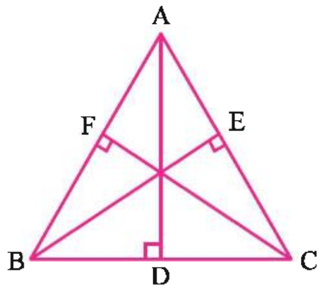


Figure 9.41

Thus, $m\angle D = m\angle E = m\angle F = 90$

Thus $\overline{BC}$ is the common hypotenuse for right angle triangles ΔBEC and ΔCFB.

Consider correspondence $BEC \leftrightarrow CFB$,

$\angle E \cong \angle F$ and $\overline{BC} \cong \overline{CB}$ and $\overline{BE} \cong \overline{CF}$.

Thus, by RHS theorem, $BEC \leftrightarrow CFB$ is a congruence.

∴ $\Delta BEC \cong \Delta CFB$

Now, the corresponding parts of congruent triangles are congruent.

Thus $\angle B \cong \angle C$

Now, the sides opposite to the congruent angles are congruent.

Hence $\overline{AC} \cong \overline{AB}$. **(i)**

Similarly, for right angle triangles ΔABD and ΔABE, $\overline{AB}$ is the common hypotenuse. Consider correspondence $ABD \leftrightarrow BAE$.

$\therefore$ $\overline{AB} \cong \overline{BA}$ and $\angle D \cong \angle E$ and $\overline{AD} \cong \overline{BE}$

Thus by RHS theorem, we get $\Delta ABD \cong \Delta BAE$

$\therefore$ $\angle B \cong \angle A$

$\therefore$ Sides opposite to $\angle B$ and $\angle A$ namely $\overline{AC}$ and $\overline{BC}$ are congruent. **(ii)**

Thus, by results (i) and (ii) we get,

$\overline{AB} \cong \overline{AC}$ and $\overline{AC} \cong \overline{BC}$ **(iii)**

$AB = BC = AC$ **(from (iii))**

$\therefore$ ΔABC is an equilateral triangle.

EXERCISE 9.5

1. In ΔABC, if $\overline{BE}$ and $\overline{CF}$ are two congruent altitudes of ΔABC. Using RHS theorem, prove that ΔABC is an isosceles triangle.

2. In ΔABC, $P \in \overline{AB}$ and $Q \in \overline{AC}$ such that $m\angle BQC = m\angle CPB = 90$ and $\overline{BP} \cong \overline{CQ}$. Then prove that $\overline{BQ} \cong \overline{CP}$.

3. In ΔABC, if $\overline{AB} \cong \overline{BC}$ and $m\angle A = 50$, then find the measure of exterior $\angle ACD$ and also find the measures of the remaining angles of ΔABC.

4. In the figure 9.42, $m\angle BAC = m\angle BDC = 90$ and $\overline{BD} \cong \overline{AC}$. Prove that $\Delta ABD \cong \Delta ACD$.

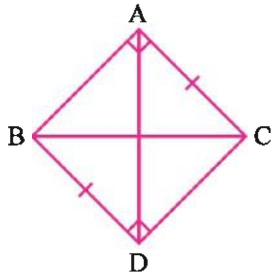


Figure 9.42

5. In the figure 9.43, if $\overline{PQ} \cong \overline{SR}$ and $\overline{QS} \cong \overline{PR}$ then prove that $\angle PQS \cong \angle SRP$ and $\angle QPS \cong \angle RSP$.

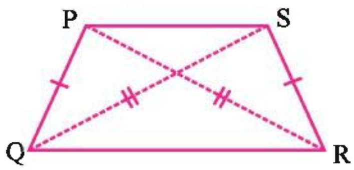


Figure 9.43

6. For ΔPQR, if M–Q–R and Q–R–N and the exterior angles $\angle PQM$ and $\angle PRN$ of ΔPQR are congruent, then prove that $\overline{PQ} \cong \overline{PR}$.

*

9.6 Inequalities in a Triangle

We know that the measures of angles and sides of triangle are real numbers. The properties of real numbers can also be applied to the measures of these quantities. We recall certain properties of real numbers.

R^+ is the set of all positive real numbers. The properties of real numbers are listed below :

(1) For each $x \in R$, one and only one of the following three possibilities holds :

$x > 0$ or $x = 0$ or $x < 0$.

Thus, if $x \neq 0$, then $x \in R^+$ or $-x \in R^+$

If $x - y > 0$, we write $x > y$

(2) If $x > 0$, $y > 0$ then $x + y > 0$ and $xy > 0$

(3) If $x > 0$, $y > z$, then $xy > xz$ and if $x < 0$, $y > z$, then $xy < xz$.

For two real numbers $a, b \in R$, one and only one of the three possibilities holds : $a > b$ or $a = b$ or $a < b$. This is called the Law of trichotomy.

If $a > b$ and $b > c$, then $a > c$.

Further if $a > b$ or $a = b$ then it can be written as $a \geq b$.

Thus, **if $a \geq b$ and $b \geq c$ then $a \geq c$ (Transitive property)**

Also if $a \geq b$ and $b > c$, then $a > c$ and

if $a > b$ and $b \geq c$, then $a > c$

We shall apply these properties of inequalities to the measures of angles and measures of sides of triangles.

Inequalities Concerning Measures of Sides and Angles of Triangle :

In a right angle triangle ΔABC, $m\angle B = 90$. Let AB = 15 units, BC = 8 units. So by the Pythagoras theorem, we get the measure of hypotenuse is 17 units. We know that two angles other than the right angle of a triangle are acute.

i.e. $m\angle A < m\angle B$ and $m\angle C < m\angle B$

The measure of the largest side of ΔABC is 17 units and the angle opposite to it is $\angle B$.

The measure of $\overline{AB}$ is 15 units which is less than 17, the measure of $\overline{AC}$.

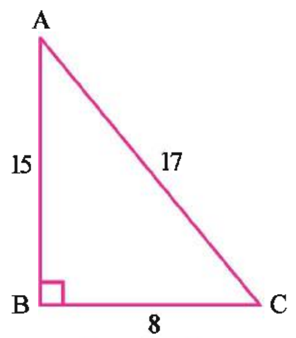


Figure 9.44

Angle opposite to $\overline{AB}$ is $\angle C$ and angle opposite to $\overline{AC}$ is $\angle B$. Also $m\angle C < m\angle B$.

Measure of $\overline{BC}$ is 8 units and less than the measure of $\overline{AC}$. Angle opposite to $\overline{BC}$ is $\angle A$ and angle opposite to $\overline{AC}$ is $\angle B$. Also $m\angle A < m\angle B$.

The same results can be concluded for any type of triangle by taking the measures of sides and angles of different triangles.

Theorem 9.9 : If the measures of two sides of a triangle are unequal, then the measure of the angle opposite to the side with larger measure is greater than the measure of the angle opposite to the side with smaller measure.

We shall call the side with greater measure a larger side and the side with smaller measure a shorter side. Thus the above theorem can be stated as follows :

In a triangle, the measure of the angle opposite to a larger side is greater than the measure of the angle opposite to a shorter side.

In figure 9.44, AC > AB. So $m\angle B > m\angle C$

In a triangle, the relation between the side opposite to an angle with larger measure and the side opposite to an angle with smaller measure is the converse of the above theorem which we accept without proof.

Theorem 9.10 : If the measures of two angles of a triangle are unequal, then the measure of the side opposite to the angle with greater measure is larger than the measure of the side opposite the the angle with smaller measure.

Let us understand the following example based on this theorem.

Example 14 : In ΔABC, $m\angle A = 35$, $m\angle B = 50$, determine the smallest and the largest side of the triangle.

Solution : Here $m\angle A = 35$ and $m\angle B = 50$

Now, $m\angle A + m\angle B + m\angle C = 180$

$\therefore$ $35 + 50 + m\angle C = 180$

$\therefore$ $m\angle C = 180 - 85$

$\therefore$ $m\angle C = 95$

$\therefore$ $m\angle A$ is the smallest and $m\angle C$ is the largest. Hence, the sides opposite to them have smallest and largest measures respectively. Thus $\overline{BC}$ has the smallest length and $\overline{AB}$ has the largest length.

$\therefore$ $\overline{BC}$ is the smallest side and $\overline{AB}$ is the largest side.

Example 15 : In ΔPQR, if PQ = PR and Q–R–T, prove that PT > PR.

Solution : $\angle PRQ$ is an exterior angle for ΔPTR.

$m\angle PRQ > m\angle PTR$

$m\angle PRQ = m\angle PQR$ as PQ = PR

$\therefore$ $m\angle PQR > m\angle PTR$

Now, because of Q–R–T,

we have $\overrightarrow{QR} = \overrightarrow{QT}$ and $\overrightarrow{TR} = \overrightarrow{TQ}$.

$\therefore$ $m\angle PQT > m\angle PTQ$

$\therefore$ PT > PQ, but PQ = PR

$\therefore$ PT > PR

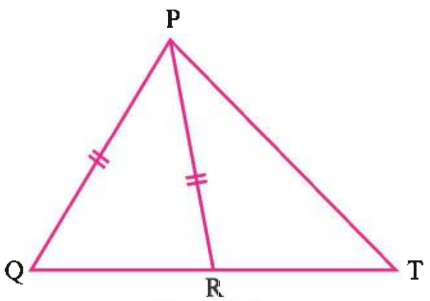


Figure 9.45

Sum of the Measures of the Sides of a Triangle :

In ΔABC, if $AB = 3.5$, $BC = 6$ and $AC = 4.5$,

then $AB + BC = 9.5$

Now, $9.5 > 4.5$, so, $BC + AB > AC$

Similarly, $BC + AC = 6 + 4.5 = 10.5 > 3.5$

So, $BC + AC > AB$.

Also, $AB + AC = 3.5 + 4.5 = 8$

and $8 > 6$. So, $AB + AC > BC$.

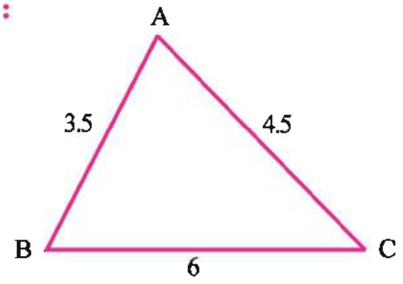


Figure 9.46

Thus, we accept the following theorem.

Theorem 9.11 : Sum of measures of any two sides of a triangle is greater than the measure of the third side.

We take a point D on $\overrightarrow{BA}$ such that $AD = AC$ (see figure 9.47)

Now show that $m\angle BCD > m\angle BDC$ and prove that $BA + AC > BC$

You will reach the proof of this theorem.

Let us understand the above theorem by applying in the following examples.

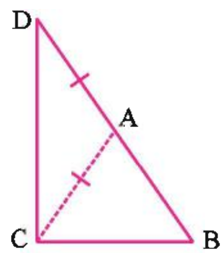


Figure 9.47

Example 16 : If D is a point on side $\overline{QR}$ of ΔPQR such that $PD = PR$, then prove that $PQ > PD$.

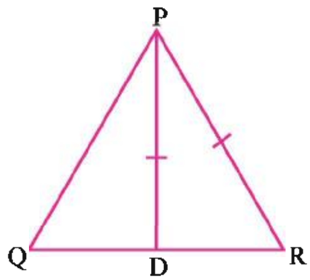


Figure 9.48

Solution : In ΔDPR, $PD = PR$ **(given)**

$\therefore$ $m\angle PRD = m\angle PDR$

(angles opposite to congruent sides)

$\therefore$ $\angle PRD \cong \angle PDR$ **(i)**

Now, $\angle PDR$ is an exterior angle of ΔPQD

$\therefore$ $m\angle PDR > m\angle PQD$ **(ii)**

$m\angle PRD > m\angle PQD$ **((i) and (ii))**

$m\angle PRQ > m\angle PQR$ **(since $\overrightarrow{RD} = \overrightarrow{RQ}$ and $\overrightarrow{QD} = \overrightarrow{QR}$)**

Now $\overline{PQ}$ is the side opposite to $\angle PRQ$ and $\overline{PR}$ is the sides opposite to $\angle PQR$.

$\therefore$ $PQ > PR$

$\therefore$ $PQ > PD$ **(PR = PD)**

Example 17 : For ΔABC, if $D \in \overline{BC}$, then prove that $AB + BC + AC > 2AD$.

Solution : Here the point $D \in \overline{BC}$ such that B–D–C.

$\therefore$ BD + DC = BC **(i)**

Now for the ΔABD, we get AB + BD > AD **(ii)**

and for ΔACD, we get AC + CD > AD **(iii)**

by (ii) and (iii) we get AB + BD + AC + CD > 2AD

AB + AC + (BD + DC) > 2AD

$\therefore$ AB + AC + BC > 2AD **(by (i))**

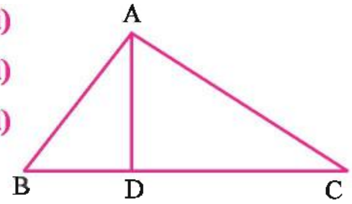


Figure 9.49

An important result : The length of each side of a triangle is greater than the positive difference of the lengths of the other two sides.

In ΔABC, we know that,

AB + BC > AC, BC + AC > AB and AC + AB > BC

Since AB + BC > AC

$\therefore$ BC > AC − AB **(i)**

and BC + AC > AB implies BC > AB − AC

$\therefore$ BC > (AC − AB) and BC > AB − AC

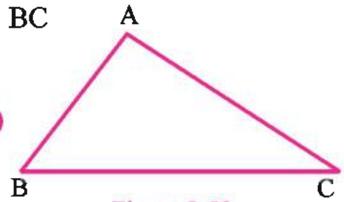


Figure 9.50

Thus we get BC > (AC − AB) and −(AC − AB)

but | AC − AB | = AC − AB or −(AC − AB) **(ii)**

$\therefore$ BC > | AC − AB | **(by (i) and (ii))**

Similarly AC > | AB − BC | and AB > | AC − BC |

$\therefore$ We say that **| AC − BC | < AB < AC + BC**

Perpendicular distance of a line from a point :

Given line $\overleftrightarrow{PQ}$ and A $\notin$ $\overleftrightarrow{PQ}$, we can draw a line passing through A and perpendicular to $\overleftrightarrow{PQ}$ in any plane containing $\overleftrightarrow{PQ}$. Such a line is unique. Thus from a point outside a line, we can draw a unique perpendicular to the line.

In figure 9.51, A is a point outside line $\overleftrightarrow{PQ}$, if we draw a perpendicular $\overline{AM}$ to $\overleftrightarrow{PQ}$ such that M $\in$ $\overleftrightarrow{PQ}$, then AM is the perpendicular distance of A from $\overleftrightarrow{PQ}$ and M is the foot of perpendicular from A to $\overleftrightarrow{PQ}$. Hence ΔAMN is a right angle triangle and the hypotenuse $\overline{AN}$ is the largest side. So AM < AN.

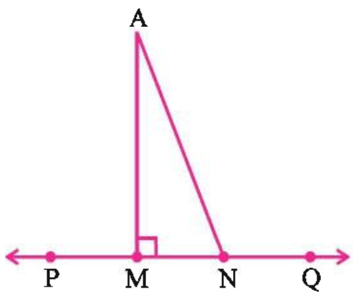


Figure 9.51

If we consider a point A outside a line *l* and we join this point with the different points on line *l*, we get many different line-segments. (see figure 9.52) but the perpendicular line-segment is at the shortest distance from the point A.

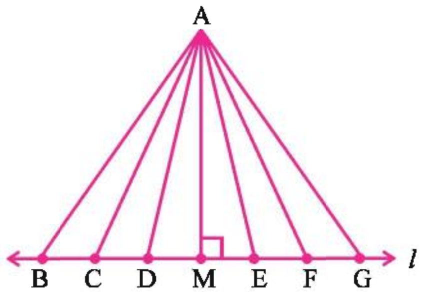


Figure 9.52

We accept the following result without proof.

Among all the line-segments obtained by joining a point outside a line to any point of the line the perpendicular line-segment from the point to the line is the shortest.

Let $l \parallel m$.

Let P and Q be any two points on *l* and let $\overline{PM}$ and $\overline{QN}$ be perpendicular to *m*, where M, N $\in$ *m*.

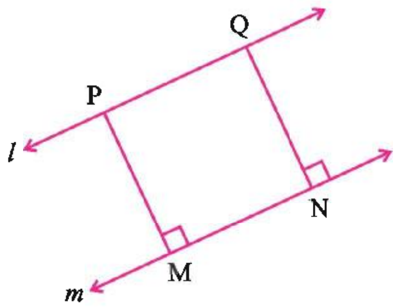


Figure 9.53

From the figure 9.53 it is apparent that PM = QN. This distance is called the distance between parallel lines. Thus, we define distance between two parallel lines as the perpendicular distance between them.

Perpendicular lines :

In the figure 9.54, lines *l* and *m* intersect at M. So we get two pairs of vertically opposite angles. i.e. $\angle PME \cong \angle DMQ$ and $\angle PMD \cong \angle EMQ$.

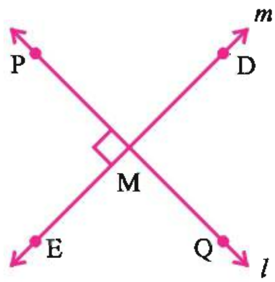


Figure 9.54

If one the above four angles formed is a right angle at M, then all the four angles formed at M are right angle and the intersecting lines are called perpendicular lines.

Perpendicular lines : If any one (hence all) of the four angles formed by two intersecting lines at the point of their intersection is a right angle, then the lines are called lines mutually perpendicular to each other.

If two lines are mutually perpendicular, then a subset of one line is said to be perpendicular to any subset of the other line. Even when these subsets do not intersect each other, if the lines containing them are perpendicular, the subsets are said to be perpendicular to each other.

Note that a ray, a line-segment are subsets of a line.

In a plane, any line-segment is given and a perpendicular to this line-segment lying in the plane which passes through the mid-point of this line-segment is called **perpendicular bisector** of the given line-segment.

In figure 9.55, $\overline{AB}$ is a given line-segment. Now P $\notin$ $\overline{AB}$ and the line-segment $\overline{PQ}$ passes through the mid-point of $\overline{AB}$ and $\overleftrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$. Then $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

Note that each point on the perpendicular bisector of $\overline{AB}$ is a equidistant from A and B. Conversely, the set of all points equidistant from A and B forms the perpendicular bisector of $\overline{AB}$.

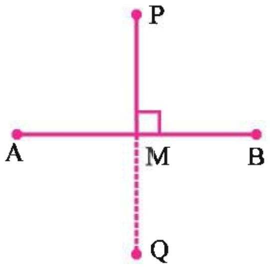


Figure 9.55

EXERCISE 9

1. For ΔABC, if $m\angle A + m\angle B = 100$ and $m\angle B + m\angle C = 130$, then find the measures of all the angles of ΔABC.

2. $\angle ACD$ is an exterior angle of ΔABC and E lies in the interior of $\angle ACD$. $\overline{AC} \perp \overline{CE}$. If $6m\angle A = 7m\angle B$, $5m\angle B = 6m\angle C$, then find the $m\angle ECD$.

3. In ΔABC, if AB = 10, BC = 18, then prove $8 < AC < 28$.

4. For triangles ΔABC and ΔDEF, $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. P is the mid-point of $\overline{BC}$ and Q is the mid-point of $\overline{EF}$. If $\overline{AP} \cong \overline{DQ}$, then prove that $\Delta ABC \cong \Delta DEF$.

5. For ΔABC, $\overline{AB} \cong \overline{BC}$ and if $m\angle A - m\angle B = 54$, find $m\angle B$.

6. For ΔABC, $\overline{AC} \cong \overline{BC}$ and B–C–D and $\overrightarrow{CP}$ is the angle bisector of $\angle ACD$. If $m\angle ACP = 35$, then find the $m\angle A$.

7. For ΔABC, BC = 5, AC = 12. Prove $7 < AB < 17$.

8. Prove that for any right angle triangle, hypotenuse is the largest side.

9. For ΔABC if D, E, F are the mid-point of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, respectively, then prove that AE + BF + CD is less than the perimeter of ΔABC.

10. In ΔABC, if AB = 8, BC = 5, then prove that $3 < AC < 13$.

11. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For ΔABC, the side opposite to $\angle A$ is ...... ☐

(a) $\overline{AB}$ (b) $\overline{BC}$ (c) $\overline{CA}$ (d) $\overrightarrow{AC}$

(2) For ΔABC ...... is included by the sides $\overline{BC}$ and $\overline{AC}$.

(a) $\angle A$ (b) $\angle B$

(c) $\angle C$ (d) exterior angle of $\angle D$

(3) If $\angle ACD$ is an exterior angle of ΔABC and $m\angle ACD = 105$, then $m\angle ACB = ...$.

(a) 105 (b) 75 (c) 100 (d) less than 75

(4) For the correspondence $BAC \leftrightarrow YXZ$ between ΔABC and ΔXYZ, the angle $\angle$ ...... corresponds to $\angle Z$.

(a) B (b) A (c) C (d) Y

(5) For ΔABC, if $D \in \overrightarrow{BC}$ such that B–C–D, then ...... is the exterior angle of ΔABC.

(a) $\angle ABC$ (b) $\angle ACB$ (c) $\angle ACD$ (d) $\angle BAD$

(6) The measure of congruent angles in ΔABC (where $\overline{AB} \cong \overline{AC}$) is ...... where $m\angle A = 60$.

(a) 35 (b) 45 (c) 60 (d) 90

(7) For ΔABC, $\angle A \cong \angle C$. If BC = 3, AC = 4, then the perimeter of ΔABC is ....

(a) 10 (b) 12 (c) 14 (d) 7

(8) ΔABC is ....

(a) $\overline{AB} \cup \overline{BC}$ (b) $\angle A \cup \angle B$

(c) $\overline{AB} \cup \overline{BC} \cup \overline{AC}$ (d) $\angle A \cup \angle B \cup \angle C$

(9) From the following which condition is not possible for the congruence of two triangles ?

(a) ASA (b) AAS (c) AAA (d) SSS

(10) For ΔABC ...... is true (if it is not a right angle).

(a) $AB^2 + BC^2 = AC^2$ (b) $AB + BC = AC$

(c) $AC > AB + BC$ (d) $AC < AB + BC$

(11) For ΔABC, if $m\angle A = 40$, $m\angle C = 50$, then the smallest side of ΔABC is ......

(a) $\overline{AB}$ (b) $\overline{BC}$ (c) AC (d) BC

(12) For ΔABC, if $\angle B \cong \angle C$, then ...... sides are congruence.

(a) $\overline{AB}$ and $\overline{BC}$ (b) $\overline{AB}$ and $\overline{AC}$ (c) $\overline{BC}$ and $\overline{AC}$ (d) any two

(13) For ΔABC, the bisectors of $\angle B$ and $\angle C$ intersect at the point P. If $m\angle A = 70$, then $m\angle BPC =$ ☐

(a) 50 (b) 75 (c) 100 (d) 125

(14) For ΔABC, if $m\angle B = 2x$, $m\angle A = x$, $m\angle C = y$ and $2x - y = 40$, the ΔABC is .... ☐

(a) scalene (b) right angled (c) isosceles (d) equilateral

(15) If the measures of the angles of ΔABC are in proportion 1 : 2 : 3, then the measure of the smallest angle is .... ☐

(a) 30 (b) 60 (c) 90 (d) 120

(16) For ΔABC, $\overline{BC}$ ...... ΔABC. ☐

(a) $\in$ (b) $\notin$ (c) $\subset$ (d) $\not\subset$

(17) In ΔABC, if $m\angle A + m\angle B = 120$, then $m\angle C =$ ...... ☐

(a) 20 (b) 40 (c) 60 (d) 80

*

Summary

In this chapter, we have studied the basic concepts of a triangle.

1. We defined terms related to a triangle like included angle between sides, angle opposite to a side of triangle, side opposite to angle of a triangle, included side between angles, partition of a plane by a triangle, interior region of a triangle.
2. An angle forming a linear pair with any angle of a triangle is known as an exterior angle of a triangle.
3. The measure of an exterior angle of a triangle is larger than the measure of each of its interior opposite angles and is equal to the sum of these two interior opposite angles.
4. Sum of the measures of three angles of any triangle is 180.
5. Correspondence between vertices and congruence
6. SAS criterion for congruence of the triangle
7. Types of the triangle according to the lengths of sides and measures of angles.
8. ASA theorem for congruence, AAS condition
9. SSS theorem for congruence
10. RHS theorem for congruence

Answers

(Answers of problems requiring some calculations only are given.)

Exercise 1.1

1. (a) (1) Singleton (2) Singleton (3) Null set (b) (1) Equivalent set (2) Equal set
2. 8, Subsets : $\emptyset$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}   3. $A \not\subset B$
4. A' = {3, 5, 7, 9, 10}   6. (1) is true; (2), (3) and (4) are false

Exercise 1.2

2. {3, 4, 6}   4. {2}   6. No; $A \cap B = \{4\}$

Exercise 1.3

1. $(A \cup B)' = \{7, 9\}$, $(A \cap B)' = \{1, 2, 3, 5, 6, 7, 8, 9\}$
3. $A' \cap B' = A'$

Exercise 1

1. $A \subset D$; $B \subset D$ 2. $A \cup B = \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$, $A \cap B = \{1, 2\}$
3. No   4. A' = {4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20}
B' = {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}
6. A' = {3, 4}   7. A' = {...–3, –1, 1, 3,...}, B' = {...–4, –2, 0, 2, 4,...}
8. A = {–1, 0, 1, 2} Subsets : $\emptyset$, {–1}, {0}, {1}, {2}, {–1, 0}, {–1, 1}, {–1, 2}, {0, 1}, {0, 2}, {1, 2}, {–1, 0, 1}, {–1, 0, 2}, {–1, 1, 2}, {0, 1, 2}, {–1, 0, 1, 2}
10. (1) c (2) a (3) a (4) b (5) b (6) a (7) d (8) d (9) d (10) c
(11) d (12) c (13) c (14) a (15) b

Exercise 2.1

1. 3, –5, –3.5 are rationals, $\frac{p}{q}$ form, $\frac{3}{1}$, $\frac{-5}{1}$, $\frac{-7}{2}$ 2. $\frac{-1}{4}$, $\frac{-9}{8}$, $\frac{-25}{16}$, $\frac{3}{2}$, $\frac{13}{4}$, $\frac{33}{8}$
3. 5, $\frac{9}{2}$, $\frac{11}{2}$ 4. $\frac{23}{63}$, $\frac{41}{126}$ 5. $\frac{29}{70}$, $\frac{57}{140}$, $\frac{59}{140}$, $\frac{113}{280}$

Exercise 2.2

2. (1) True; (2), (3), (4) False

Exercise 2.3

1. (1), (3) and (6) are rational numbers; (2), (4), (5) are irrational numbers
2. (1) 0.043 (2) 6.6 (3) $0.8\overline{3}$ (4) $1.\overline{285714}$ (5) $0.52\overline{3}$ (6) $1.\overline{27}$
(1), (2) terminating; (3), (4), (5), (6) non-terminating recurring
3. $\frac{32}{99} = 0.\overline{32}$, $\frac{80}{99} = 0.\overline{80}$ 4. $\frac{3}{7} = 0.\overline{428571}$, $\frac{5}{7} = 0.\overline{714285}$

5. (1) $\frac{23}{99}$ (2) $\frac{1437}{9999}$ (3) $\frac{344}{99}$

Exercise 2.4

1. (1), (4) and (6) are irrational numbers; (2), (3) and (5) are rational numbers

2. (1) $15 - 5\sqrt{7} + 3\sqrt{3} - \sqrt{21}$ (2) $9 + 6\sqrt{2}$ (3) $6 - 3\sqrt{30} - \sqrt{10} + 5\sqrt{3}$ (4) -7 (5) $14 - 6\sqrt{5}$ (6) $5\sqrt{5} - 2\sqrt{2}$ **4.** (1) $\sqrt{2}$ (2) $\sqrt{5}$ (3) $(8 - \sqrt{7})$ (4) $(-\sqrt{3} + \sqrt{2})$ (5) $(4 + \sqrt{11})$

5. (1) $\frac{\sqrt{15}}{5}$ (2) $\frac{1}{9}(4 + \sqrt{7})$ (3) $-2 + \sqrt{3}$ (4) $\frac{1}{10}(\sqrt{11} + 1)$ (5) $\frac{1}{7}(\sqrt{14} + \sqrt{7})$

Exercise 2.5

1. (1) 15 (2) 3 (3) 25 (4) 2 **2.** (1) 243 (2) 3125 (3) $\frac{1}{8}$ (4) $\frac{1}{27}$

3. (1) $3^{\frac{23}{10}}$ (2) $4^{\frac{10}{3}}$ (3) 6 (4) $(5)^{\frac{-11}{5}}$

Exercise 2

5. $\frac{-3}{13} = -0.\overline{230769}$, non-terminating recurring; $\frac{15}{4} = 3.75$ terminating

6. (1) $\frac{29}{90}$ (2) $\frac{1459}{990}$ (3) $\frac{271}{999}$ (4) $\frac{35}{99}$ **9.** (1) $3\sqrt{3} - 3\sqrt{7} + \sqrt{15} - \sqrt{35}$ (2) $20 - 10\sqrt{3}$ (3) $7\sqrt{2} - 2\sqrt{14} + 2\sqrt{7} - 4$ **10.** (1) $\frac{-1}{12}(\sqrt{3} + \sqrt{15})$ (2) $\frac{5}{7}(3 - \sqrt{2})$ (3) $3(\sqrt{5} + 2)$ (4) $\frac{1}{58}(-8 + \sqrt{6})$ **11.** a^2b^3 **13.** $(ab)^{\frac{3}{8}}$ **14.** $\frac{3}{5}$

15. (1) 2 (2) $2^{\frac{1}{3}}$ **16.** (1) $\frac{1}{2}(47 + 21\sqrt{5})$ (2) $\frac{1}{19}(9\sqrt{6} - 12 - 3\sqrt{15} + 2\sqrt{10})$

18. (1) a (2) b (3) c (4) d (5) b (6) d (7) a (8) b (9) a (10) d
(11) c (12) b (13) c (14) a (15) b (16) d (17) c (18) a (19) c (20) c
(21) c (22) b (23) c (24) b (25) b (26) b (27) b (28) a (29) b (30) d
(31) b (32) c (33) a (34) d (35) c (36) b

Exercise 3.1

1. (1) 7 (2) 200 (3) 4 (4) 2 **2.** (1) 4 (2) 0 (3) $-\sqrt{3}$

3. (1) quadratic (2) linear (3) cubic (4) quadratic **4.** (1) Yes (2) No (3) No

Exercise 3.2

1. 0 **2.** (1) 35 (2) $-9, -23$ (3) 14 **3.** (1) P(0) = 0, P(1) = 1, P(2) = 128 (2) P(0) = -3, P(1) = 0, P(2) = 5 (3) P(0) = 0, P(1) = -1, P(2) = 0

4. (1) $\frac{-2}{3}$ (2) $\frac{3}{5}$ (3) Impossible

Exercise 3.3

1. (1) Quotient : $x^4+x^3+x^2+x+1$; Remainder : 0
   (2) Quotient : x^3+5x^2+2x+1; Remainder : 2
2. (1) Quotient : $2t^3-5t^2-18t+45$; Remainder : 0
   (2) Quotient : $2t^3-t^2-16t+15$; Remainder : 0
   (3) Quotient : t^3-t^2-9t+9; Remainder : 0
   (4) Quotient : $2t^3-13t^2+26t-15$; Remainder : 0
   (5) Quotient : t^3-5t^2+t+30; Remainder : -135
3. Remainder 41 **4.** 4 **5.** 166 **6.** x^2-x-6 **7.** (1) 0 (2) 23 (3) 6 **8.** 3

Exercise 3.4

1. (1) and (3) $x-1$ is a factor; (2) and (4) $x-1$ is not a factor
2. (1) $7x+4$ (2) $x^2+9x+14$ (3) x^2-3x+2
3. 1 **4.** (1) $(x+1)(3x+4)$ (2) $(3x+2)(5x+2)$ (3) $-(x-1)(21x+5)$
5. (1) $(x-1)$ is a factor (2) $(x+1)$ is a factor (3) $(x-1)$ is a factor
   (4) $(x+1)$ is a factor **6.** 2

Exercise 3.5

1. (1) $x^2-19x+84$ (2) $16x^2-48x+35$ (3) $\frac{1}{6}(12x^2+28x+15)$
   (4) $\frac{1}{4}(36x^2+48x+15)$ **2.** (1) 9991 (2) 3591 (3) 884
3. (1) $(4x-5y)^2$ (2) $\left(\frac{x}{3}+\frac{2y}{5}\right)^2$ (3) $(3a-5b-7c)^2$ or $(-3a+5b+7c)^2$
   (4) $(2a-5b)(2a+5b)(4a^2+25b^2)$
   (5) $\left(\frac{2x}{3}+\frac{3y}{4}+\frac{4z}{5}\right)\left[\frac{4x^2}{9}+\frac{9y^2}{16}+\frac{16z^2}{25}-\frac{xy}{2}-\frac{3yz}{5}-\frac{8zx}{15}\right]$
   (6) $(5a+8b)^3$ (7) $(4a-3b)^3$ **4.** (1) 10710 (2) 8464 (3) 448
5. -16380

Exercise 3

1. $\frac{-20}{9}$ **2.** (1) Quotient : $x^3+7x-13$; Remainder : 21
   (2) Quotient : $2x^2-9x+29$; Remainder : -39
   (3) Quotient : $5x^2-x+10$; Remainder : 0
3. Number of Chocolates : 74; Number of Chocolates to each friend : 8,
   Number of friends : 6
4. Number of students : x^2-x-1 **5.** x^2+3 **6.** x^2-2x-4
7. 11449 **8.** 1260 **9.** $(2x+3y-5z)^2$ or $(-2x-3y+5z)^2$

10. (1) Quotient : $2x^2 + 5x + 19$; Remainder : 55
(2) Quotient : $x^4 - x^3 + x^2 - x + 1$; Remainder : 0
(3) Quotient : $3x^3 + 10x^2 + 4x + 9$; Remainder : 0
(4) Quotient : $7x - 18$; Remainder : 5

11. −12; 432

12. (1) a (2) b (3) d (4) d (5) b (6) b (7) d (8) c (9) d (10) c
(11) b (12) c (13) c (14) b (15) c (16) c

Exercise 4.1

1. (1) X-axis; Y-axis (2) quadrant I, II, III, IV (3) Yes, Origin (0, 0)

2. (1) P(4, 1); Q(−2, 4) (2) Q (3) −3 (4) −2
(5) A(−3, −4), B(2, −1), C(2, 3), D(−4, 3), E(−3, 1), F(−2, −1), G(2, −3), H(3, −4), I(3, 2), J(5, 4), R(−4, −2), S,(−1, −3), T(4, −2)

Exercise 4.2

3. P × Q = {(0, −3), (0, 2), (1, −3), (1, 2), (−1, −3), (−1, 2)}
Q × P = {(−3, 0), (2, 0), (−3, 1), (2, 1), (−3, −1), (2, −1)}

4. (1) A × B = {(−2, −1), (−2, 1), (−2, 4), (3, −1), (3, 1), (3, 4)}
(2) B × A = {(−1, −2), (−1, 3), (1, −2), (1, 3), (4, −2), (4, 3)}
(3) A × A = {(−2, −2), (−2, 3), (3, −2), (3, 3)}
(4) B × B = {(−1, −1), (−1, 1), (−1, 4), (1, −1), (1, 1), (1, 4), (4, −1), (4, 1), (4, 4)}

5. (1, 2); (1, 2) **6.** (1, 2)

Exercise 4

1. (1) First (2) Fourth (3) Third (4) Second (5) Fourth (6) First

5. $(x + y, z + w) \rightarrow$ IV quadrant, $(y - z, w + x) \rightarrow$ IV quadrant,
$(x - w, y + z) \rightarrow$ I quadrant

6. (1) c (2) b (3) c (4) b (5) d (6) d (7) d (8) c (9) b (10) b
(11) a (12) d (13) a (14) b (15) d (16) d (17) c (18) b (19) c (20) c
(21) d (22) b (23) d (24) a (25) c

Exercise 5.1

1. $x = 2y$; ₹ x is the cost of a notebook and ₹ y is the cost of a pen.

2. (1) Yes; $5x - 6y + 0 = 0$; $a = 5$, $b = -6$, $c = 0$ (2) No
(3) Yes; $7x + 0y + 0 = 0$; $a = 7$, $b = 0$, $c = 0$
(4) Yes; $-4x + 6y - 3 = 0$; $a = -4$, $b = 6$, $c = -3$
(5) Yes; $3x + 4.5y - 8.2 = 0$; $a = 3$, $b = 4.5$, $c = -8.2$
(6) Yes; $\frac{-x}{4} + y - 3 = 0$; $a = \frac{-1}{4}$, $b = 1$, $c = -3$
(7) Yes; $9x + 0y - 3 = 0$; $a = 9$, $b = 0$, $c = -3$

(8) Yes; $3x - 2y + 4 = 0$; $a = 3$, $b = -2$, $c = 4$

(9) Yes; $2x - y + 5 = 0$; $a = 2$, $b = -1$, $c = 5$

(10) Yes; $\frac{3}{2}x + \frac{7}{2}y - 1 = 0$; $a = \frac{3}{2}$, $b = \frac{7}{2}$, $c = -1$ (11) No (12) No

Exercise 5.2

4. (3); Linear equation in two variables has infinite solutions

5. (1) is a solution, (2) is a solution, (3) is not a solution, (4) is not a solution, (5) is a solution, (6) is not a solution, (7) is not a solution, (8) is not a solution (9) is a solution, (10) is not a solution

6. (1) $k = -1$ (2) $k = 2$ (3) $k = 4$ (4) $k = 1$

Exercise 5.3

2. $a = -2$ **4.** (2) 86° F (3) 35° C (4) 32° F; $-\frac{160°}{9}$ C (5) Yes; −40° C

Exercise 5.4

1. (1) Point −4; Horizontal line $y = -4$ (2) Point $-\frac{9}{2}$; Vertical line $x = -\frac{9}{2}$

3. (1) Point −3; Vertical line $x = -3$ (2) Point 4; Horizontal line $y = 4$ **4.** (1, 2)

Exercise 5

1. (1) Yes (2) No (3) Yes (4) No (5) Yes (6) Yes

3. (1) $k = \frac{5}{6}$ (2) $k = \frac{3}{2}$ (3) $k = 2$ (4) $k = \frac{1}{7}$ (5) $k = 1$

4. (1) (0, 0) (2) (0, 0) (3) (0, 2), (2, 0) (4) (3, 0), (0, −3) (5) (−4, 0), (0, −3) (6) (2, 0), (0, −3) (7) (2, 0), (0, 3) (8) (−4, 0), (0, 3) (9) $(-\frac{5}{2}, 0)$ (10) (0, 2)

6. (4, 4), (−4, 4), (−4, −4), (4, −4), (0, 0) **7.** (3, 1)

9. (1) c (2) d (3) c (4) c (5) d (6) c (7) b (8) c (9) c (10) c (11) c (12) b (13) b (14) a (15) a

Exercise 6.1

3. False **4.** (1) a (2) c (3) d (4) b (5) b (6) d (7) c (8) c

Exercise 6

4. (1) Data : X ⊂ Y and Y ⊂ X, To prove : X = Y

(2) Data : Triangle, To prove : Sum of measures of three angles of a triangle is 180

(3) Data : B is not null set, To prove : B has at least two subsets.

(4) Data : Today is Sunday. To prove : Today is holiday in school.

5. (2) $x = 2$ is necessary condition for $x + 5 = 7$,

$x + 5 = 7$ is sufficient condition for $x = 2$.

(3) Parts of theorem : Hypothesis, Conclusion and Proof

(4) Direct proof and indirect proof

(5) Exhausting alternatives and Reductio ad Absurdum

(6) Defined terms, Undefined terms, Postulates and Theorems

Exercise 7.1

1. (3) One (4) $P \notin \overleftrightarrow{QR}$ (5) Six lines, one line

Exercise 7.2

3. (1) $\emptyset$ (2) $\emptyset$ (3) {Y} (4) {Z} (5) {Y} (6) {R} (7) {X} (8) {Q} (9) {Y} (10) {P}

Exercise 7.3

1. (2) Points X, Y and Z are linear; Point Y is in between X and Z
2. XY = 9, YZ = 2, ZX = 7 **3.** Q–R–P
4. 2 or −8 corresponds to B **5.** AB = 6 **6.** (1) 3 (2) 11 (3) 9 (4) 2 (5) $8\frac{1}{2}$ (6) 13
7. (1) 5 or −5 (2) 9 or −1 (3) −3 or 17 (4) −2 or 20 (5) 3 or −6

Exercise 7.4

1. S–T–U; ST = TU **2.** (1) $\overline{XY} \cong \overline{PQ}$ (2) $\overline{SB}$ (3) $\emptyset$ (4) $\emptyset$ (5) {R}
3. 0.5 **4.** (1) 1 (2) −1 (3) −4.5 (4) $\sqrt{2}$

Exercise 7

1. (1) One (2) $\overrightarrow{XY} = \overline{XY} \cup \{P \mid X\text{–}Y\text{–}P\}$, $\overrightarrow{AB} = \overline{AB} \cup \{P \mid A\text{–}B\text{–}P\}$
   (3) $\overrightarrow{YX}$ and $\overrightarrow{YZ}$
3. (1) {E} (2) $\emptyset$ (3) $\overrightarrow{EF}$ (4) $\overrightarrow{DC}$ (5) C, D, E, F
4. (1) Data : P, Q, R, S are linear points and PR = PS; To prove : PQ = RS
   (2) Data : P, Q, R are linear points and PQ = QR; To prove : Q is a midpoint of $\overline{PR}$.
5. (1) 1 (2) AB = 9, BC = 5, AC = 4 (3) 2 and 12 (4) 1
   (5) Y : 3, −13; M : −1, −9 **6.** Z; Y–Z–X
8. (1) b (2) a (3) c (4) c (5) b (6) a (7) c (8) c (9) b

Exercise 8.1

2. (1) P, Q, R, S, T, U (2) $l \subset \beta$ (3) P, Q, R; U, T, S
5. (1) α, β, γ (2) P, Q, R, S, T (3) Yes (4) No (5) skew (6) α

Exercise 8.2

1. (1) $\overrightarrow{BA}$, $\overrightarrow{BC}$ (2) Interior points of ∠ABC (3) B (5) $\overrightarrow{BJ}$, $\overrightarrow{BD}$, $\overrightarrow{BG}$
3. ∠PQR, ∠PRQ **5.** Exterior points of ∠ABC

Exercise 8.3

1. (1) 50 (2) 15 **2.** Adjacent angles : ∠AOD, ∠BOD : ∠AOD, ∠DOC; ∠DOC, ∠COB
   Linear pair of angles : ∠AOD, ∠DOB; ∠AOC, ∠COB

3. 126 **4.** 105, 75 **6.** (1) 90 (2) 73, 107 (3) 30
(4) (i) 48 (ii) 53 (iii) 80 − x (iv) 9 (5) (i) 80 (ii) 91 (iii) 210 − y (iv) 131

Exercise 8.4

2. $m\angle CGF = 115$, $m\angle DGF = 65$ **4.** $m\angle EFB = 100$, $m\angle FGD = 100$

Exercise 8.5

1. Alternate angles : ∠NQR, ∠ARQ; ∠MQR, ∠QRB
Corresponding angles : ∠PQN, ∠QRB; ∠NQR, ∠BRS
∠PQM, ∠QRA; ∠MQR, ∠ARS

2. 60 **3.** $m\angle RTA = 125$, $m\angle RSB = 125$, $m\angle PRS = 35$

4. $m\angle YXZ = 77$, $m\angle XYZ = 50$ **6.** $m\angle XZL = 60$, $m\angle MNZ = 48$, $m\angle ZNR = 60$

Exercise 8

1. $m\angle ROQ = 150$, $m\angle QOS = 30$, $m\angle ROP = 30$, $m\angle TOS = 75$

4. $x = 37$, $y = 53$ **5.** 90 **6.** 72, 108

7. (1) c (2) d (3) d (4) c (5) b (6) b (7) c (8) a (9) c (10) a (11) b (12) c

Exercise 9.1

1. 80, $m\angle C = 60$ **2.** $m\angle BCE = 50$, $m\angle ADC = 100$ **3.** 36, 54, 90

4. (i) 35 (ii) 25 **5.** $m\angle A = 120$, $m\angle B = 50$, $m\angle C = 10$

6. $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$ **8.** $m\angle ACB = 45$

Exercise 9.2

1. $\angle D \cong \angle P$, $\angle E \cong \angle Q$, $\angle F \cong \angle R$, $\overline{DE} \cong \overline{PQ}$, $\overline{EF} \cong \overline{QR}$ $\overline{DF} \cong \overline{PR}$

2. ABC ↔ XYZ **3.** ABC ↔ FDE **4.** DEF ↔ XYZ

Exercise 9.3

4. (i) ABC ↔ DEF (ii) DEF ↔ RQP (iii) DEF ↔ RPQ **6.** Right angle

8. $m\angle ABC = 65$, $m\angle ACB = 65$ **9.** $m\angle A = 20$, $m\angle C = 80$, $m\angle ABC = 80$,

10. 40, 40, 100

Exercise 9.4

2. $m\angle A = 40$, $m\angle B = m\angle C = 70$, $m\angle CAP = 140$

Exercise 9.5

3. $m\angle B = 80$, $m\angle C = 50$, $m\angle ACD = 130$

Exercise 9

1. $m\angle C = 80$, $m\angle B = 50$, $m\angle A = 50$ **2.** $m\angle ECD = 40$ **5.** $m\angle B = 24$

6. $m\angle A = 35$

11. (1) b (2) c (3) b (4) c (5) c (6) c (7) a (8) c (9) c (10) d
(11) b (12) b (13) d (14) a (15) a (16) c (17) c