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MATHEMATICS

Standard 11

(Semester II)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place before the sudents this textbook of **Mathematics** for **Standard 11 (Semester II)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions from people interested in education, to improve the quality of the textbook.

Dr. Bharat Pandit
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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

*Constitution of India : Section 51 A

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About This Textbook...

We have created a background in the book of Mathematics for standard 11, semester I about formation of new syllabus and writing textbooks following curriculum of NCERT.

First of all, this book was written in English. It was reviewed by teachers and professors teaching in English medium schools and colleges. According to the suggestions made by experts, necessary amendments were made and the manuscript was translated in Gujarati. It was again reviewed by experts teaching in Gujarati medium; considering their suggestions, the necessary changes were made.

Thus, the manuscript prepared was compeletely read by the authors in workshops and the authors gave final touches to the manuscript.

In chapter 1, mathematical induction which is a tool to prove many properties about statements related to natural numbers is studied. Also, we have shown the use of mathematical induction in various fields using various formats. Chapter 2 gives an introduction to complex number system. Fundamental theorem of algebra, square roots and cube roots of complex numbers, Argand diagrams, inequalities etc. have been presented in a very lucid manner in this chapter. Any algebraic n degree equation with real coefficients can be solved using complex numbers and thus complex numbers are very useful. Chapter 3 introduces binomial theorem which is an extension of expansions of the squares and the cubes studied at secondary school level. Binomial theorem for positive index is useful while using polynomials. Chapters 4, 5 and 6 advance the study of trigonometry studied in semester I. These chapters are useful to study properties of triangles and for studying general solution of trigonometric equations.

In chapter 7, there are arithmetic progression, geometric progression and power series (index 1, 2 and 3). In chapter 8, elementary study of conics and primary information have been given. We mention intersection of cones and general second degree curves. In chapter 9, there is a study of three dimensional geometry. To study this, vector is an important tool. So in the beginning of the chapter, we have given introduction of vectors. The study of three dimensional geometry is limited to section of a line segment.

Chapter 10 and 11 suggest the beginning of the calculus. Only intuitive concept of limit has been taken and then limit has been defined. We have stressed how to obtain limit using lemmas and theorems. The concept of limit has been explained with the help of graphs but students are not supposed to draw the graphs. Having defined differentiation, we have explained how to obtain derivatives of elementary functions. There are ample number of examples so that a student can understand all the concepts by himself / herself and a teacher can lead a student to self study. At the end of every chapter enough number of multiple choice questions have been given so that understanding of concept can be evaluated. We intend to render a student enough study material from the textbook itself. Attractive four colour printing is an additional attraction of the book. We have given some information about contribution of Indian Mathematicians at the end of some chapters.

Enough care has been taken to make the textbook maximally interesting and errorfree. However all constructive suggestions regarding further improvement in the textbook are most welcome.

We hope teachers and students both will find this book useful and valuable.

– Authors

Please consider following points while teaching textbook.

**Following is necessary for study by students and teachers.
But it will not be asked in the board examination.**

Chapter	Exercise	Examples
Chapter 1	Exercise 1 : Ex. No. 21	21, 24
Chapter 2	Exercise 2 : Ex. No. 16	–
Chapter 5	Exercise 5 : Ex. No. 19 to 22	–
Chapter 8	–	13, 14, 19, 32
Chapter 10	Article 10.3 Exercise 10 : From statements of examples 1, 2, 3 remove the word 'definition'.	14, 15, 16
Chapter 11	Exercise 11 : Ex. No. 6, 20(4)	17, 26 In Example 19 Let P(n) : $\frac{d}{dx}\, sin^n x = n sin^{n-1} x \; cos x$

**Following is useful for higher studies and competitive examinations,
but not for board examination.**

Chapter	Exercise	Examples
Chapter 1	Exercise 1 : Ex. No. 9, 24, 29	23
Chapter 2	Exercise 2.3 : Ex. No. 3	–
Chapter 8	Exercise 8.3 : Ex. No. 3, 4 Exercise 8.4 : Ex. No. 8, 9 Exercise 8 : Ex. No. 6	–
Chapter 10	Exercise 10 : Ex. No. 9	
Chapter 11	Exercise 11 : Ex. No.20(23)	

Chapter 1

PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematics is the queen of science and number theory is the queen of mathematics.

– Gauss

Mathematics passes not only truth but also supreme beauty !

– Bertrand Russell

1.1 Introduction

We have studied one method of reasoning, deductive reasoning.

For example, consider the following statements :

(1) $1 + 2 + 3 + ... + 100 = 5050$

(2) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(3) Let $n = 100$ in (2). $1 + 2 + 3 + ... + 100 = \frac{(100)(101)}{2} = (50)(101) = 5050$

Here we want to prove that sum of all integers from 1 to 100 is 5050. We have a general result $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. We take $n = 100$ in it and get the required result. Here, we apply a general principle to deduce a particular result.

Consider (1) If 3 divides product ab, then 3 divides a or 3 divides b. (2) If p is a prime and p divides ab then p divides a or p divides b. (3) Let $p = 3$ in (2) as 3 is a prime. Hence, if 3 divides product ab, then 3 divides a or 3 divides b.

Here also we apply a general principle to deduce a particular result.

(1) Amitabh Bachchan is a good actor.

(2) Actors are awarded national *Padma* honour in their category, if selected.

(3) Amitabh Bachchan was selected and got *Padma* honour.

Here also a similar situation occurs.

But consider the following against this deductive reasoning,

$4 - 1 = 3$ is divisible by 3.

$4^2 - 1 = 15$ is divisible by 3.

$4^3 - 1 = 63$ is divisible by 3.

Here we observe a pattern and we make a conjecture that for every positive integer n, $4^n - 1$ is divisible by 3. So from a particular case, we conjecture a general result. This is not a proof. This inductive assumption has to be proved. All conjectures may not be true. For example, $n^2 - n + 41$ is a prime for $n = 1, 2, 3, ...39$. But for $n = 41$, $41^2 - 41 + 41 = 41^2$ is obviously not a prime. Hence we cannot deduce that $n^2 - n + 41$ is a prime by observing values for $n = 1, 2, 3, ...39$.

So, inductive argument starts from a particular case and by rigorous deduction the conjecture is proved.

The history of this dates back to **Plato**. In 370 B.C. Plato's *parmenides* (Discussions or Dialogues) contained an early example of implicit inductive proof. The early traces of mathematical induction can be found in Euclid's proof that number of primes is infinite. Bhaskara II's *cyclic* method (*Chakravala*) also introduces mathematical induction.

Sorites paradox used the method of descent. He said 10,00,000 grains of sand form a heap. Removing one grain from the heap does not change the situation. So continuing the argument even one grain or no grain also forms a heap !

Around 1000 A.D., **Al-Karaji** introduced mathematical induction for arithmetic sequences in **Al-Fakhri** and proved the binomial theorem and properties of **Pascal's** triangle.

The first explicit formulation of the principle of mathematical induction was given by **Pascal** in *Traité-du-triangle arithmetique* (1665). French mathematician **Fermat** and Swiss mathematician **Jacob Bernoulli** used the principle. The modern rigorous and systematic treatment came only in 19th century with **George Boole, Sanders Peirce, Peano** and **Dedekind.**

1.2 Induction Principle

We start with following principle :

Principle of Induction : If a statement $P(n)$ of natural variable n is true for $n = 1$ and if $P(k)$ is true $\Rightarrow P(k + 1)$ is true, $k \in N$, then $P(n)$ is true, $\forall n \in N$.

Let us be given a statement $P(n)$ involving a natural variable to be true for all natural numbers n. We prove it in two stages :

(1) The basis : We prove it for $n = 1$ (or 0 or the lowest value).

(2) Inductive step : Assuming that the statement holds for some natural number k, prove it for $n = k + 1$.

Then $P(n)$ is true for all $n \in N$.

Domino effect : We are presented with a 'long' row of dominos such that,

(1) The first domino will fall.

(2) Whenever a domino falls, its next neighbour will fall.

So it is concluded that all of the dominos will fall.

So the proof is like this. The first statement in an infinite sequence of statements is true and if it is true for some $k \in$ N, it is true for the next value of the variable, then the given sequence of statements is true for all $n \in$ N.

In logical symbols, $(\forall P)\ [P(1) \wedge (\forall k \in N)\ (P(k) \Rightarrow P(k+1))] \Rightarrow (\forall n \in N)[P(n)]$

This can be proved by using **well-ordering principle** which states that every non-empty subset of N has a least element.

Proof : Let S be the set of natural numbers for which P(n) is false. $1 \notin$ S as P(1) is true. If S is non-empty, it has a least element t which is not 1. Let $t = n + 1$. Since t is the least element for which P(t) is false, P(n) is true. Also $P(n) \Rightarrow P(n+1)$. Hence $P(n+1) = P(t)$ is true, a contradiction. Hence S = $\emptyset$.

$\therefore$ P(n) is true, $\forall n \in$ N.

Sometimes paradoxes are created by misuse of the principle.

There is a famous **Polya's** proof that there is no horse of different colour.

Basis : If there is only one horse, there is only one colour and hence P(1) is true.

Induction step : Assume that in any set of n horses, all have the same colour. Consider a set of $n + 1$ horses numbered 1, 2, 3,... $n + 1$. Consider the subsets $\{1, 2, 3,..., n\}$ and $\{2, 3, 4,...n + 1\}$. Each is a set of n horses and therefore they have the same colour and since they are overlapping sets, all $n + 1$ horses have same colour. This argument is true for 1 horse and $n \geq 3$ horses. But for 2 horses the set {1} and {2} are disjoint and the argument falls flat.

1.3 Examples

Now we will apply the principle of mathematical induction to some examples.

Example 1 : Prove $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

For $n = 1$, L.H.S. = 1 and R.H.S. = $\frac{1 \times 2}{2} = 1$. Hence, P(1) is true.

Let P(k) be true i.e. P(n) is true for $n = k$, $k \in$ N.

$$\therefore \quad 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} \qquad \text{(i)}$$

For $n = k + 1$ we have to prove,

$$1 + 2 + 3 + ... + (k + 1) = \frac{(k+1)(k+2)}{2}$$

Now, $1 + 2 + 3 + ... + (k + 1) = (1 + 2 + 3 + ... + k) + (k + 1)$

$$= \frac{k(k+1)}{2} + (k + 1) \qquad \text{by (i)}$$

$$= (k + 1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$$

Hence, P(k + 1) is true.

∴ P(1) is true and P(k) is true, ⇒ P(k + 1) is true.

∴ P(n) is true, $\forall n \in N$ by principle of mathematical induction.

Note : This example has historical importance.

Obviously, $1 + 2 + 3 + ... + 100 = 5050$ according to this formula. When this formula was not known, Gauss, at very young age, calculated this by the following method and surprised his teacher Buttner and assistant teacher Bartels.

Let $S = 1 + 2 + 3 + ... + 100$ **(i)**

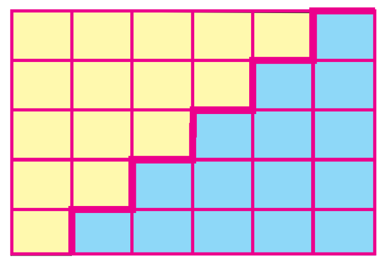
∴ $S = 100 + 99 + 98 + ... + 1$ **(ii)**

Adding (i) and (ii)

∴ $2S = (101) + (101) + ...$ 100 times **((i) + (ii))**

∴ $S = \frac{101 \times 100}{2} = 5050$. This was done in no time !

Let us review a geometric 'proof'.



Consider a rectangle of sides n and $n + 1$ divided into subrectangles of unit sides as shown. The portion under the dark ladder has area $1 + 2 + 3 + ... + n$.

By symmetry the rectangle has area

$$2(1 + 2 + 3 + ... + n) = n(n + 1)$$

∴ $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Example 2 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in N$

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in N$

Let $n = 1$. L.H.S. $= 1^2 = 1$ and R.H.S. $= \frac{1 \times 2 \times 3}{6} = 1$.

∴ P(1) is true.

Let P(k) be true, $k \in N$.

∴ $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Let $n = k + 1$.

$$\therefore \text{L.H.S.} = 1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k+1)\left(\frac{2k^2 + k + 6k + 6)}{6}\right)$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by principle of mathematical induction.

Example 3 : Prove $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}, \quad n \in N$

Solution : Let P(n) : $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}, \quad n \in N$

For $n = 1$, L.H.S. = $1^3 = 1$ and R.H.S. = $\frac{1^2 \times 2^2}{4} = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad 1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Let $n = k + 1$.

$$\text{L.H.S.} = 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2(k+1+1)^2}{4} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

(Now onwards we shall abbreviate Principle of Mathematical Induction as P.M.I.)

Example 4 : Prove $1 + 3 + 5 + ... + (2n - 1) = n^2, \quad n \in N$

Solution : Let P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2, \quad n \in N$

Let $n = 1$. L.H.S. = 1 and R.H.S. = $1^2 = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad 1 + 3 + 5 + ... + (2k - 1) = k^2$

Let $n = k + 1$.

L.H.S. = $1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$

$= k^2 + 2k + 1$

$= (k + 1)^2$ = R.H.S.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 5 : Prove $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$, $n \in N$

Solution : Let P(n) : $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$, $n \in N$

Let $n = 1$. L.H.S. $= \frac{1}{1\cdot 2} = \frac{1}{2}$ and R.H.S. $= \frac{1}{2}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Let $n = k + 1$.

$$\text{L.H.S.} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Example 6 : Prove $1\cdot 1! + 2\cdot 2! + ... + n\cdot n! = (n + 1)! - 1$, $n \in N$

Solution : Let P(n) : $1\cdot 1! + 2\cdot 2! + ... + n\cdot n! = (n + 1)! - 1$, $n \in N$

Let $n = 1$. L.H.S. $= 1\cdot 1! = 1$, R.H.S. $= (1 + 1)! - 1 = 2! - 1 = 1$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1\cdot 1! + 2\cdot 2! + ... + k\cdot k! = (k + 1)! - 1$

Let $n = k + 1$.

$$\text{L.H.S.} = 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + ... + k\cdot k! + (k + 1)(k + 1)!$$

$$= (k + 1)! - 1 + (k + 1)(k + 1)!$$

$$= (k + 1)!\ [1 + (k + 1)] - 1$$

$$= (k + 1)!\ (k + 2) - 1$$

$$= (k + 2)! - 1 = \text{R.H.S.}$$

$\therefore$ P$(k+1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $n \cdot n! = (n+1-1)\, n! = (n+1)\, n! - n!$
$= (n+1)! - n!$

Let $n = 1, 2, 3, \ldots$ etc. and add

$1 \cdot 1! + 2 \cdot 2! + 33! + \ldots + n \cdot n! = (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)$
$= (n+1)! - 1$

Example 7 : Prove $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \ldots \left(1+\frac{2n+1}{n^2}\right) = (n+1)^2, \ n \in$ N

Solution : Let P(n) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \ldots \left(1+\frac{2n+1}{n^2}\right) = (n+1)^2, \ n \in$ N

Let $n = 1$. L.H.S. $= 1 + \frac{3}{1} = 4$ and R.H.S. $= (1+1)^2 = 2^2 = 4$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right) \ldots \left(1+\frac{2k+1}{k^2}\right) = (k+1)^2$

Let $n = k + 1$.

$$\text{L.H.S.} = \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \ldots \left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$= (k+1)^2 \times \left(\frac{k^2+2k+1+2k+3}{(k+1)^2}\right)$$

$$= k^2 + 4k + 4$$

$$= (k+2)^2 = \text{R.H.S.}$$

$\therefore$ P$(k+1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \ldots \left(1+\frac{2n+1}{n^2}\right)$

$$= \frac{4}{1} \cdot \frac{9}{4} \cdot \frac{16}{9} \ldots \frac{(n+1)^2}{n^2} = (n+1)^2$$

Example 8 : Prove $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2, \ n \in$ N

(This type of series is called **arithmetico geometric** series.)

Solution : Let P(n) : $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2, \ n \in$ N

Let $n = 1$. L.H.S. $= 2$ and R.H.S. $= 0 + 2 = 2$

$\therefore$ P(1) is true.

Let P(k) be true.

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Hence, $1 \cdot 2 + 2 \cdot 2^2 + ... + k \cdot 2^k = (k - 1)2^{k+1} + 2$

Let $n = k + 1$.

$$\begin{aligned} \text{L.H.S.} &= 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + k \cdot 2^k + (k + 1)2^{k+1} \\ &= (k - 1)2^{k+1} + 2 + (k + 1)2^{k+1} \\ &= (k - 1 + k + 1)2^{k+1} + 2 \\ &= 2k \cdot 2^{k+1} + 2 \\ &= k \cdot 2^{k+2} + 2 = \text{R.H.S.} \end{aligned}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 9 : Prove $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$ **($r \neq 1$), $n \in$ N**

Solution : Let P(n) : $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$ **($r \neq 1$), $n \in$ N**

Let $n = 1$. L.H.S. = a and R.H.S. = $\dfrac{a(r - 1)}{r - 1} = a$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}$

Let $n = k + 1$.

$$\begin{aligned} \text{L.H.S.} &= a + ar + ar^2 + ... + ar^{k-1} + ar^k \\ &= \frac{a(r^k - 1)}{r - 1} + ar^k \\ &= a\left(\frac{r^k - 1}{r - 1} + r^k\right) \\ &= a\,\frac{r^k - 1 + r^k(r - 1)}{r - 1} \\ &= a\,\frac{(r^k - 1 + r^{k+1} - r^k)}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} = \text{R.H.S.} \end{aligned}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 10 : Prove $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Solution : Let P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Let $n = 1$. $3^4 - 8 - 9 = 81 - 8 - 9 = 64$ is divisible by 8.

Let P(k) be true. Hence $3^{2k+2} - 8k - 9$ is divisible by 8.

Let $n = k + 1$.

Now, $3^{2k+4} - 8(k+1) - 9$ **($2(k+1) + 2 = 2k + 4$)**

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 3^{2k+2}(8 + 1) - 8k - 8 - 9$ **($3^2 = 9 = 8 + 1$)**

$= 8 \cdot 3^{2k+2} + 3^{2k+2} - 8k - 8 - 9$

$= 3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

Now, 8 divides $3^{2k+2} - 8k - 9$ **by $p(k)$**

Also, 8 divides $8(3^{2k+2} - 1)$

$\therefore$ 8 divides $3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Obviously,

$3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 1 - 8n - 8$

$= (3^2 - 1)((3^2)^n + (3^2)^{n-1} + ... + 1) - 8n - 8$ **(Example 9)**

$= 8(3^{2n} + 3^{2n-2} + ... + 1) - 8n - 8$ is divisible by 8.

Another Method :

P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

For $n = 1$, $3^{2+2} - 8(1) - 9 = 64$ is divisible by 8.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore$ $3^{2k+2} - 8k - 9 = 8m$ where $m \in$ N **(i)**

Now, Let $n = k + 1$,

$3^{2(k+1)+2} - 8(k+1) - 9 = 3^{2k+2} \times 3^2 - 8k - 8 - 9$

$= (8k + 9 + 8m)9 - 8k - 8 - 9$ **(From (i))**

$= 72k + 81 + 72m - 8k - 8 - 9$

$= 64k + 72m + 64$

$= 8(8k + 9m + 8)$ is divisible by 8.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 11 : Prove $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Solution : Let P(n) : $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Let $n = 1$.

$2002^3 + 2003^3 = (2002 + 2003)\,[(2002)^2 - (2002)(2003) + (2003)^2]$

$= (4005)\,[(2002)^2 - (2002)(2003) + (2003)^2]$

$\therefore$ $(2002)^3 + (2003)^3$ is divisible by 4005.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $2002^{2k+1} + 2003^{2k+1}$ is divisible by 4005.

Let $n = k + 1$.

Now, $2002^{2(k+1)+1} + 2003^{2(k+1)+1}$

$= 2002^{2k+3} - 2002^{2k+1}(2003)^2 + (2002)^{2k+1} \cdot (2003)^2 + (2003)^{2k+3}$

$= (2002)^{2k+1}\,[(2002)^2 - (2003)^2] + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$

$= -(4005)\,(2002)^{2k+1} + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$

Now, $(2002)^{2k+1}\ (2003)^{2k+1}$ is divisible by 4005. **(P(k))**

$\therefore$ $(2002)^{2(k+1)+1} + (2003)^{2(k+1)+1}$ is divisible by 4005.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Example 12 : Prove $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Solution : Let P(n) : $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Let $n = 1$.

Then $x^2 - y^2 = (x - y)(x + y)$ and so $x^2 - y^2$ is divisible by $x + y$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $x^{2k} - y^{2k}$ is divisible by $x + y$.

Let $n = k + 1$.

$x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - x^{2k}y^2 + x^{2k}y^2 - y^{2k+2}$

$= x^{2k}(x^2 - y^2) + y^2(x^{2k} - y^{2k})$

$= x^{2k}(x - y)(x + y) + y^2(x^{2k} - y^{2k})$

Now, $x^{2k} - y^{2k}$ is divisible by $(x + y)$. **(P(k))**

$\therefore$ $x^{2(k+1)} - y^{2(k+1)}$ is divisible by $(x + y)$.

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$\therefore$ P$(k+1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 13 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Let $n = 1$. L.H.S. $= 1^2 = 1$, R.H.S. $= \frac{1}{3}$ and $1 > \frac{1}{3}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1^2 + 2^2 + 3^2 + ... + k^2 > \frac{k^3}{3}$

Let $n = k + 1$.

Now, $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 > \frac{k^3}{3} + (k+1)^2$ **(i)**

Now, $\frac{k^3}{3} + (k+1)^2 = \frac{1}{3}(k^3 + 3k^2 + 6k + 3)$

$= \frac{1}{3}(k^3 + 3k^2 + 3k + 1 + 3k + 2)$

$> \frac{1}{3}(k^3 + 3k^2 + 3k + 1)$ as $\frac{1}{3}(3k+2) \geq \frac{5}{3} > 0$

$\therefore$ $\frac{k^3}{3} + (k+1)^2 > \frac{1}{3}(k+1)^3$ **(ii)**

$\therefore$ $1^2 + 2^2 + 3^2 + ... + (k+1)^2 > \frac{1}{3}(k+1)^3$ **(by (i) and (ii))**

$\therefore$ P$(k+1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6} > \frac{2n^3}{6} = \frac{n^3}{3}$, $n \in$ N

Example 14 : Prove $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$, $n \in$ N

Let $n = 1$. L.H.S. $= 1$, R.H.S. $= \frac{1}{8}(3)^2 = \frac{9}{8}$ and $1 < \frac{9}{8}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1 + 2 + 3 + ... + k < \frac{1}{8}(2k+1)^2$

Add $k + 1$ on both the sides.

$\therefore$ $1 + 2 + 3 + ... + k + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$ **(i)**

Now, $\frac{1}{8}(2k + 1)^2 + (k + 1) = \frac{1}{8}(4k^2 + 4k + 1 + 8k + 8)$

$= \frac{1}{8}(4k^2 + 12k + 9)$

$\therefore \quad \frac{1}{8}(2k + 1)^2 + (k + 1) = \frac{1}{8}(2k + 3)^2$ **(ii)**

$\therefore \quad 1 + 2 + 3 + ... + (k + 1) < \frac{1}{8}(2k + 3)^2$ **(by (i) and (ii))**

$\therefore$ P$(k + 1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k + 1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} = \frac{4n^2 + 4n}{8} < \frac{4n^2 + 4n + 1}{8} = \frac{1}{8}(2n + 1)^2$

Example 15 : Prove $(1 + x)^n \geq 1 + nx, \; n \in$ N **$(x > -1)$**

Solution : Let P(n) : $(1 + x)^n \geq 1 + nx, \; n \in$ N

Let $n = 1$. $(1 + x)^1 = 1 + x \geq 1 + 1 \cdot x$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad (1 + x)^k \geq 1 + kx$

Let $n = k + 1$.

Now, $(1 + x)^{k+1} = (1 + x)^k (1 + x)$

$\geq (1 + kx)(1 + x)$ **(by P(k) and as $x > -1$)**

$\therefore \quad (1 + x)^{k+1} \geq 1 + kx + x + kx^2 \geq 1 + kx + x$ as $k \in$ N, $x^2 \geq 0$

$\therefore \quad (1 + x)^{k+1} \geq 1 + (k + 1)x$

$\therefore$ P$(k + 1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k + 1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 16 : Prove $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \; n \in$ N

Solution : Let P(n) : $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \; n \in$ N

Let $n = 1$, L.H.S. = 1, R.H.S. = $2 - 1 = 1$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Add $\frac{1}{(k+1)^2}$ on both the sides.

Hence, $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ **(i)**

Now, $2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{1}{k} + \frac{1}{(k+1)^2} + \frac{1}{k+1} - \frac{1}{k+1}$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{-k-1+k}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} - \frac{1}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{k-k-1}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}$$

$\therefore \quad 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ **$\left(k \in \mathrm{N} \text{ gives } \frac{1}{k(k+1)^2} > 0\right)$ (ii)**

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{(k+1)^2} < 2 - \frac{1}{(k+1)}$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true $\forall n \in$ N by P.M.I.

Note : Thus however large n, sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$ is 'bounded' and less than < 2.

Example 17 : Prove $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n, \quad n \in \mathrm{N}$

Solution : Let P(n) : $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n, \quad n \in \mathrm{N}$

Let $n = 1$. L.H.S. $= \binom{1}{0} + \binom{1}{1} = 2$, R.H.S. $= 2^1 = 2$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad \binom{k}{0} + \binom{k}{1} + ... + \binom{k}{k} = 2^k$

Let $n = k + 1$.

L.H.S. $= \binom{k+1}{0} + \binom{k+1}{1} + ... + \binom{k+1}{k} + \binom{k+1}{k+1}$

$$= \binom{k}{0} + \left(\binom{k}{0} + \binom{k}{1}\right) + \left(\binom{k}{1} + \binom{k}{2}\right) + ... + \left(\binom{k}{k-1} + \binom{k}{k}\right) + \binom{k}{k}$$

$$\left(\text{as } \binom{k}{0} = \binom{k+1}{0} = 1, \binom{k}{k} = \binom{k+1}{k+1} = 1, \binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}\right)$$

$$= 2\left[\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right]$$

$$= 2 \cdot 2^k$$

$$= 2^{k+1} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

1.4 Some Variants of P.M.I.

Variant 1 : If P(n) is a statement involving natural variable n and if P(k_0) is true for some positive integer k_0 and if the truth of P(k) for some integer $k \geq k_0$ implies the truth of P(k + 1), then P(n) is true $\forall n \in$ N, such that $n \geq k_0$.

Example 18 : Prove $2^n > n^2$; $n \geq 5$, $n \in$ N

Solution : Let P(n) : $2^n > n^2$; $n \geq 5$, $n \in$ N

Let $n = 5$. ($k_0 = 5$), $2^5 = 32$, $5^2 = 25$ and $32 > 25$.

$\therefore$ P(5) is true.

Let P(k) be true for $k \geq 5$. Hence, $2^k > k^2$

Let $n = k + 1$.

Now, $2^{k+1} = 2 \cdot 2^k > 2k^2$ **($2^k > k^2$) (i)**

Now, $2k^2 - (k + 1)^2 = 2k^2 - k^2 - 2k - 1$

$= k^2 - 2k + 1 - 2$

$= (k + 1)^2 - 2 > 0$ as $k \geq 5$

$\therefore$ $2k^2 > (k + 1)^2$ **(ii)**

$\therefore$ $2^{k+1} > (k + 1)^2$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Variant 2 : Let P(n) be a statement of integer variable n.

If P(1) and P(2) are true and if P(k) and P(k + 1) are true for some positive integer k implies P(k + 2) is also true, then P(n) is true for all $n \in$ N.

Example 19 : Let a_n be a sequence of natural numbers with $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$. Prove $a_n = 2^n + 3^n$, $\forall n \in$ N.

Solution : Let P(n) : If $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$, $a_1 = 5$, $a_2 = 13$, then $a_n = 2^n + 3^n$, $\forall n \in$ N.

Let $n = 1$. $a_1 = 5$ and $2^1 + 3^1 = 2 + 3 = 5$. Hence, P(1) is true.

Let $n = 2$. $a_2 = 13$ and $2^2 + 3^2 = 4 + 9 = 13$. Hence, P(2) is true.

Let $a_k = 2^k + 3^k$, $a_{k+1} = 2^{k+1} + 3^{k+1}$ for $k \geq 1$

Now, $a_{k+2} = 5a_{k+1} - 6a_k$

$= 5(2^{k+1} + 3^{k+1}) - 6 \cdot 2^k - 6 \cdot 3^k$

$= 5 \cdot 2^k \cdot 2 + 5 \cdot 3^k \cdot 3 - 6 \cdot 2^k - 6 \cdot 3^k$

$= 2^k (10 - 6) + 3^k (15 - 6)$

$= 2^k \cdot 2^2 + 3^k \cdot 3^2$

$= 2^{k+2} + 3^{k+2}$

$\therefore$ P(k + 2) is true.

$\therefore$ P(k) is true and P(k + 1) is true $\Rightarrow$ P(k + 2) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $a_{n+2} = 5a_{n+1} - 6a_n$ is called a **recurrence relation**. Its solution is $a_n = A\alpha^n + B\beta^n$ where α, β are roots of $x^2 - 5x + 6 = 0$ (5 is co-efficient of a_{n+1}, -6 is co-efficient of a_n)

$\therefore$ $\alpha = 3$, $\beta = 2$

$\therefore$ $a_n = A3^n + B2^n$

$\therefore$ $a_1 = 3A + 2B = 5$; $a_2 = 9A + 4B = 13$. Hence, A = B = 1

$\therefore$ $a_n = 3^n + 2^n$. If $a_{n+2} = l \cdot a_{n+1} - m \cdot a_n$, then α and β are the roots of equation $x^2 - lx + m = 0$.

Miscellaneous Problems :

Example 20 : Prove that any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only.

Solution : Let P(n) : Any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only. $n \in$ N

For $n = 4$, we require two coins of ₹ 2 to pay ₹ 4. Let the statement be true for $k \geq 4$.

Let $n = k + 1$.

Consider two cases :

(1) If the payment for ₹ k contains a ₹ 5 coin, take it back and give 3, ₹ 2 coins. Hence $k + 6 - 5 = k + 1$ rupees are paid.

(2) If the payment for ₹ k does not contain any ₹ 5 coin, since $k \geq 4$, he must have paid at least two ₹ 2 coins. Take them back and pay one ₹ 5 coin. Hence ₹ $k + 5 - 4 = k + 1$ are paid.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) be true, for $\forall n \in$ N by P.M.I.

Example 21 : Prove that any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in N \cup \{0\}$, $y \in N \cup \{0\}$.

Solution : Let P(n) : Any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in N \cup \{0\}$, $y \in N \cup \{0\}$.

Let $n = 24$. Then $7 \cdot 2 + 5 \cdot 2 = 24$ is the required form with $x = y = 2$.

Let $7x + 5y = k$ for $k \geq 24$, $x \in N \cup \{0\}$, $y \in N \cup \{0\}$. **(i)**

Now, $5 \cdot 3 - 7 \cdot 2 = 1$ **(ii)**

$\therefore \quad 7(x - 2) + 5(y + 3) = k + 1$ **(Adding (i) and (ii))**

Here $y + 3 \in N \cup \{0\}$ and $x - 2 \in N \cup \{0\}$ if $x \neq 0$ or 1.

Let $x = 0$. Then $5y = k \geq 24$. Thus $y \geq 5$, using (i).

$7 \cdot 3 - 5 \cdot 4 = 1$ and $5y = k$ gives on adding. **(iii)**

$7 \cdot 3 + 5(y - 4) = k + 1$

Here $x = 3 \geq 0$, $y - 4 \geq 0$ **($y \geq 5$)**

$\therefore \quad P(k + 1)$ is true, if $x = 0$

Let $x = 1$. Hence, $7 + 5y = k$, using (i).

Then $5y = k - 7 \geq 17$. Thus $y \geq 4$

$\therefore \quad 7 \cdot 3 - 5 \cdot 4 = 1$ and $7 + 5y = k$ gives on adding. **(iv)**

$7(4) + 5(y - 4) = k + 1$ with $y - 4 \geq 0$ and $x = 4$ **(Adding in (iv))**

$\therefore \quad P(k + 1)$ is true.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true for $\forall n \in N$ by P.M.I.

Example 22 : (Tower of Hanoi) We have three pegs and a collection of disks of different sizes. Initially they are on top on each other according to their size on the first peg, the largest being on the bottom and the smallest on the top. A move in this game consists of moving disks from one peg to another such that larger disk can never rest on a smaller one. Prove that the number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Solution : Let $P(n)$: The number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Let $n = 1$, obviously there is only one move.

$\therefore \quad P(1)$ is true. $2^1 - 1 = 1$. **($p(k)$)**

Suppose there are $2^k - 1$ moves to transfer k disks as required.

First we move top k disks to the second peg using the third peg as the intermediate one. This will take $2^k - 1$ moves. Now move the last disk to the third peg. This is one move. Now move k disks from second peg to the third peg in $2^k - 1$ moves.

$\therefore \quad$ The total number moves is $2^k - 1 + 1 + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$\therefore \quad P(k + 1)$ is proved.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true, $\forall n \in N$ by P.M.I.

Example 23 : Prove $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in N$, $n \in N$ (to be done after chapter 3)

Solution : Let P(n) : $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in N$, $n \in N$

For $n = 1$, $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} = \frac{15+33+55+62}{165} = \frac{165}{165} = 1$

$\therefore$ P(1) is true.

Let P(k) be true. Hence, $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in N$

Let $n = k + 1$.

Consider $\left(\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}\right) - \left(\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}\right)$

$$= \frac{1}{11}((k+1)^{11} - k^{11}) + \frac{1}{5}((k+1)^5 - k^5) + \frac{1}{3}((k+1)^3 - k^3) + \frac{62}{165}$$

$$= \frac{1}{11}\left(1 + \binom{11}{1}k + \binom{11}{2}k^2 + ... + \binom{11}{10}k^{10}\right) + \frac{1}{5}\left(1 + \binom{5}{1}k + \binom{5}{2}k^2 + ... + \binom{5}{4}k^4\right)$$

$$+ \frac{1}{3}\left(1 + \binom{3}{1}k + \binom{3}{2}k^2\right) + \frac{62}{165}$$

$$= \frac{1}{11}\binom{11}{1}k + \frac{1}{11}\binom{11}{2}k^2 + ... + \frac{1}{11}\binom{11}{10}k^{10} + \frac{1}{5}\binom{5}{1}k + \frac{1}{5}\binom{5}{2}k^2 + ... + \frac{1}{5}\binom{5}{4}k^4$$

$$+ \frac{1}{3}\binom{3}{1}k + \frac{1}{3}\binom{3}{2}k^2 + \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} \quad \text{(i)}$$

Now, 11, 5, 3 being primes, 11 divides $\binom{11}{r}$ for $r = 1, 2, ..., 10$

5 divides $\binom{5}{r}$ for $r = 1, 2, 3, 4$

3 divides $\binom{3}{r}$ for $r = 1, 2$

and $\frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} = 1$

$\therefore$ The R.H.S. in (1) represents a natural number.

Also $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in N$

$\therefore$ $\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}$

$= \frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ + a natural number $\in N$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true for $\forall n \in N$ by P.M.I.

Example 24 : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Prove that the number of handshakes is at most n^2.

Solution : Let P(n) : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Then the number of handshakes is at most n^2.

For $n = 1$, there are two persons. Hence there is at most $1 = 1^2$ handshake.

$\therefore$ P(1) is true.

Let P(k) be true.

Let $n = k + 1$.

Now there are $2k + 2$ persons. Choose two persons A and B who have had a handshake.

(If there are no two such persons, number of handshakes is zero which is at most $(k + 1)^2$.

Now the remaining $2k$ persons had at most k^2 handshakes (P(k) is true). A and B have one handshake.

Each of $2k$ persons could shake hands with A or B only as no three persons had handshakes with each other. Hence the number of handshakes is at most

$$k^2 + 1 + 2k = (k + 1)^2$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true for $\forall n \in$ N by P.M.I.

A paradox :

[**Note :** A paradox is the misinterpretation of a result to arrive at a contradictory result.]

P(n) : A thirsty man can drink n drops of water.

For $n = 1$, obviously a thirsty man would like to drink one drop of water.

If he can drink k drops of water, he can definitely drink $k + 1$ drops of water.

So he can drink any amount water to exhaust all resources of water on the earth !

Exercise 1

Prove the following by the principle of mathematical induction : **(1 to 19) ($n \in$ N)**

1. $1^2 \cdot 2 + 2^2 \cdot 3 + ... + n^2(n + 1) = \frac{n(n+1)(n+2)(3n+1)}{12}$

2. $a + (a + d) + (a + 2d) + ... + (a + (n - 1)d) = \frac{1}{2}n(2a + (n - 1)d)$

3. $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

4. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

5. $1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$

6. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

7. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+3+...+n} = \frac{2n}{n+1}$

8. $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + ... + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

9. $1^4 + 2^4 + 3^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

10. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, then $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$.

11. $41^n - 1$ is divisible by 40.

12. $4007^n - 1$ is divisible by 2003.

13. $7^n - 6n - 1$ is divisible by 36.

14. $2 \cdot 7^n + 3 \cdot 5^n - 5$ is a multiple of 24.

15. $11^{n+2} + 12^{2n+1}$ is divisible by 133.

16. $n(n + 1)(2n + 1)$ is divisible by 6.

17. $1 \cdot 3^1 + 2 \cdot 3^2 + 3 \cdot 3^3 + ... + n \cdot 3^n = \frac{(2n-1)\,3^{n+1}+3}{4}$

18. $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.

19. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \in \mathrm{N}$

20. Prove $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$

21. For Lucas' sequence $a_n = a_{n-1} + a_{n-2}$ $(n \geq 3)$; $a_1 = 1$, $a_2 = 3$, prove $a_n \leq (1.75)^n$.

22. Prove $2^n > n^3$, if $n \geq 10$

23. Prove a polygon of n sides has $\frac{n(n-3)}{2}$ diagonals, $n > 3$

24. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, then prove that

$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ (This $\{a_n\}$ is called **Fibonacci** sequence.)

25. If $f : \mathrm{N} \to \mathrm{N}$, $f(1) = 1$, $f(2) = 5$, $f(n + 1) = f(n) + 2f(n - 1)$, $n \geq 2$

then prove that $f(n) = 2^n + (-1)^n$

26. If $f : \mathrm{N} \to \mathrm{N}$, $f(1) = 1$, $f(n + 1) - f(n) = 2^n$

then prove that $f(n) = 2^n - 1$

27. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$; $n \geq 3$

then prove that $a_2 + a_4 + a_6 + ... + a_{2n} = a_{2n+1} - 1$

28. If $a_1 = 1$, $a_2 = 11$ and $a_n = 2a_{n-1} + 3a_{n-2}$; $n \geq 3$

then prove that $a_n = 2(-1)^n + 3^n$ for $n \in \mathrm{N}$

29. Prove that every integer $n \geq 12$ can be written in the form $7x + 3y = n$,

$x \in \mathrm{N} \cup \{0\}$, $y \in \mathrm{N} \cup \{0\}$

30. Prove that $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is even, $n \in N$.

31. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For $P(n) : 2^n < n!$, ... is true. ☐

(a) P(1) (b) P(2) (c) any $P(n)$, $n \in N$ (d) P(4)

(2) For $P(n) : 2^n = 0$, ... is true. ☐

(a) P(1) (b) P(3)

(c) P(10) (d) $P(k) \Rightarrow P(k + 1)$, $k \in N$

(3) $P(n) : 1 + 2 + 3 + ... + (n + 1) = \frac{(n+1)(n+2)}{2}$, $n \in N$ ☐

(a) P(1) requires L.H.S. = 7 = R.H.S.

(b) P(1) requires L.H.S. = 3 = R.H.S.

(c) $P(k) \Rightarrow P(k + 1)$ is not true for $k \in N$

(d) It is false that $P(n)$ is true, $\forall n \in N$ by P.M.I.

(4) If ...... is true and $P(k)$ is true $\Rightarrow$ $P(k + 1)$ is true for $k \geq -1$, then $P(n)$ is true for all $n \in N \cup \{0, -1\}$. ☐

(a) P(−1) (b) P(0) (c) P(1) (d) P(2)

(5) $P(n)$: Every prime is of the form $2^{2^n} + 1$ is not true, for n = ... ☐

(a) 1 (b) 2 (c) 0 (d) 5

(6) $P(n) : 2^n - 1$ is a prime for n = ... ☐

(a) 1 (b) 3 (c) 4 (d) 8

(7) $P(n) : n^2 - n + 41$ is a prime, is false for n = ... ☐

(a) 1 (b) 2 (c) 3 (d) 41

(8) $P(n) : 2n + 1$ is a prime, is false for n = ... ☐

(a) 1 (b) 2 (c) 3 (d) 4

(9) $P(n) : 4n + 1$ is a prime, is false for n = ... ☐

(a) 1 (b) 3 (c) 7 (d) 11

(10) $P(n) : 2^n > n^2$ is true for n = ... ☐

(a) 2 (b) 3 (c) 4 (d) 5

*

Summary

We studied the following points in this chapter :

1. Principle of Induction and Examples
2. Different variants of P.M.I. and applications

Puzzle

There are n people in a room each being put on a hat from amongst at least n white hats and $n - 1$ black hats. They stand in a queue, so that every one can see the colour of the hat of the person standing in front of him. Starting from back we ask the persons in turn, 'Do you know what is the colour of your hat ?' If the first $(n - 1)$ persons say no, the person in the front will say 'Yes the colour of my hat is white.' Prove.

Solution : Let P(n) : If the first $(n - 1)$ persons say no, the person in the front will say yes.

For $n = 1$, there is no black hat $(1 - 1 = 0)$. Hence the first person will say, 'yes, my hat is white.' Suppose the statement is true for $n = k$. Let $n = k + 1$.

See how the man standing in the front would think. Suppose my hat is black. Then excluding me there are k people with at least k white hats and $k - 1$ black hats. By P(k), since the first $(k - 1)$ persons said no, the person behind me must say yes. 'I know the colour of my hat.'

But he said no. So the colour of my hat cannot be black. Hence it is white.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Explanation : If $n = 2$, there is one black hat and at least two white hats. If the last person sees a black hat put on by the person in front of him, he would definitely say, 'Yes, colour of my hat is white,' as there is only one black hat. But he is not able to answer. So the first person logically thinks he has put on a white hat and the person behind might have put on a black or a white hat.

Srinivasa Ramanujan (1887-1920) was one of India's greatest mathematical geniuses. He made substantial contributions to the analytical theory of numbers and worked on elliptical functions, continued fractions and infinite series.

In 1990 he began to work on his own on mathematics summing geometric and arithmetic series.

Ramanujan had shown how to solve cubic equations in 1902 and he went to find his own method to solve the quartic.

In 1904 Ramanujan had begun to undertake deep research. He investigated the series $\Sigma\left(\frac{1}{n}\right)$ and calculated Euler's constant to 15 decimal places.

Continuing his mathematical work Ramanujan studied continued fractions and divergent series in 1908.

Chapter 2

COMPLEX NUMBERS

A mathematician is a device for turning coffee into theorems.

– Paul Erdos

As far as the laws of mathematics refer to reality, they are not certain and as far as they are certain they do not refer to reality.

– Albert Einstein

2.1 Introduction

In previous classes, we have studied the number sets N, Z, Q and R. We know that the set of rational numbers and the set of irrational numbers constitute the set of real numbers. We also studied properties of numbers and solutions of linear equations in one variable and two variables. We also discussed the solutions of quadratic equations in one variable. We observed that if the discriminant $b^2 - 4ac < 0$, the quadratic equation $ax^2 + bx + c = 0$, $a, b, c \in$ R, $a \neq 0$ has no solution in R. For example $x^2 + 1 = 0$ has no solution in R. To allow the square root of negative numbers, the real number system has to be extended to a larger system. In fact, Greeks were the first to recognize the fact that square root of a negative number does not exist in the real number system. The Indian mathematician **Mahavira** or **Maviracharya** (850 A.D.) too mentions this difficulty in his work ***'Ganitasara Sangraha'***. The extension of real number system should be in such a way that the algebraic operations such as addition, subtraction, multiplication and division can be defined properly. This new set is called the set of **Complex Numbers** and is denoted by **C**.

2.2 The Set R × R and the Set of Complex Numbers

We begin with the set R of real numbers to obtain the set C of complex numbers. R × R is the set of all ordered pairs of real numbers.

$$\text{R} \times \text{R} = \{(a, b) \mid a \in \text{R}, b \in \text{R}\}$$

We shall define the equality, addition and multiplication of two elements of R × R.

(1) Equality : Two elements (a, b) and (c, d) of R × R are defined to be equal if $a = c$ and $b = d$. Thus $a = c, b = d \Rightarrow (a, b) = (c, d)$

For example, $(1, 0) = (sin^2x + cos^2x, \log 1)$ but $(1, 4) \neq (4, 1)$

(2) Addition : The sum of two elements (a, b) and (c, d) of R × R is defined as follows :

$(a, b) + (c, d) = (a + c, b + d)$

For example, $(5, 2) + (2, 3) = (5 + 2, 2 + 3) = (7, 5)$

(3) Multiplication : The product of two elements (a, b) and (c, d) of R × R is defined as follows :

$(a, b)(c, d) = (ac - bd, ad + bc)$

For example, $(5, 2)(2, 3) = (5 \times 2 - 2 \times 3, 5 \times 3 + 2 \times 2) = (4, 19)$

The set R × R with these rules is called the set of complex numbers and it is denoted by C. Generally, we denote a complex number by z.

2.3 Basic Algebraic Properties of Complex Numbers

We have discussed the properties of closure, commutativity, associativity and distributivity with respect to operations of addition and multiplication on R. We shall see that these properties hold good in C too.

The operation of addition satisfies the following properties :

(1) The closure property : The sum of two complex numbers is a complex number.

i.e. $z_1 + z_2 \in C \quad \forall z_1, z_2 \in C$

We also say that the **addition is a binary operation on C.**

(2) The commutative property : $z_1 + z_2 = z_2 + z_1 \quad \forall z_1, z_2 \in C$

(3) The associative property : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \forall z_1, z_2, z_3 \in C$

(4) The existence of additive identity : There exists a complex number O = (0, 0), called an additive identity or the zero complex number, such that

$z + O = z = O + z \quad \forall z \in C$

It can be proved that **the additive identity O is unique.**

In fact if $(a, b) + (x, y) = (a, b)$ for all $(a, b) \in C$,

then $a + x = a$, $\quad b + y = b$,

$\therefore \quad x = 0$, $\quad y = 0$.

Thus, $(x, y) = (0, 0)$

Also $(a, b) + (0, 0) = (a, b)$.

(5) The existence of additive inverse : To every complex number $z = (a, b)$, there corresponds a complex number $(-a, -b)$, denoted by $-z$, called the additive inverse (or negative) of z such that $z + (-a, -b) = (0, 0) = O$.

We observe that,
$$z + (-z) = (a, b) + (-a, -b) = (a + (-a), b + (-b)) = (0, 0) = O \text{ (O is the additive identity.)}$$

Also, $(-z) + z = O$

We can prove that **for $z \in C$, its additive inverse $-z$ is unique.**

Note : $(a, b) + (x, y) = (0, 0)$ requires $a + x = 0 = b + y$

$\therefore \quad x = -a, y = -b$

$\therefore$ **$(-a, -b)$ is the additive inverse of (a, b).**

The operation of multiplication satisfies following properties :

(1) The closure property : The product of two complex numbers is a complex number.

i.e. $z_1z_2 \in C$, $\forall z_1, z_2 \in C$

We also say that the multiplication is a binary operation on C.

(2) The commutative property : $z_1z_2 = z_2z_1$ $\forall z_1, z_2 \in C$

(3) The associative property : $(z_1z_2)z_3 = z_1(z_2z_3)$ $\forall z_1, z_2, z_3 \in C$

(4) The existence of multiplicative identity : There exists a complex number $(1, 0)$, called a multiplicative identity such that $z(1, 0) = z = (1, 0)z$, $\forall z \in C$

By taking $z = (a, b)$, $z. (1, 0) = (a, b)(1, 0) = (a - 0, 0 + b) = (a, b) = z$

Also, $(1, 0)z = z(1, 0) = z$

The multiplicative identity $(1, 0)$ is unique.

Note : If $(a, b)(x, y) = (a, b)$. $\forall (a, b) \in C$, then $ax - by = a$ and $ay + bx = b$, $\forall a, b \in R$. In particular $a = 1$, $b = 0$ gives $x = 1$, $y = 0$. Then $(a, b)(1, 0) = (a, b)$, $\forall (a, b) \in C$.

(5) The existence of multiplicative inverse : To each non-zero complex number $z = (a, b)$, there corresponds a complex number $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ (denoted by z^{-1}), called a multiplicative inverse of z such that

$z \cdot z^{-1} = (1, 0) = z^{-1} \cdot z$ **((1, 0) is the multiplicative identity)**

Since $(a, b) \neq (0, 0)$, $a^2 + b^2 \neq 0$ and hence $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) \in C$ and

$$z \cdot z^{-1} = (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(a \cdot \frac{a}{a^2+b^2} - b \cdot \frac{-b}{a^2+b^2},\ a \cdot \frac{-b}{a^2+b^2} + b \cdot \frac{a}{a^2+b^2}\right)$$

$$= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ab}{a^2+b^2}\right) = (1, 0)$$

Also, $z^{-1} \cdot z = (1, 0)$

Note that for each non-zero $z \in C$, its **multiplicative inverse z^{-1} is unique.**

z^{-1} is also denoted in $\frac{1}{z}$.

Note : Let z' be a complex number such that $zz' = (1, 0)$

Let $z' = (x, y)$

$\therefore$ $zz' = (a, b)(x, y) = (1, 0)$

$\therefore$ $(ax - by, ay + bx) = (1, 0)$

$\therefore$ $ax - by = 1$, $ay + bx = 0$

Solving these equations we get $x = \frac{a}{a^2+b^2}$, $y = \frac{-b}{a^2+b^2}$

$\therefore \quad z' = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

As $z = (a, b) \neq (0, 0)$ we have $a^2 + b^2 \neq 0$.

$\therefore \quad z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

The existence of multiplicative inverse enables us to show that if a product $z_1 z_2$ is zero, then at least one of the factors z_1 and z_2 is zero. (why ?)

(6) The distributive laws : For any three complex numbers z_1, z_2, z_3

(a) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

(b) $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

2.4 R as a Subset of C

By definition, every complex number is an ordered pair of real numbers. Let us denote by R' the set of those complex numbers (a, b) in which $b = 0$. So, R' = $\{(a, 0) \mid a \in$ R$\}$. Obviously R' $\subset$ C. Let $(a, 0)$, $(b, 0)$ be two elements of R'. Note that,

(1) $(a, 0) = (b, 0) \Leftrightarrow a = b$

(2) $(a, 0) + (b, 0) = (a + b, 0) \in$ R'

(3) $(a, 0)(b, 0) = (ab, 0) \in$ R'

Thus, the sum as well as the product of two elements of R' is again an element of R'. Moreover, the first component of the sum or product of two numbers $(a, 0)$ and $(b, 0)$ is obtained merely by adding or multiplying respectively the first components a and b, while the second component remains zero. Infact R' is closed for addition and multiplication as in C. So as far as equality, sum and multiplication are concerned, the complex numbers of the form $(a, 0)$ behave exactly like real number a. Hence we identify complex numbers of the form $(a, 0)$ with a and write a for $(a, 0)$. Thus $(4, 0) = 4$, $(0, 0) = 0$ etc. In this way we look upon every real number a as the complex number $(a, 0)$, which allows us to identify R with R' and so R' = R $\subset$ C. Thus we have **N $\subset$ Z $\subset$ Q $\subset$ R $\subset$ C**. Now O = (0, 0) = 0, the additive identity (1, 0) = 1, the multiplicative identity.

2.5 Representation of a Complex Number in the form *a* + *ib*

By writing a for $(a, 0)$ we are able to represent a complex number (a, b) in another form.

Firstly, let us get familiar with a special complex number (0, 1). We use the symbol i for this complex number. Thus, $i = (0, 1)$.

Now, $i^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1$. In the year 1737 Euler was the first person to introduce the symbol i for the complex number (0, 1), satisfying $i^2 = -1$. $i = (0, 1)$ is called **an imaginary number**.

Now, $(a, b) = (a, 0) + (0, b)$

$= (a, 0) + (0, 1)(b, 0)$ **$((0, 1)(b, 0) = (0 - 0, 0 + b) = (0, b))$**

$= a + ib$

$\therefore \quad (a, b) = a + ib$

Hence, every complex number (a, b) can be expressed in the form $a + ib$, where $a, b \in$ R and $i^2 = -1$.

Thus, $\mathbf{C} = \{a + ib \mid a, b \in \mathbf{R}\}$

According to the commutative law for multiplication, $ib = bi$.

Hence, $a + ib = a + bi$

For example, $(3, 5) = 3 + 5i$, $(0, 7) = 0 + 7i = 7i$, $(5, 0) = 5 + 0i = 5$

For the complex number $z = a + bi$, a is called the **real part** of z and is denoted by $Re(z)$ and b is called the **imaginary part** of z and is denoted by $Im(z)$.

So, $z = a + ib = Re(z) + iIm(z)$. For example, if $z = 3 + 2i$, then $Re(z) = 3$ and $Im(z) = 2$.

Note that both the real and imaginary parts of a complex number are real numbers.

A complex number, whose real part is zero and whose imaginary part is non-zero is called a **purely imaginary number**. For example, $9i = 0 + 9i$ is a purely imaginary number.

Let us now revert to the algebraic operations on complex numbers which are in the form $a + bi$.

Equality of two complex numbers :

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal i.e. $(a, b) = (c, d)$ if $a = c$ and $b = d$.

If $z = a + bi = 0$ then $a = 0$ and $b = 0$. **$(0 = 0 + 0i)$**

Example 1 : if $3x + (3x - y)i = 4 + (-6)i$, where x and y are real numbers, then find the values of x and y.

Solution : We have $3x + (3x - y)i = 4 + (-6)i$. Since $a + bi = c + di \Rightarrow a = c$ and $b = d$, we get $3x = 4$, $3x - y = -6$. On solving, we get $x = \frac{4}{3}$, $y = 10$.

Addition of two complex numbers :

Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then the sum of $z_1 = (a, b)$, $z_2 = (c, d)$ is as follows :

$$z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d) = a + c + (b + d)i$$

For example, $(2 + 2\sqrt{2}\,i) + (-3 + \sqrt{2}\,i) = (2 - 3) + (2\sqrt{2} + \sqrt{2})i$

$$= -1 + 3\sqrt{2}\,i$$

Difference of two complex numbers :

Let z_1 and z_2 be any two complex numbers. The difference $z_1 - z_2$ is defined by,

$$z_1 - z_2 = z_1 + (-z_2)$$

Let $z_1 = (a, b)$, $z_2 = (c, d)$

Then $-z_2 = (-c, -d)$

$$\begin{aligned} \therefore\quad z_1 - z_2 &= z_1 + (-z_2) \\ &= (a, b) + (-c, -d) \\ &= (a - c, b - d) \\ &= (a - c) + (b - d)i \end{aligned}$$

For example, $(2 + \sqrt{3}i) - (-3 + 2\sqrt{3}i) = 2 - (-3) + (\sqrt{3} - 2\sqrt{3})\,i$

$$= 5 - \sqrt{3}i$$

Multiplication of two complex numbers :

Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers.

$\therefore \quad z_1 = (a, b),\ z_2 = (c, d)$

$\therefore \quad z_1z_2 = (ac - bd, ad + bc)$

$\therefore \quad \mathbf{z_1z_2 = (ac - bd) + (ad + bc)i}$

For example, $(2 + \sqrt{3}i)(-3 + \sqrt{3}i) = (2 \times (-3) - \sqrt{3}\sqrt{3}) + (2\sqrt{3} + \sqrt{3} \times (-3))i$

$= (-6 - 3) + (2\sqrt{3} - 3\sqrt{3})i = -9 - \sqrt{3}i$

We can open the bracket and multiply them because of the distributive laws.

Quotient of two complex numbers :

Let z_1 and z_2 be any two complex numbers where, $z_2 \neq 0$. The quotient $\frac{z_1}{z_2}$ is defined as $\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \frac{1}{z_2}$.

In fact, $\frac{1}{z} = z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$

For example, $\frac{6+3i}{10+8i} = (6 + 3i)(10 + 8i)^{-1}$

$= (6 + 3i)\left(\frac{10}{164} - \frac{8i}{164}\right)$

$= \frac{60+30i-48i-24i^2}{164}$

$= \frac{84-18i}{164}$ **($i^2 = -1$)**

$= \frac{21}{41} + \frac{(-9)}{82}i$

Powers of *i* :

We shall assume that the usual laws of indices hold good for integral powers of z.

We know that $i^2 = -1$, $i^3 = i^2i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$, $i^5 = i$, $i^6 = -1$ etc.

Remember, $\mathbf{i^2 = -1,\ i^3 = -i,\ i^4 = 1}$

Also, we have $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$ etc.

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

> In mathematics, **trichotomy** is the property of an order relation. For any real numbers x and y, exactly one of the following holds : $x < y$, $x = y$ or $x > y$. This law of trichotomy holds for comparison of real numbers. This property is no more valid for complex numbers as C is not ordered.

Example 2 : Evaluate (i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2$ (ii) $i^1 + i^2 + i^3 + i^4 + ... + i^{1000}$

Solution : (i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{16}\,i^3 + \left(\frac{1}{i}\right)^{24}\left(\frac{1}{i}\right)\right]^2$

$= (-i - i)^2$

$= (-2i)^2 = -4$

(ii) $i^1 + i^2 + i^3 + i^4 + ... + i^{997} + i^{998} + i^{999} + i^{1000}$

$= (i - 1 - i + 1) + (i - 1 - i + 1) + ... + (i - 1 - i + 1)$ (250 brackets)

$= 0 + 0 + ... + 0 = 0$

Conjugate of a Complex Number :

If $z = (a, b) = a + bi$, then its conjugate complex number is defined to be the complex number $a - bi = (a, -b)$ and is denoted by $\bar{z}$.

We note that $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$. So just like a surd $\bar{z}$ acts like a 'rationalising' factor. Since $z\bar{z} = a^2 + b^2$ is real, we express complex number $\frac{p}{q}$ as $\frac{p\bar{q}}{q\bar{q}}$ so that the denominator $q\bar{q}$ is real. Let us understand this concept by a few examples.

Example 3 : Express the following in the form of $a + ib$, where $a, b \in$ R

(1) $\frac{(2-8i)(7+8i)}{1+i}$ (2) $(3 + 4i)^{-1}$ (3) $\frac{(1+i)^3}{4+3i}$ (4) $\frac{1}{1+\cos\theta - i\sin\theta}$

Solution : (1) $\frac{(2-8i)(7+8i)}{1+i} = \frac{14+16i-56i-64i^2}{1+i}$

$= \frac{14-40i+64}{1+i}$ **($i^2 = -1$)**

$= \frac{78-40i}{1+i}$

$= \frac{78-40i}{1+i} \times \frac{1-i}{1-i}$ **(multiply and divide by conjugate of $1 + i$)**

$= \frac{78-78i-40i+40i^2}{1-i^2}$

$= \frac{38-118i}{2}$ **($i^2 = -1$)**

$= 19 - 59i$

(2) $(3 + 4i)^{-1} = \frac{1}{3+4i} = \frac{1}{3+4i} \times \frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \frac{3}{25} + i\left(-\frac{4}{25}\right)$

or directly $(3 + 4i)^{-1} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4i}{25}$ **(formula of z^{-1})**

(3) $\frac{(1+i)^3}{4+3i} = \frac{1^3+3\cdot1^2\cdot i+3\cdot1\cdot i^2+i^3}{4+3i}$

$= \frac{1+3i-3-i}{4+3i}$

$= \frac{-2+2i}{4+3i}$

$= \frac{(-2+2i)(4-3i)}{(4+3i)(4-3i)}$

$= \frac{-8+8i+6i-6i^2}{16+9}$

$= -\frac{2}{25} + \frac{14}{25}i$ **($i^2 = -1$)**

(4) $\dfrac{1}{1+\cos\theta - i\sin\theta} = \dfrac{1}{1+\cos\theta - i\sin\theta} \times \dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta + i\sin\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{(1+\cos\theta)^2 + \sin^2\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{2+2\cos\theta}$

$= \dfrac{1+\cos\theta}{2(1+\cos\theta)} + i\,\dfrac{\sin\theta}{2(1+\cos\theta)}$

$= \dfrac{1}{2} + i\,\dfrac{\sin\theta}{2(1+\cos\theta)}$

Note : We will see in chapter 5 that $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\dfrac{\theta}{2}$.

Example 4 : Find the real values of x and y so that

(1) $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$ (2) $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$

Solution : (1) $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$

$\therefore$ $[x+(x-2)i](3-i) + [2y+(1-3y)i](3+i) = (3+i)(3-i)i$

(Multiplying both sides by $(3+i)(3-i)$)

$\therefore$ $3x+(x-2)+[3(x-2)-x]i+6y-(1-3y)+[2y+3(1-3y)]i = (9+1)i$

$\therefore$ $(4x+9y-3)+(2x-7y-3)i = 10i$

$\therefore$ $4x+9y-3 = 0$ and $2x-7y-3 = 10$ **(Equality of two complex numbers)**

$\therefore$ $4x+9y-3 = 0$ and $2x-7y-13 = 0$

Solving the above simultaneous equations, we get $x = 3$, $y = -1$.

(2) $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$

$\therefore$ $iy(3x+y)-(3y+4i)(ix+1) = 0$ **(Multiplying both sides by $(ix+1)(3x+y)$)**

$\therefore$ $(-3y+4x)+i(3xy+y^2-3xy-4) = 0+i0$

$\therefore$ $(-3y+4x)+i(y^2-4) = 0+i0$

$\therefore$ $-3y+4x = 0$ and $y^2-4 = 0$ **($a+bi = 0 \Rightarrow a = 0, b = 0$)**

$y^2-4 = 0$ gives us $y = \pm 2$

For $y = 2$ we get $x = \frac{3}{2}$ and for $y = -2$ we get $x = -\frac{3}{2}$.

$\therefore$ The solution set is $\left\{\left(\frac{3}{2}, 2\right), \left(-\frac{3}{2}, -2\right)\right\}$.

Exercise 2.1

1. Express the following complex numbers in the form $a+bi$:

(1) $(\sqrt{2}-i)-i(1-\sqrt{2}\,i)$ (2) $(2-3i)(-2+i)$

(3) $(3+i)(3-i)\left(\frac{1}{5}+\frac{1}{10}i\right)$ (4) $\dfrac{4+i}{2-3i}$ (use $(2-3i)^{-1}$)

(5) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$ (6) $\frac{5i}{(1-i)(2-i)(3-i)}$

(7) $(1 - i)^4$ (8) $\left[i^{17} - \left(\frac{1}{i}\right)^{34}\right]^2$

(9) $\left(\frac{4i^3-1}{2i+1}\right)^2$ (10) $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

2. Find the real values of x and y, if

(1) $x + 4yi = xi + y + 3$

(2) $(4 + 5i)x + (3 - 2i)y + i^2 + 6i = 0$

(3) $\frac{x}{1-i} + \frac{y}{1+i} = 1 + 3i$

(4) $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$

(5) $(3x - 2yi)(2 + i)^2 = 10(1 + i)$

3. Find the multiplicative inverse of :

(1) $3 - 2i$ (2) $-1 + i\sqrt{3}$ (3) $\frac{4+3i}{5-3i}$ (4) $(2 - 3i)^2$ (5) $-i$

4. Show that, (1) $Re(iz) = -Im(z)$ (2) $Im(iz) = Re(z)$

5. Verify that each of the two complex numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

*

2.6 Conjugate and Modulus of a Complex Number

Complex Conjugate : We know if $z = a + bi$, $\bar{z} = a - bi$.

As an example,

(1) If $z = 3 + 5i$, then $\bar{z} = 3 - 5i$

(2) If $z = 5 - 3i$, then $\bar{z} = 5 + 3i$

(3) If $z = 3 = 3 + 0i$, then $\bar{z} = 3 - 0i = 3$

(4) If $z = 3i = 0 + 3i$, then $\bar{z} = 0 - 3i = -3i$

Here are some basic facts about conjugates.

For any three complex numbers z, z_1, z_2 we have the following properties :

1. $\overline{(\bar{z})} = z$ **2.** $\frac{z+\bar{z}}{2} = Re(z)$

3. $\frac{z-\bar{z}}{2i} = Im(z)$ **4.** $z = \bar{z}$ if and only if z is real.

5. $\bar{z} = -z$ if and only if z is purely imaginary.

6. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ **7.** $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

8. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, where $z_2 \neq 0$

The above properties are easy to verify. Let us verify some of them.

Let $z = a + ib$

1. $\bar{z} = a - ib$

$\therefore \overline{(\bar{z})} = \overline{a - ib} = a + ib = z$

2. $z + \bar{z} = a + ib + a - ib = 2a = 2Re(z)$ as $Re(z) = a$

$\therefore \quad \frac{z+\bar{z}}{2} = Re(z)$

3. $z - \bar{z} = a + ib - a + ib = 2ib = 2i\,Im(z)$ as $Im(z) = b$

$\therefore \quad \frac{z-\bar{z}}{2i} = Im(z)$

4. $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow b = -b \Leftrightarrow 2b = 0 \Leftrightarrow b = 0.$

Thus, $z = \bar{z}$ if and only if z is real.

Modulus of a complex number :

Modulus of a complex number $z = a + ib$ is defined as $\sqrt{a^2+b^2}$ and is denoted by $|z|$.

Thus, $|z| = \sqrt{a^2+b^2}$

Note that $|z|$ is a real number and $|z| \geq 0, \quad \forall z \in C$.

As an example, if $z = 3 + 4i$, then $|z| = \sqrt{9+16} = \sqrt{25} = 5$

Notice that if z is a real number (i.e. $z = a + 0i$) then, $|z| = \sqrt{a^2} = |a|$, where $|z|$ is the modulus of the complex number and $|a|$ is the absolute value of the real number (recall that for any real number a we have $\sqrt{a^2} = |a|$).

Properties of modulus :

1. $|z| = 0$ if and only if $z = 0$

2. $|z| \geq |Re(z)|, |z| \geq |Im(z)|$

3. $z\bar{z} = |z|^2$

4. $|z| = |\bar{z}|$

5. $|z| = |-z|$

6. $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}$, where $z_2 \neq 0$

7. $|z_1z_2| = |z_1||z_2|$

8. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ where $z_2 \neq 0$

9. $|z_1 + z_2| \leq |z_1| + |z_2|$ **(Triangular inequality)** (Why triangular ?)

10. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Let us verify some of the above properties :

1. $|z| = 0 \Leftrightarrow \sqrt{a^2+b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0 \Leftrightarrow z = 0$

2. $|z|^2 = a^2 + b^2 = (Re(z))^2 + (Im(z))^2 \geq (Re(z))^2$

$\therefore \quad |z| \geq |Re(z)|$ Similarly, $|z| \geq |Im(z)|$

3. $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$

4. $|z| = |a + ib| = \sqrt{a^2+b^2}$ and $|\bar{z}| = |a - bi| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

so, $|z| = |\bar{z}|$

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7. $|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2})$

$= (z_1z_2)(\overline{z_1}\,\overline{z_2})$

$= (z_1\overline{z_1})(z_2\overline{z_2})$

$= |z_1|^2 |z_2|^2$

$\therefore |z_1z_2| = |z_1||z_2|$

9. $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}}$ $(\overline{z_1\overline{z_2}} = \overline{z_1}\,\overline{\overline{z_2}} = \overline{z_1}z_2)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z_2}|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$= (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$

10. $z_1 - z_2 + z_2 = z_1$

$\therefore\ |z_1 - z_2 + z_2| = |z_1| \leq |z_1 - z_2| + |z_2|$

$\therefore\ |z_1| - |z_2| \leq |z_1 - z_2|$

Similarly, $|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$

But, $|z_1| - |z_2|$ or $|z_2| - |z_1| = \big||z_1| - |z_2|\big|$ (If $a \in$ R then $|a| = a$ or $-a$)

$\therefore\ \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$ or $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$.

Example 5 : Find the conjugate and modulus of (1) $(2 - 3i)^2$ (2) $\frac{-3+7i}{1+i}$

Solution : (1) $(2 - 3i)^2 = 4 - 12i - 9 = -5 - 12i$

$\therefore$ Complex conjugate of $(2 - 3i)^2$ is $-5 + 12i$ and

$|(2 - 3i)^2| = |2 - 3i|^2 = 4 + 9 = 13$

(2) Let $z = \frac{-3+7i}{1+i}$

$= \frac{-3+7i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-3+3i+7i-7i^2}{1-i^2}$

$= \frac{4+10i}{2} = 2 + 5i$

$\therefore\ \overline{z} = 2 - 5i$ and $|z| = \sqrt{2^2+5^2} = \sqrt{29}$

or $|z| = \frac{|-3+7i|}{|1+i|} = \frac{\sqrt{49+9}}{\sqrt{2}} = \sqrt{29}$

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Example 6 : If $z = x + yi$ and $| 3z | = | z - 4 |$, then prove that $x^2 + y^2 + x = 2$.

Solution : We have $| 3z | = | z - 4 |$

$\therefore \quad | 3x + 3yi | = | (x - 4) + yi |$

$\therefore \quad 3\sqrt{x^2 + y^2} = \sqrt{(x-4)^2 + y^2}$

$\therefore \quad 9(x^2 + y^2) = (x - 4)^2 + y^2$

$\therefore \quad 9x^2 + 9y^2 = x^2 - 8x + 16 + y^2$

$\therefore \quad 8x^2 + 8x + 8y^2 = 16$

$\therefore \quad x^2 + y^2 + x = 2$

Example 7 : If $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$, verify the following :

(1) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (2) $| z_1 + z_2 | < | z_1 | + | z_2 |$ (3) $| z_1 z_2 | = | z_1 || z_2 |$

Solution : We have $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$

(1) $z_1 z_2 = (3 + 4i)(12 - 5i) = 36 - 15i + 48i - 20i^2$

$= 36 - 15i + 48i + 20$

$= 56 + 33i$

$\therefore \quad \overline{z_1 z_2} = 56 - 33i$

Now, $\overline{z_1}\,\overline{z_2} = (3 - 4i)(12 + 5i) = 36 - 48i + 15i - 20i^2 = 56 - 33i$

Hence, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ is verified.

(2) $z_1 + z_2 = 3 + 4i + 12 - 5i = 15 - i$

$\therefore \quad | z_1 + z_2 | = \sqrt{225 + 1} = \sqrt{226}$

Also, $| z_1 | = \sqrt{9 + 16} = 5$, $| z_2 | = \sqrt{144 + 25} = 13$

Also, $| z_1 | + | z_2 | = 5 + 13 = 18 = \sqrt{324}$

Clearly, $\sqrt{226} < \sqrt{324}$

Hence, $| z_1 + z_2 | < | z_1 | + | z_2 |$ is verified.

(3) $| z_1 z_2 | = \sqrt{56^2 + 33^2} = \sqrt{3136 + 1089} = \sqrt{4225} = 65$ **(by (1))**

Also, $| z_1 || z_2 | = 5 \cdot 13 = 65$

Hence, $| z_1 z_2 | = | z_1 || z_2 |$ is verified.

Example 8 : (1) If $z \in C$ and $| z + 3 | \le 8$, find the maximum and minimum values of $| z - 2 |$.

(2) If $z \in C$ and $| z - 4 | \le 4$, find the maximum and minimum values of $| z + 1 |$.

Solution : (1) We have $| z + 3 | \le 8$

$| z - 2 | = | (z + 3) - 5 | \le | z + 3 | + | -5 |$ **(Triangular Inequality)**

$\le 8 + 5 = 13$

$\therefore \quad | z - 2 | \le 13$

If we take $z = -11$ then $|z + 3| = |-11 + 3| = 8$ and $|z - 2| = 13$

So the maximum value of $|z - 2|$ subject to $|z + 3| \leq 8$ is 13.

Now, $|z - 2| \geq 0$ is always true.

For $z = 2$, $|z + 3| \leq 8$ is true and $|z - 2| = 0$.

So the minimum value of $|z - 2|$ subject to $|z + 3| \leq 8$ is 0.

(2) We have $|z - 4| \leq 4$

$|z + 1| = |(z - 4) + 5| \leq |z - 4| + |5|$ **(Triangular Inequality)**

$\leq 4 + 5 = 9$

$\therefore \ |z + 1| \leq 9$

If we take $z = 8$ then $|z - 4| = 4$ and $|z + 1| = 9$.

So the maximum value of $|z + 1|$ subject to $|z - 4| \leq 4$ is 9.

$|z + 1| \geq 0$. If we let $z = -1$, $|z + 1|$ would be zero.

But, $|z - 4| = |-1 - 4| = 5 \nleq 4$.

Thus the condition $|z - 4| \leq 4$ is violated if $z = -1$.

Now, $|z + 1| = |(z - 4) + 5| = |(z - 4) - (-5)| \geq ||z - 4| - |-5||$

$(|z_1 - z_2| \geq ||z_1| - |z_2||)$

$\geq 5 - 4 = 1$

$\therefore \ |z + 1| \geq 1$

If we take $z = 0$ then $|z - 4| = 4$ and $|z + 1| = 1$.

So, the minimum value of $|z + 1|$ subject to $|z - 4| \leq 4$ is 1.

Example 9 : If $z (\neq -1)$ is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then show that $|z| = 1$.

Solution : Let $z = x + iy$.

Then $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$

Since $\frac{z-1}{z+1}$ is purely imaginary, we have $Re\left(\frac{z-1}{z+1}\right) = 0$

$\therefore \ \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0$

$\therefore \ x^2 + y^2 = 1$

$\therefore \ |z| = 1$ $\left(|z| = \sqrt{x^2 + y^2}\right)$

2.7 Argand Plane and Polar representation

Historically, the geometric representation of a complex number as a point in the plane is useful because it relates the whole idea of a complex number as an ordered pair in R^2. We know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa. (Figure 2.1)

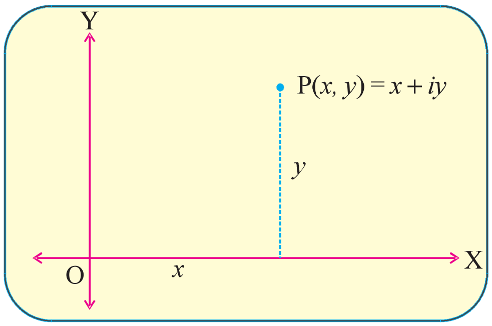

Figure 2.1

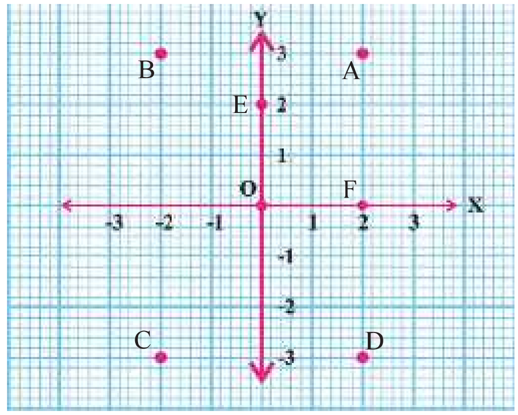

Figure 2.2

Some complex numbers such as $2 + 3i$, $-2 + 3i$, $-2 - 3i$, $2 - 3i$, $0 + 2i$, $2 + i0$ which correspond to the ordered pairs (2, 3), (−2, 3), (−2, −3), (2, −3), (0, 2), (2, 0) are represented geometrically by the points A, B, C, D, E, F respectively in the figure 2.2.

The plane having a complex number assigned to each of its point is called the **Complex Plane** or the **Argand Plane**. The points on the x-axis correspond to the complex numbers of the form $a + i0$ (real numbers) and the points on the y-axis correspond to the complex numbers of the form $0 + ib$ (purely imaginary numbers). The X-axis and Y-axis in the Argand plane are called the real axis and the imaginary axis respectively.

(**Jean-Robert Argand** (1768 – 1822) was a gifted amateur mathematician. In 1806, while managing a bookstore in Paris, he published the idea of geometrical interpretation of complex numbers known as the Argand diagram.)

Geometrical representation of modulus of a complex number :

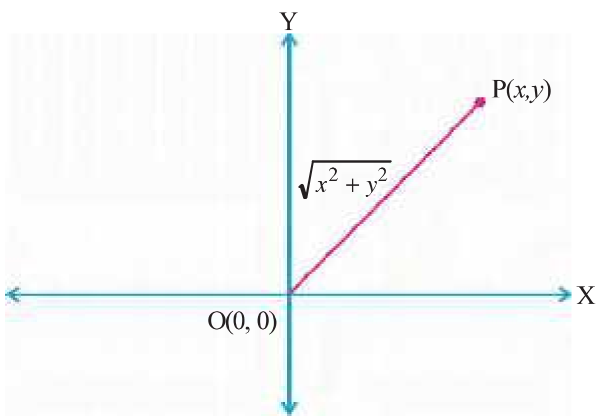

Figure 2.3

In the Argand plane, the modulus of the complex number $x + iy$ is the distance between the point P(x,y) and the origin O(0,0). (Figure 2.3)

Geometrical representation of the conjugate of a complex number :

The representations of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ in the Argand plane are the points P(x,y) and Q($x,-y$) respectively. Geometrically, the point Q($x,-y$) is called the **mirror image** of the point P(x,y) with respect to the real axis. (Figure 2.4)

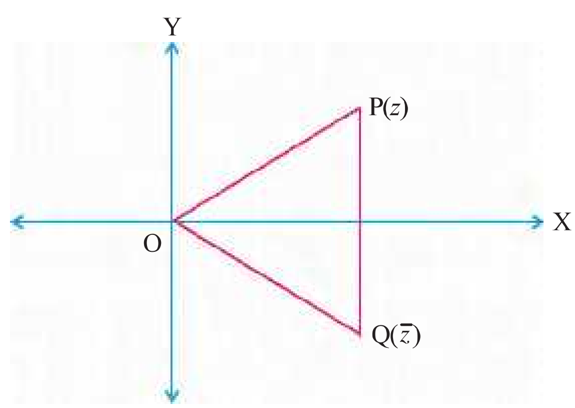

Figure 2.4

Downloaded from https:// www.studiestoday.com

Geometrical representation of the sum of two complex numbers :

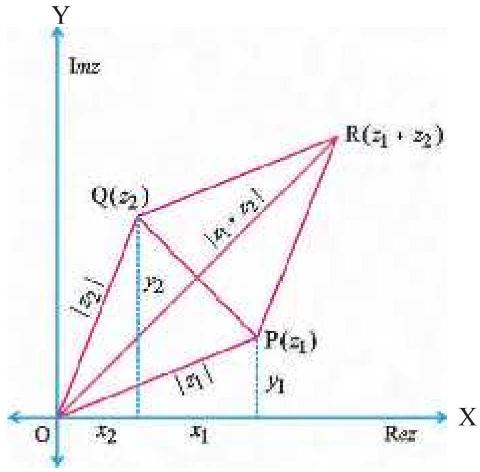

Figure 2.5

From the figure 2.5, in the argand plane P, Q and R represent z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Mid-point of $\overline{OR}$ and $\overline{PQ}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$\therefore$ $\overline{OR}$ and $\overline{PQ}$ bisect each other.

Here, we have assumed that O, P and Q are non-collinear points.

The absolute values of z_1, z_2 and $z_1 + z_2$ are geometrically given by $|z_1| = OP$, $|z_2| = OQ = PR$ and $|z_1 + z_2| = OR$. We know that the sum of any two sides of a triangle is greater than the third side. Hence, in ΔORP, we have $OR < OP + PR$ implying $|z_1 + z_2| < |z_1| + |z_2|$. That is why this inequality for the absolute values of complex numbers is called the triangular inequality. (When does equality occur in $|z_1 + z_2| \leq |z_1| + |z_2|$?)

Polar representation of a complex number :

There is an alternate form to represent a complex number $z = x + iy$ which is known as polar representation. Let us understand how we can express any complex number into polar form. Let $z = x + iy$ be a non-zero complex number represented by the point P(*x*, *y*). (Figure 2.6) Draw $\overline{PM} \perp \overleftrightarrow{OX}$. Then $OM = x$ and $PM = y$. Draw $\overline{OP}$. Let $OP = r$ and $m\angle MOP = \theta$. Then $x = r\cos\theta$ and $y = r\sin\theta$.

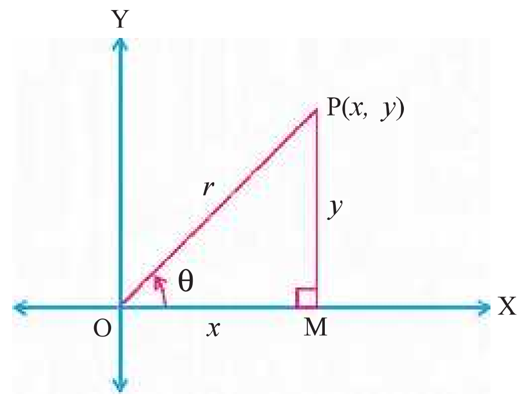

Figure 2.6

Therefore $z = x + iy = r(\cos\theta + i\sin\theta)$

Note : Here P lies in the first-quadrant.

$\therefore$ $x > 0$, $y > 0$. But if P(*x*, *y*) lies anywhere in the Argand plane except for origin, then also $x = r\cos\theta$, $y = r\sin\theta$ are true.

$\therefore$ $z = x + iy = r(\cos\theta + i\sin\theta)$

Here, $r^2 = x^2 + y^2$ **(r = OP > 0)**

$\therefore$ $r = \sqrt{x^2 + y^2}$ **(r > 0)**

$\therefore$ $r = \sqrt{x^2 + y^2} = |z|$ and $\tan\theta = \frac{y}{x}$

The form $z = r(\cos\theta + i\sin\theta)$ is called the **polar form** of the complex number *z*. Also θ is known as **amplitude** or **argument of *z***, written as **arg(*z*)**. Since *sine* and *cosine* functions are periodic, there are many values of θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Each of these θ is an argument of *z*. The unique value of θ such that $-\pi < \theta \leq \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$ is known as the

principal value of arg(z). While reducing a complex number to polar form, we always take the principal value of arg(z). Unless specified the notation arg(z) means principal value of arg(z). To find the value of arg(z), one has to take care of the position of the point in the plane. **Argument of the complex number 0 is not defined (Why ?)**

$$arg(x + i0) = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases} \qquad arg(0 + iy) = \begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ -\frac{\pi}{2}, & \text{if } y < 0 \end{cases}$$

∴ Argument of positive real number is 0 and that of negative real number is π. Similarly argument of purely imaginary number yi is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as $y > 0$ or $y < 0$ respectively.

Also, $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$ and $-\pi < \theta \le \pi$.

(i) If $x > 0$, $y > 0$, then we can get θ, $0 < \theta < \frac{\pi}{2}$, such that $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

(ii) If $x < 0$, $y > 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = \pi - \alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

(iii) If $x < 0$, $y < 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = -\pi + \alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$

(iv) If $x > 0$, $y < 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = -\alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

Example 10 : Write the following complex numbers in polar form. Determine the modulus and the principal value of the argument in each case :

(1) $1 + i$ (2) $-1 + \sqrt{3}i$ (3) $-\sqrt{3} - i$ (4) $1 - i$

(5) -3 (6) $-2i$ (7) 1 (8) $2i$

Solution : (1) Let $z = 1 + i = x + iy$

∴ $x = 1$, $y = 1$

∴ $r = \sqrt{x^2 + y^2} = \sqrt{2}$

$cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$ and $sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$

∴ P(θ) lies in the first quadrant.

∴ $\theta = \frac{\pi}{4}$

∴ The polar form of z is $\sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$.

$|z| = r = \sqrt{2}$, $arg\, z = \theta = \frac{\pi}{4}$.

(2) Let $z = -1 + \sqrt{3}i = x + iy$

∴ $x = -1$, $y = \sqrt{3}$

∴ $r = |z| = \sqrt{1+3} = 2$

$cos\theta = \frac{-1}{r} = \frac{-1}{2}$ and $sin\theta = \frac{\sqrt{3}}{r} = \frac{\sqrt{3}}{2}$

$\therefore \quad cos\alpha = \frac{1}{2},\ sin\alpha = \frac{\sqrt{3}}{2}$ **($|x| = 1,\ |y| = \sqrt{3}$)**

$\therefore \quad \alpha = \frac{\pi}{3}$

Since $x < 0$, $y > 0$, P(θ) lies in the second quadrant.

$\therefore \quad \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ The polar form of z is $2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$.

Also, $|z| = r = 2$, $arg\ z = \theta = \frac{2\pi}{3}$.

(3) Let $z = -\sqrt{3} - i = x + iy$

$\therefore \quad x = -\sqrt{3},\ y = -1$

$\therefore \quad r = |z| = \sqrt{3+1} = 2$

$cos\theta = \frac{-\sqrt{3}}{2}$ and $sin\theta = \frac{-1}{2}$

$\therefore \quad cos\alpha = \frac{\sqrt{3}}{2},\ sin\alpha = \frac{1}{2}$ **($|x| = \sqrt{3},\ |y| = 1$)**

$\therefore \quad \alpha = \frac{\pi}{6}$

Since $x < 0$, $y < 0$, P(θ) lies in the third quadrant.

$\therefore \quad \theta = -\pi + \alpha = -\pi + \frac{\pi}{6} = \frac{-5\pi}{6}$

$\therefore$ The polar form of z is $2\left(cos\left(-\frac{5\pi}{6}\right) + isin\left(-\frac{5\pi}{6}\right)\right)$.

Also, $|z| = r = 2$, $arg\ z = \theta = \frac{-5\pi}{6}$.

(4) Let $z = 1 - i = x + iy$

$\therefore \quad x = 1,\ y = -1$

$\therefore \quad r = |z| = \sqrt{1+1} = \sqrt{2}$

$cos\theta = \frac{1}{r} = \frac{1}{\sqrt{2}}$ and $sin\theta = \frac{-1}{r} = \frac{-1}{\sqrt{2}}$

$\therefore \quad cos\alpha = \frac{1}{\sqrt{2}},\ sin\alpha = \frac{1}{\sqrt{2}}$ **($|x| = 1,\ |y| = 1$)**

$\therefore \quad \alpha = \frac{\pi}{4}$

Since $x > 0$, $y < 0$, P(θ) lies in the fourth quadrant.

$\therefore \quad \theta = -\alpha = -\frac{\pi}{4}$

$\therefore$ The polar form of z is $\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right)$.

Also, $|z| = r = \sqrt{2}$, $arg\ z = \theta = -\frac{\pi}{4}$.

(5) Let $z = -3$. Here $z = x + i0$ and $x < 0$.

$\therefore$ Its polar form is $3(cos\pi + i\,sin\pi)$

Also, $|z| = 3$, $arg\ z = \theta = \pi$.

(6) Let $z = -2i$. Here $z = 0 + iy$ and $y < 0$.

$\therefore$ Its polar form is $2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$.

Also, $|z| = 2$, $\arg z = \theta = -\frac{\pi}{2}$.

(7) Let $z = 1$. Here $z = x + i0$ and $x > 0$. So Its polar form is $1(\cos 0 + i\sin 0)$.

Also, $|z| = 1$, $\arg z = \theta = 0$.

(8) Let $z = 2i$. Here $z = 0 + iy$ and $y > 0$. So its polar form is $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Also, $|z| = 2$, $\arg z = \theta = \frac{\pi}{2}$.

Exercise 2.2

1. Find the absolute value and the principal argument of the following complex numbers :

(1) $\frac{1+7i}{(2-i)^2}$ (2) $\left(\frac{2+i}{3-i}\right)^2$ (3) $\sqrt{3} - i$ (4) $\frac{(1+i)(1+\sqrt{3}i)}{1-i}$ (5) $-3\sqrt{2} + 3\sqrt{2}\,i$

2. If $z = 3 + 2i$, then verify the following :

(1) $|z| = |\bar{z}|$ (2) $-|z| \le Re(z) \le |z|$ (3) $z^{-1} = \frac{\bar{z}}{|z|^2}$

3. If $z_1 = 3 + 2i$ and $z_2 = 2 - i$, then verify the following :

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

4. If z is a non-zero complex number, show that $\overline{(z^{-1})} = (\bar{z})^{-1}$.

5. If $(a + ib)^2 = \frac{1+i}{1-i}$, show that $a^2 + b^2 = 1$.

6. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, then is it necessary that $z_1 = z_2$? Justify your answer.

7. A complex number $z = a + ib$ is such that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$. Show that $a^2 + b^2 - 2b = 1$.

8. Find the maximum value of $|1 + z + z^2 + z^3|$, if $z \in$ C and $|z| \le 3$.

9. (1) If $z = a + ib$ and $2|z - 1| = |z - 2|$, prove that $3(a^2 + b^2) = 4a$.

(2) If $z \in$ C such that $|2z - 3| = |3z - 2|$, prove that $|z| = 1$.

(3) If $z \in$ C such that $|2z - 1| = |z - 2|$, prove that $|z| = 1$.

10. Show that complex number $-3 + 2i$ is closer to the origin than $1 + 4i$.

11. Represent the points $-2 + 3i$, $-2 - i$ and $4 - i$ in the Argand diagram and prove that they are vertices of a right angled triangle.

12. Find the complex number z whose modulus is 4 and argument is $\frac{5\pi}{6}$.

13. If $(1 - 5i)z_1 - 2z_2 = 3 - 7i$, find z_1 and z_2, where z_1 and z_2 are conjugate complex numbers.

14. If $(a + ib)^2 = x + iy$ prove that $x^2 + y^2 = (a^2 + b^2)^2$.

15. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

*

2.8 Square Roots of a Complex Number

If $(a + ib)^2 = z = x + iy$, we say that $a + ib$ is a square root of z.

Let $z = x + iy$ and let a square root of z be the complex number $a + ib$, if it exists.

$\therefore \quad x + iy = (a + ib)^2$

$\therefore \quad x + iy = (a^2 - b^2) + (2ab)i$

$\therefore \quad a^2 - b^2 = x$ and $2ab = y$ **(i)**

Now, $a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = \sqrt{x^2 + y^2} = |z|$ **(by (i)) (ii)**

From (i) and (ii) we get $2a^2 = |z| + x$ i.e. $a = \pm\sqrt{\frac{|z| + x}{2}}$ and $b = \pm\sqrt{\frac{|z| - x}{2}}$

If $y > 0$, then a and b both positive or both negative as $y = 2ab$.

Therefore, the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right)$.

If $y < 0$, then out of a and b, one is positive and another is negative.

Therefore, the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right)$.

Now, we have proved that every complex number has two square roots.

Example 11 : Find the square roots of (1) $\sqrt{3} - i$ (2) $7 + 24i$

Solution : (1) Let $z = \sqrt{3} - i$. Here $x = \sqrt{3}$, $y = -1 < 0$

$|z| = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$

We know that if $y < 0$, then the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right)$.

Hence the square roots of $\sqrt{3} - i$ are $\pm\left(\sqrt{\frac{2 + \sqrt{3}}{2}} - i\sqrt{\frac{2 - \sqrt{3}}{2}}\right)$.

Now $2 + \sqrt{3} = \frac{4 + 2\sqrt{3}}{2} = \frac{(\sqrt{3} + 1)^2}{2}$

$\therefore$ The square roots of $z = \sqrt{3} - i$ are $\pm\left(\frac{\sqrt{3} + 1}{2} - i\frac{\sqrt{3} - 1}{2}\right)$. $(\sqrt{2}\sqrt{2} = 2)$

(2) Let $z = 7 + 24i$. Here $x = 7$, $y = 24 > 0$

$|z| = \sqrt{x^2 + y^2} = \sqrt{49 + 576} = 25$

We know that if $y > 0$, then the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right)$.

Hence the square roots of $7 + 24i$ are $\pm\left(\sqrt{\frac{25 + 7}{2}} + i\sqrt{\frac{25 - 7}{2}}\right) = \pm(4 + 3i)$.

Example 12 : Find the square roots of (1) 1 (2) -1 (3) i (4) $-i$

(1) Let $z = 1$

$\therefore \quad |z| = 1$. Let the square roots of z be $a + ib$.

$\therefore \quad (a + ib)^2 = 1$

$\therefore \quad a^2 - b^2 + 2abi = 1 = 1 + 0i$

$\therefore$ $a^2 - b^2 = 1$, $2ab = 0$. From $2ab = 0$ we have $a = 0$ or $b = 0$.

From $a = 0$, we have $-b^2 = 1$ which is not possible as $b \in$ R. So $a \neq 0$.

$\therefore$ $2ab = 0$ gives $b = 0$

$\therefore$ $a^2 = 1$

$\therefore$ $a = \pm 1$

$\therefore$ $a + ib = \pm 1$

$\therefore$ Square roots of 1 are ± 1.

Note : In R, we know square roots of 1 are ± 1.

(2) Let $z = -1$. Let the square root of z be $a + ib$.

$\therefore$ $(a + ib)^2 = -1$

$\therefore$ $a^2 - b^2 + 2abi = -1$

$\therefore$ $a^2 - b^2 = -1$, $2ab = 0$

$2ab = 0$ gives $a = 0$ or $b = 0$

But $b = 0$ gives $a^2 = -1$ which is not possible as $a \in$ R. So $b \neq 0$.

$\therefore$ $a = 0$ and $b^2 = 1$

$\therefore$ $b = \pm 1$

$\therefore$ Square roots of -1 are $\pm i$. (as we expected since $i^2 = -1$)

Remember $i^2 = -1$.

Similarly the square roots of -4 are $\pm 2i$,

the square roots of -3 are $\pm\sqrt{3}\,i$.

(3) Let $z = a + ib$ be a square root of i.

$\therefore$ $(a + ib)^2 = i$

$\therefore$ $a^2 - b^2 + 2iab = i$

$\therefore$ $a^2 - b^2 = 0$ and $2ab = 1$

$\therefore$ $a = b$ or $a = -b$

But $a = -b$ gives $-2a^2 = 1$ using $2ab = 1$.

This is not possible.

$\therefore$ $a = b$ and $2a^2 = 1$

$\therefore$ $a = \pm\frac{1}{\sqrt{2}}$. Since $a = b$ we have $b = \pm\frac{1}{\sqrt{2}}$.

$\therefore$ Square roots of i are $\pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$.

(4) Let $z = -i$. From (3) above $a^2 - b^2 = 0$, $2ab = -1$

$\therefore$ $a = b$ or $a = -b$

If $a = b$, then $2a^2 = -1$ which is not possible.

$\therefore$ $a = -b$ and $2a^2 = 1$

$\therefore$ $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ and $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

$\therefore$ The square roots of $-i$ are $\pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$.

2.9 Quadratic Equations having Complex Roots

We have studied quadratic equations and solved them in the set of real numbers when the value of discriminant is non-negative. i.e. when D $\geq$ 0. Now we can answer the unanswered question, 'What happens when D < 0 ?'

Now let us try to solve quadratic equation $ax^2 + bx + c = 0$, $a, b\ c \in R$, $a \neq 0$, where $D = b^2 - 4ac < 0$.

$$ax^2 + bx + c = \frac{1}{a}(a^2x^2 + abx + ac)$$

$$= \frac{1}{a}\left[\left(ax+\frac{b}{2}\right)^2 + ac - \frac{b^2}{4}\right]$$

$$= \frac{1}{a}\left[\left(ax+\frac{b}{2}\right)^2 + \frac{4ac-b^2}{4}\right]$$

If $ax^2 + bx + c = 0$, then $\left(ax+\frac{b}{2}\right)^2 = \frac{b^2-4ac}{4}$.

Now, $b^2 - 4ac < 0$

$\therefore$ Square root of $ax + \frac{b}{2}$ = Square root of $\frac{b^2-4ac}{4}$ that is $\frac{\pm i\sqrt{4ac-b^2}}{2}$.

$\therefore \quad ax + \frac{b}{2} = \frac{\pm i\sqrt{4ac-b^2}}{2}$

$\therefore \quad x = \frac{-b \pm i\sqrt{4ac-b^2}}{2a}$ **($a \neq 0$)**

If D < 0, roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm i\sqrt{-D}}{2a}$.

Fundamental Theorem of Algebra :

Every polynomial equation having complex coefficients and degree $\geq$ 1 has at least one complex root.

Example 13 : Solve (1) $x^2 + 3 = 0$ (2) $2x^2 + x + 1 = 0$ (3) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Solution :

(1) $x^2 + 3 = 0$

$\therefore \quad x^2 = -3$

$\therefore \quad x = \pm\sqrt{3}i$

(2) Here, $a = 2$, $b = 1$, $c = 1$

$\therefore \quad b^2 - 4ac = 1 - 4 \cdot 2 \cdot 1 = -7 < 0$

Therefore, the solutions are given by $x = \frac{-b \pm i\sqrt{-D}}{2a} = \frac{-1 \pm \sqrt{7}i}{4}$

(3) Here, $a = \sqrt{3}$, $b = -\sqrt{2}$, $c = 3\sqrt{3}$

$\therefore \quad b^2 - 4ac = 2 - 4\sqrt{3} \cdot 3\sqrt{3} = 2 - 36 = -34 < 0$

Therefore, the solutions are given by $x = \frac{-b \pm i\sqrt{-D}}{2a} = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}} = \frac{1 \pm \sqrt{17}i}{\sqrt{6}}$

2.10 Cube Roots of Unity

Let z be a cube roots of unity.

Then, $z^3 = 1$

$\therefore \quad z^3 - 1 = 0$

$\therefore \quad (z - 1)(z^2 + z + 1) = 0$

$\therefore \quad z = 1$ or $z^2 + z + 1 = 0$

$\therefore \quad z = 1, \frac{-1 \pm \sqrt{3}i}{2}$ **($a = 1, b = 1, c = 1, D = -3$)**

Hence, the cube roots of unity are $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$.

Properties of Cube Roots of Unity :

(1) Each of the two non-real cube roots of unity is the square of each other.

Let $\omega = \frac{-1+\sqrt{3}i}{2}$. Then $\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{1}{4}(1 - 2\sqrt{3}i + 3i^2) = \frac{-1-\sqrt{3}i}{2}$

Also, $(\omega^2)^2 = \omega^4 = \omega^3\omega = \omega$. Hence cube roots of unity are $1, \omega, \omega^2$.

(2) We observe that sum of the cube roots of unity is 0. i.e. $1 + \omega + \omega^2 = 0$

(3) It can easily verify that product of cube roots of unity is 1. i.e. $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

(4) Representing $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$ in the Argand plane as A, B, C respectively then A is (1, 0), B is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and C is $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Note that AB = BC = AC = $\sqrt{3}$. Thus A, B, C are the vertices of an equilateral triangle. (Figure 2.7)

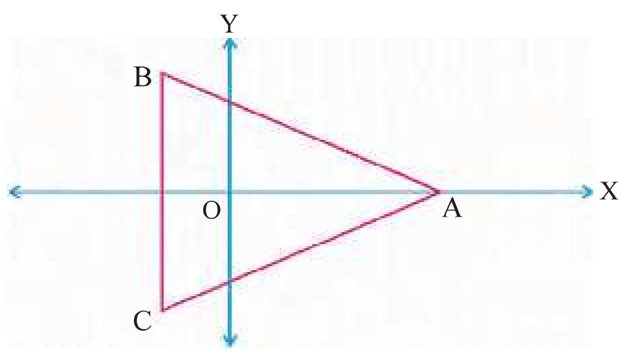

Figure 2.7

Exercise 2.3

1. Solve :

(1) $x^2 + 2 = 0$ (2) $x^2 + x + 1 = 0$ (3) $\sqrt{5}x^2 + x + \sqrt{5} = 0$

(4) $x^2 + x + \frac{1}{\sqrt{2}} = 0$ (5) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$ (6) $3x^2 - 4x + \frac{20}{3} = 0$

2. Find the square roots of :

(1) $4 + 4\sqrt{3}i$ (2) $5 - 12i$ (3) $-48 + 14i$ (4) $3 - 4\sqrt{10}i$

(5) $\frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4} + \frac{1}{i^5} + \frac{1}{i^6}$ (6) $4i$ (7) $-16i$ (8) -25 (9) -10

3. When do we have $|z_1 + z_2| = |z_1| + |z_2|$? Prove your contention.

4. Prove that in the Argand plane if P represents z and Q represents iz, then OP = OQ and $m\angle POQ = \frac{\pi}{2}$. State geometrical meaning.

5. Prove points representing $z, iz, -z$ and $-iz$ in Argand plane form a square.

6. What is the relation between representation of z and $\bar{z}$ in the Argand plane ?

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Miscellaneous Problems :

Example 14 : Find all the complex numbers z satisfying the condition $\bar{z} = z^2$.

Solution : Let $z = x + iy$ be such that $\bar{z} = z^2$.

$\therefore \quad x - iy = (x^2 - y^2) + i(2xy)$

By definition of equality of complex numbers, we have $x = x^2 - y^2$ and $-y = 2xy$.

From the second result we have either $y = 0$ or $x = -\frac{1}{2}$.

Assume first $y = 0$. Then from $x = x^2 - y^2$, we have $x = x^2$

$\therefore \quad x = 0$ or $x = 1$ **($y = 0$)**

So in this case $z = 0$ or $z = 1$

Now, if $x = -\frac{1}{2}$, then $-\frac{1}{2} = \frac{1}{4} - y^2$ **($x = x^2 - y^2$)**

$\therefore \quad y = \pm\frac{\sqrt{3}}{2}$

So in the second case $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ or $z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Consequently, there are four complex numbers $0, 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ satisfying the equation $\bar{z} = z^2$.

Example 15 : Find real θ such that $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is real. Also find the number.

Solution : We have, $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} = \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \times \frac{1 + 2i\sin\theta}{1 + 2i\sin\theta}$

$$= \frac{3 + 6i\sin\theta + 2i\sin\theta + 4i^2\sin^2\theta}{1 + 4\sin^2\theta}$$

$$= \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} + i\frac{8\sin\theta}{1 + 4\sin^2\theta}$$

If the given complex number is real, its imaginary part is zero.

Therefore, $\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0$

$\therefore \quad \sin\theta = 0$

$\therefore \quad \theta = k\pi, k \in Z$

This number is $\frac{3 + 0}{1 - 0} = 3$

Exercise 2

1. Reduce : (1) $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$ (2) $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

2. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

3. For any two complex numbers z_1 and z_2, prove that $Re(z_1z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$.

4. Find the value of $Re(f(z))$ and $Im(f(z))$ for $f(z) = \frac{1}{1-z}$ at $z = 7 + 2i$.

5. Show that the point set of the equation $|z - 1| = |z + i|$ represents a line through the origin whose slope is -1.

6. Prove that $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.

7. If z_1 and z_2 are distinct complex numbers with $|z_2| = 1$, then find the value of $\left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|$.

8. If $\frac{1}{\alpha + i\beta} + \frac{1}{a + ib} = 1$, where α, β, a and b real, express b in terms of α and β.

9. If $(x + iy)^3 = a + ib$, prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$.

10. Solve : (1) $x^2 - 2x + \frac{3}{2} = 0$ (2) $27x^2 - 10x + 1 = 0$ (3) $21x^2 - 28x + 10 = 0$

11. If $z \in C$ and $|z| \le 2$, find the maximum and minimum values of $|z - 3|$.

12. For $z = 3 - 2i$ show that $z^2 - 6z + 13 = 0$. Hence obtain the value of $z^4 - 4z^3 + 6z^2 - 4z + 17$.

13. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

14. If $(x - iy)^2 = \frac{a - ib}{c - id}$, prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$

15. Find the value of z which satisfies the equation $|z| - z = 1 + 2i$.

16. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then show that $z_1 + z_2 + z_3 = 0$.

17. Show that the area of the triangle in the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\frac{1}{2}|z|^2$.

18. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that $|w| = 1 \Rightarrow z$ is real.

19. If $z = -5 + 4i$, show that $z^4 + 9z^3 + 35z^2 - z + 164 = 0$.

20. If $z = x + iy$, prove that $|x| + |y| \le \sqrt{2}|z|$.

21. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Solution of $|z - 4| < |z - 2|$ is given by ... ☐

(a) $Re(z) > 0$ (b) $Re(z) < 0$ (c) $Re(z) > 3$ (d) $Re(z) > 2$

(2) If $|z - 1|^2 = |z|^2 + 1$, then z lies on...... in the Argand diagram. ☐

(a) $x^2 + y^2 = 1$ (b) the imaginary axis

(c) the real axis (d) $2x + 3 = 0$

(3) If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is ... ☐

(a) 6 (b) 0 (c) 4 (d) 10

(4) The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is ... ☐

(a) $\frac{1}{i-1}$ (b) $\frac{-1}{i-1}$ (c) $\frac{1}{i+1}$ (d) $\frac{-1}{i+1}$

(5) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to ... ☐

(a) 1 (b) -1 (c) 0 (d) i^n

(6) The multiplicative inverse of $\frac{3+4i}{4-5i}$ is ...

(a) $-\frac{8}{25} + \frac{31}{25}i$ (b) $\frac{8}{25} - \frac{31}{25}i$ (c) $-\frac{8}{25} - \frac{31}{25}i$ (d) $\frac{8}{25} + \frac{31}{25}i$

(7) If $x + iy = \frac{u+iv}{u-iv}$, then $x^2 + y^2 =$ ....

(a) 1 (b) -1 (c) 0 (d) 2

(8) The smallest positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$ is ...

(a) 4 (b) 8 (c) 2 (d) 12

(9) On the Argand plane the complex number $\frac{1+2i}{1-i}$ lies in the ...... quadrant.

(a) first (b) second (c) third (d) fourth

(10) $arg(-1) =$ ......

(a) 0 (b) π (c) $\frac{\pi}{2}$ (d) $-\pi$

(11) The complex numbers $sinx + icos2x$ and $cosx - isin2x$ are conjugate of each other, for ...

(a) $x = k\pi,\ k \in Z$ (b) $x = 0$

(c) $x = \left(k+\frac{1}{2}\right)\pi,\ k \in Z$ (d) no value of x

(12) If a complex number lies in the third quadrant, then its conjugate lies in the ...... quadrant.

(a) first (b) second (c) third (d) fourth

(13) The complex number with modulus 2 and argument $\frac{2\pi}{3}$ is ...

(a) $-1 + i\sqrt{3}$ (b) $-1 - i\sqrt{3}$ (c) $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (d) $\frac{1}{2} - \frac{i\sqrt{3}}{2}$

(14) Argument of $1 - i\sqrt{3}$ is ...

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

(15) If the cube roots of unity are $1, \omega, \omega^2$, then $1 + \omega + \omega^2 =$ ......

(a) 1 (b) 0 (c) -1 (d) ω

*

Summary

We studied following points in this chapter :

1. A number of the form $a + ib$, where a and b are real numbers, is called a complex number where $i^2 = -1$.

2. Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers.

$z_1 + z_2 = (a + c) + i(b + d)$, $z_1z_2 = (ac - bd) + i(ad + bc)$

3. $(a + ib)(a - ib) = a^2 + b^2$

4. Multiplicative inverse of a non-zero complex number $z = a + ib$ is $\frac{1}{z} = z^{-1} = \frac{a}{a^2+b^2} + \frac{-bi}{a^2+b^2}$.

5. $i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i$

6. For complex number $z = a + bi$, its complex conjugate is $\bar{z} = a - bi$.

7. Modulus of a complex number $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

8. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

9. Square roots of $x + iy$ are $\begin{cases} \pm\left(\sqrt{\dfrac{|z|+x}{2}} + i\sqrt{\dfrac{|z|-x}{2}}\right), & y > 0 \\ \pm\left(\sqrt{\dfrac{|z|+x}{2}} - i\sqrt{\dfrac{|z|-x}{2}}\right), & y < 0 \end{cases}$

10. The cube roots of unity are 1, $\omega = \dfrac{-1+\sqrt{3}i}{2}$, $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$

11. If $b^2 - 4ac < 0$, the solutions of $ax^2 + bx + c = 0$ where $a, b, c \in R$, $a \neq 0$ are $\dfrac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$.

— ❖ —

Brahmagupta was the first to use zero as a number. He gave rules to compute with zero. Negative numbers did not appear in *Brahmaphuta siddhanta* but in the Nine Chapters on the Mathematical Art (Jiu zhang suan-shu) around 200 BC. Brahmagupta's most famous work is his *Brahmasphutasiddhanta*.

Brahmagupta gave the solution of the general linear equation in chapter eighteen of *Brahmasphutasiddhanta*.

The difference between *rupas*, when inverted and divided by the difference of the unknowns, is the unknown in the equation. The *rupas* are [subtracted on the side] below that from which the square and the unknown are to be subtracted which is a solution equivalent to $x = \dfrac{e-c}{b-d}$, where *rupas* represents constants. He further gave two equivalent solutions to the general quadratic equation.

Diminish by the middle [number] the square root of the *rupas* multiplied by four times the square and increased by the square of the middle; divide the remainder by twice the square. the middle.

Whatever is the square root of the *rupas* multiplied by the square [and] increased by the square of half the unknown, diminish that by half the unknown [and] divide [the remainder] by its square. [The result is] the unknown which are, respectively, solutions equivalent to, $x = \dfrac{\sqrt{4ac+b^2} - b}{2a}$.

Brahmagupta then goes on to give the sum of the squares and cubes of the first n integers.

The sum of the squares is that [sum] multiplied by twice the [number of] step[s] increased by one [and] divided by three. The sum of the cubes is the square of that [sum] Piles of these with identical balls [can also be computed].

It is important to note here Brahmagupta found the result in terms of the sum of the first n integers.

He gives the sum of the squares of the first n natural numbers as $n(n + 1)(2n + 1)/6$ and the sum of the cubes of the first n natural numbers as $\left(\dfrac{n(n+1)}{2}\right)^2$.

Chapter **3**

BINOMIAL THEOREM

The laws of nature are but the mathematical thoughts of God.

– Euclid

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I like mathematics because it is not human and has nothing particular to do with this planet or with the whole accidental universe, because like Spinoza's God, it won't love us in return.

– Bertrand Russell

*

If there is God, he is a great mathematician.

– Paul Dirac

3.1 Introduction

In earlier classes, we have learnt about expansions like,

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and even $(a + b)^4$ as a product of $(a + b)^3$ with $(a + b)$

i.e. $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

However, the expansions of $(a + b)^5$, $(a + b)^6$, ... become difficult by using multiplication.

It is believed that in the eleventh century, Persian poet and mathematician **Omar Khayyám** gave the general formula for $(a + b)^n$, where n is a positive integer. This formula or expansion is called the **Binomial Theorem.**

Euclid (Fourth B.C.) a **Greek mathematician** gave a specific example of Binomial Expansion for $n = 2$. An **Indian mathematician Pingla** (Third Century B.C.) had given the idea about the higher order expansions. In the tenth century an **Indian mathematician Halayadha** was aware of general binomial theorem and Pascal's Triangle. Persian mathematician Al-Karaji and in 13th century Chinese mathematician Yang hui have also obtained such results.

The coefficients of the consecutive terms in the expansion of $(a + b)^n$, for $n = 1, 2, 3, ...$ can also be obtained from a row from triangular arrangement of numbers, known as **Pascal's Triangle** named after **French mathematician Blaise Pascal** (1623-1662).

Index	Coefficients
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
.	. . . .
.	. . . .
.	. . . .

In Pascal's Triangle first and last element of any row is 1, while the other elements are obtained by adding the numbers of the upper row which are at the beginning of the arrows.

Pascal's Triangle : The first row is 1 1

i.e. $\binom{1}{0}$ $\binom{1}{1}$

The second row is 1 2 1

Here the first and last entry is 1 and the middle term is obtained as sum of the two terms of 1st row, because $\binom{1}{0} + \binom{1}{1} = \binom{2}{1}$ $\left(\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}\right)$.

Similarly, the third row is 1 3 3 1, the first and last term is 1, the second term is obtained as the sum of 1st and 2nd term of 2nd row i.e. $1 + 2 = 3$ as $\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$ and 3rd term is obtained as the sum of 2nd and 3rd terms of 2nd row i.e. $2 + 1 = 3$ as $\binom{2}{1} + \binom{2}{2} = \binom{3}{2}$.

In the same manner, let us check 5th row in the light of above discussion.

4th row : 1 4 6 4 1

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

the 5th row : 1 (1 + 4) (4 + 6) (6 + 4) (4 + 1) 1

1 5 10 10 5 1

i.e. $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Here, $\binom{4}{0} + \binom{4}{1} = \binom{5}{1}$; $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$; $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$; $\binom{4}{3} + \binom{4}{4} = \binom{5}{4}$.

$$\left(\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}\right)$$

By using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $0 \le r \le n$, and also $\binom{n}{0} = 1 = \binom{n}{n}$, the Pascal's triangle can be written as,

Index					Coefficients				
1				$\binom{1}{0}$		$\binom{1}{1}$			
2			$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$		
3		$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$	
4	$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Observing above array, we can write the coefficients of the terms in the expansion of $(a + b)^n$, for any index n, without writing the earlier rows. For example, for index 7,

we have the coefficients of the terms as $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \binom{7}{4}, \binom{7}{5}, \binom{7}{6}, \binom{7}{7}$.

Now, we are in a position to write the binomial expansion of $(a + b)^n$ for any positive integral value of n.

3.2 Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 + ... + \binom{n}{r}a^{n-r}\cdot b^r + ... + \binom{n}{n}b^n,\ n \in \text{N}$$

We shall prove this theorem using the principle of mathematical induction.

Let, P(n) : $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 + ... + \binom{n}{r}a^{n-r}\cdot b^r + ... + \binom{n}{n}b^n,\ n \in \text{N}$

Let $n = 1$

L.H.S. $= (a + b)^1 = a + b$

R.H.S. $= \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}\cdot b = a + b$

$\therefore$ P(1) is true.

Let P(k) be true.

$$\therefore\ (a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}\cdot b + \binom{k}{2}a^{k-2}\cdot b^2 + ...$$
$$+ \binom{k}{r-1}a^{k-(r-1)}\cdot b^{r-1} + \binom{k}{r}a^{k-r}\cdot b^r + ... + \binom{k}{k}b^k$$

Now, $(a+b)^{k+1} = (a+b)(a+b)^k$

$$= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}\cdot b + \binom{k}{2}a^{k-2}\cdot b^2 + ...\right.$$
$$\left. + \binom{k}{r-1}a^{k-(r-1)}\cdot b^{r-1} + \binom{k}{r}a^{k-r}\cdot b^r + ... + \binom{k}{k}b^k\right]$$

On multiplying both the factors and rearranging the terms, we get,

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k\cdot b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}\cdot b^2 + ...$$
$$+ \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k-(r-1)}\cdot b^r + ... + \binom{k}{k}b^{k+1}$$

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Now, we know that; $\binom{n}{0} = 1 = \binom{n}{n}$ and $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, $1 \le r \le n$

$$\therefore \quad (a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{(k+1)-1}\cdot b + \binom{k+1}{2}a^{(k+1)-2}\cdot b^2 + \ldots + \binom{k+1}{r}a^{(k+1)-r}\cdot b^r + \ldots + \binom{k+1}{k+1}b^{k+1}$$

$\therefore$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction, P(n) is true, $\forall n \in$ N.

Some Corollaries :

(1) Substituting $a = 1$, $b = x$ in the binomial expansion of $(a + b)^n$, we have,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n, \ \forall n \in \text{N}$$

(2) Replacing b by $-b$, we obtain

$$(a-b)^n = \binom{n}{0}a^n - \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 - \binom{n}{3}a^{n-3}\cdot b^3 + \ldots + (-1)^r\cdot\binom{n}{r}a^{n-r}\cdot b^r + \ldots + (-1)^n\cdot\binom{n}{n}b^n$$

(3) Taking $x = 1$ in (1), we get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n}$$

$$\therefore \quad \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n} = 2^n$$

(4) Substituting $x = -1$ in (1), we have

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\cdot\binom{n}{n} \qquad \textbf{(i)}$$

Also, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$ **(ii)**

$\therefore$ Adding respective terms of (i) and (ii), we have,

$$2^n = 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\right]$$

$$\therefore \quad \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + = 2^{n-1} \qquad \textbf{(iii)}$$

$$\therefore \quad \binom{n}{1} + \binom{n}{3} + \ldots = \binom{n}{0} + \binom{n}{2} + \ldots = 2^{n-1} \qquad \textbf{(From (i) and (iii)) (iv)}$$

Note : From the expansion of $(a + b)^n$, we observe the following points :

(1) There are $(n + 1)$ terms in the expansion.

(2) The index of 'a' in the first term is n and the index of 'a' decreases by 1 in the successive terms and simultaneously the index of b is zero in the first term and the index of b increases by 1 in the successive terms.

(3) Degree of each term (i.e. the sum of indices of a and b) is n, the index of $(a + b)$.

(4) The coefficients of the terms in order are $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$.

(5) As we know that, $\binom{n}{r} = \binom{n}{n-r}$, so the co-efficients of terms in the expansion are symmetrically situated successively from left or right i.e. $\binom{n}{0} = \binom{n}{n}$; $\binom{n}{1} = \binom{n}{n-1}$; $\binom{n}{2} = \binom{n}{n-2}$, ...

Example 1 : Expand : $\left(\frac{x}{2}+\frac{1}{x}\right)^5$, $x \neq 0$

Solution : Here $a = \frac{x}{2}$, $b = \frac{1}{x}$, $n = 5$

Substituting these values in the binomial theorem, we get,

$$\left(\frac{x}{2}+\frac{1}{x}\right)^5 = \binom{5}{0}\left(\frac{x}{2}\right)^5 + \binom{5}{1}\left(\frac{x}{2}\right)^4\left(\frac{1}{x}\right) + \binom{5}{2}\left(\frac{x}{2}\right)^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}\left(\frac{x}{2}\right)^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}\left(\frac{x}{2}\right)\left(\frac{1}{x}\right)^4 + \binom{5}{5}\left(\frac{1}{x}\right)^5$$

$$= 1\left(\frac{x^5}{32}\right) + 5\left(\frac{x^4}{16}\right)\left(\frac{1}{x}\right) + \frac{5\cdot 4}{1\cdot 2}\left(\frac{x^3}{8}\right)\left(\frac{1}{x^2}\right) + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\left(\frac{x^2}{4}\right)\left(\frac{1}{x^3}\right)$$

$$+ \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}\left(\frac{x}{2}\right)\left(\frac{1}{x^4}\right) + 1\cdot\left(\frac{1}{x^5}\right)$$

$$= \frac{x^5}{32} + \frac{5}{16}x^3 + \frac{5}{4}x + \frac{5}{2x} + \frac{5}{2x^3} + \frac{1}{x^5}$$

Example 2 : Expand : $\left(2x-1+\frac{1}{x}\right)^4$, $x \neq 0$

Solution : Taking $a = 2x$, $b = 1 - \frac{1}{x}$, $n = 4$ in the corollary (2).

$$\left(2x-1+\frac{1}{x}\right)^4 = \left[2x-\left(1-\frac{1}{x}\right)\right]^4$$

$$= \binom{4}{0}(2x)^4 - \binom{4}{1}(2x)^3\left(1-\frac{1}{x}\right) + \binom{4}{2}(2x)^2\left(1-\frac{1}{x}\right)^2 - \binom{4}{3}(2x)\left(1-\frac{1}{x}\right)^3 + \binom{4}{4}\left(1-\frac{1}{x}\right)^4$$

$$= 16x^4 - 4(8x^3)\left(1-\frac{1}{x}\right) + \frac{4\cdot 3}{1\cdot 2}(4x^2)\left(1-\frac{2}{x}+\frac{1}{x^2}\right) - \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}(2x)\left(1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}\right)$$

$$+ \left[\binom{4}{0} - \binom{4}{1}\left(\frac{1}{x}\right) + \binom{4}{2}\left(\frac{1}{x^2}\right) - \binom{4}{3}\left(\frac{1}{x^3}\right) + \binom{4}{4}\left(\frac{1}{x^4}\right)\right]$$

$$= 16x^4 - 32x^3 + 32x^2 + 24x^2 - 48x + 24 - 8x + 24 - \frac{24}{x} + \frac{8}{x^2}$$

$$+ 1 - \frac{4}{x} + \frac{6}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$$

$$= 16x^4 - 32x^3 + 56x^2 - 56x + 49 - \frac{28}{x} + \frac{14}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$$

Example 3 : Evaluate $(0.99)^5$ using binomial theorem.

Solution :

$$(0.99)^5 = (1 - 0.01)^5$$

$$= \binom{5}{0} - \binom{5}{1}(0.01) + \binom{5}{2}(0.01)^2 - \binom{5}{3}(0.01)^3 + \binom{5}{4}(0.01)^4 - \binom{5}{5}(0.01)^5$$

$$= 1 - 5(0.01) + 10(0.0001) - 10(0.000001) + 5(0.00000001) - (0.0000000001)$$

$$= 0.9509900499$$

Example 4 : Which is smaller ? $(1.1)^{100000}$ or 100000

Solution : $(1.1)^{100000} = (1 + 0.1)^{100000}$

$$= \binom{100000}{0} + \binom{100000}{1}(0.1) + \text{Sum of some positive terms}$$

$$= 1 + 10000 + \text{Sum of positive terms}$$

$$> 10000$$

$\therefore$ 10000 is smaller out of $(1 \cdot 1)^{10000}$ and 10000.

Excercise 3.1

1. Expand the following :

(1) $\left(x^2 + \frac{1}{x}\right)^5$, $(x \neq 0)$ (2) $(1 - 2x)^4$ (3) $(3x - 2)^6$ (4) $\left(x - \frac{1}{2x}\right)^5$, $(x \neq 0)$

2. Expand : (1) $(1 + x + x^2)^4$ (2) $(1 - x + x^2)^3$

3. Evaluate by using binomial theorem :

(1) $(0.98)^4$ (2) $(99)^4$ (3) $(101)^6$

4. Using binomial theorem, indicate which one is larger ? $(1.01)^{10000}$ or 100

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3.3 General and Middle Term

1. The expansion of $(a + b)^n$ contains $(n + 1)$ terms. If we consider $T_1, T_2, T_3, \ldots, T_{n+1}$ as the first, second, third, ... $(n + 1)$th terms respectively in the expansion of $(a + b)^n$, then $T_1 = \binom{n}{0}a^n$, $T_2 = \binom{n}{1}a^{n-1} \cdot b$, $T_3 = \binom{n}{2}a^{n-2} \cdot b^2, \ldots, T_{n+1} = \binom{n}{n}b^n$.

   We may take the **general term as $T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$, $0 \leq r \leq n$**

2. If in $(a + b)^n$; n is even, then $n + 1$ is odd. So the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

   So $\left(\frac{n}{2}+1\right)^{th}$ term $= \left(\frac{n+2}{2}\right)^{th}$ term is the **middle term.**

   For example, in the expansion of $(2x + y)^{10}$, the middle term is $\frac{10+2}{2} = 6$th term. If n is odd, then $n + 1$ is even, so there are **two middle terms : $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.**

   For example, in the expansion of $(2x + y)^9$, the middle terms are $\frac{9+1}{2} = 5$th term and $\frac{9+3}{2} = 6$th term.

Example 5 : Find the fourth term in the expansion of $(3x - y)^7$.

Solution : Here, $a = 3x$, $b = -y$, $n = 7$

Now, $T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

To find T_4, we let $r = 3$ **$(r + 1 = 4)$**

$\therefore \quad T_4 = T_{3+1} = \binom{7}{3}(3x)^{7-3} \cdot (-y)^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}(81x^4)(-y)^3$

$= -2835x^4y^3$

Example 6 : Find the coefficient of x^{-2} in the expansion of $\left(x - \frac{1}{x^2}\right)^{16}$, $(x \neq 0)$

Solution : Here, $a = x$, $b = -\frac{1}{x^2}$, $n = 16$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$= \binom{16}{r}(x)^{16-r} \cdot \left(-\frac{1}{x^2}\right)^r = \binom{16}{r}(-1)^r \cdot x^{16-3r}$

For the index of x to be -2, we must have $16 - 3r = -2$ i.e. $r = 6$.

$\therefore \quad T_{6+1} = \binom{16}{6}(-1)^6 \cdot x^{16-3(6)}$

$\therefore \quad T_7 = \binom{16}{6} \cdot 1 \cdot x^{-2}$

$\therefore$ Coefficient of x^{-2} is $\binom{16}{6}$ or 8008.

Example 7 : Find the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{11}$, if it exists. $(x \neq 0)$

Solution : Suppose the constant term (i.e. term in which index of x is zero) exists and it is $(r + 1)$th term.

Here, $a = 2x^2$, $b = -\frac{1}{x}$, $n = 11$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$= \binom{11}{r}(2x^2)^{11-r} \cdot \left(-\frac{1}{x}\right)^r = \binom{11}{r}(2)^{11-r} \cdot (-1)^r \cdot x^{22-3r}$

For the constant term, index of x is zero.

$\therefore \quad 22 - 3r = 0$

$\therefore \quad r = \frac{22}{3} \notin N$

$\therefore$ Our assumption is wrong.

$\therefore$ Constant term does not exist in the expansion.

Example 8 : Find the middle term / terms in the expansion of $\left(\frac{x}{2} + 3y\right)^9$.

Solution : As $n = 9$ is odd, so we have two middle terms namely,

$\frac{n+1}{2} = \frac{9+1}{2} = 5$th term and $\frac{n+3}{2} = \frac{9+3}{2} = 6$th term

Here, $a = \frac{x}{2}$, $b = 3y$, $n = 9$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$\therefore \quad T_5 = T_{4+1} = \binom{9}{4} \cdot \left(\frac{x}{2}\right)^{9-4} \cdot (3y)^4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{x^5}{32}\right)(81y^4) = \frac{5103}{16}x^5y^4$

And $T_6 = \binom{9}{5} \cdot \left(\frac{x}{2}\right)^{9-5} \cdot (3y)^5$ **$(r + 1 = 6)$**

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\left(\frac{x^4}{16}\right)(243y^5) = \frac{15309}{8}\ x^4y^5$$

$\therefore$ Middle terms are $\frac{5103}{16}\ x^5y^4$ and $\frac{15309}{8}\ x^4y^5$.

Example 9 : Obtain the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right)^{12}$. **$(x > 0)$**

Solution : Here, $T_{r+1} = \binom{12}{r}\left(\sqrt{\frac{x}{3}}\right)^{12-r} \cdot \left(\sqrt{\frac{3}{2x^2}}\right)^{r}$

$$= \binom{12}{r} \cdot \frac{\sqrt{3}^r}{(\sqrt{3})^{12-r}} \times \frac{1}{(\sqrt{2})^r} \cdot x^{6 - \frac{r}{2} - r}$$

$$= \binom{12}{r} \cdot \frac{1}{(\sqrt{3})^{12-2r}} \times \frac{1}{(\sqrt{2})^r} \times x^{6 - \frac{3r}{2}}$$

For the term independent of x, we let $6 - \frac{3r}{2} = 0$

$\therefore\quad r = 4$

$$T_5 = \binom{12}{4} \cdot \frac{1}{(\sqrt{3})^4} \cdot \frac{1}{(\sqrt{2})^4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{36} = \frac{55}{4}$$

Exercise 3.2

1. Find the coefficient of : (1) x^6 in $(x + 2)^9$ (2) x^{32} in $\left(x^4 - \frac{1}{x^3}\right)^{15}$, $(x \neq 0)$

2. Find the constant term in the expansion of :

(1) $\left(\frac{3}{x^2} + \frac{\sqrt{x}}{3}\right)^{10}$, $(x > 0)$ (2) $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, $(x \neq 0)$

3. The coefficients of x^7 and x^8 in the expansion of $\left(2 + \frac{x}{3}\right)^n$ are equal, find n.

4. Find the middle term or terms in the expansion of :

(1) $\left(2 - \frac{x^3}{3}\right)^7$ (2) $\left(\frac{x}{2} + 3y\right)^8$ (3) $\left(\frac{3}{2x} - \frac{2x^2}{3}\right)^{20}$, $(x \neq 0)$ (4) $(3x + 2y)^5$

5. If the coefficient of x^3 in the expansion of $(1 + x)^n$ is 20, find n.

6. If the coefficients of fifth, sixth and seventh terms in the expansion of $(1 + x)^n$ are in arithmetic progression, find n.

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Miscellaneous Problems :

Example 10 : Find the coefficient of x^3 in the expansion of the product $(1 - x)^{15} \cdot (1 + 3x)^4$.

Solution : Applying binomial theorem to get $(1 - x)^{15}$ and $(1 + 3x)^4$, we have

$(1-x)^{15} = \binom{15}{0} - \binom{15}{1}x + \binom{15}{2}x^2 - \binom{15}{3}x^3 + \ldots - \binom{15}{15}x^{15}$ and

$(1+3x)^4 = (3x+1)^4 = \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3 + \binom{4}{2}(3x)^2 + \binom{4}{3}(3x) + \binom{4}{4} \cdot 1$

Now, we want to find the coefficient of x^3 in the product $(1-x)^{15} \cdot (1+3x)^4$, we shall simply collect the terms containing x^3 from the product, without finding complete product.

They are, $\binom{15}{0} \cdot \binom{4}{1}(27x^3) - \binom{15}{1}x \cdot \binom{4}{2}(9x^2) + \binom{15}{2}x^2 \cdot \binom{4}{3}(3x) - \binom{15}{3}x^3 \cdot \binom{4}{4}$

$= 1 \cdot 4 \cdot 27x^3 - 15 \cdot x \cdot \frac{4 \cdot 3}{1 \cdot 2} \cdot 9x^2 + \frac{15 \cdot 14}{1 \cdot 2}x^2 \cdot 4 \cdot 3x - \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3}x^3 \cdot 1$

$= (108 - 810 + 1260 - 455)x^3 = 103x^3$

$\therefore$ Coefficient of x^3 in $(1-x)^{15} \cdot (1+3x)^4$ is 103.

Example 11 : If the middle term in the expansion of $\left(\frac{x}{3}+3\right)^{10}$ is 8064, find x.

Solution : Here $n = 10$

$\therefore$ n is even, so middle term is $\frac{n+2}{2} = \frac{10+2}{2} = 6$th term

$\therefore$ $T_6 = T_{5+1} = \binom{10}{5} \cdot \left(\frac{x}{3}\right)^{10-5} \cdot (3)^5$

$\therefore$ $8064 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{x^5}{3^5} \cdot 3^5$

$\therefore$ $\frac{8064}{252} = x^5$

$\therefore$ $x^5 = 32 = 2^5$

$\therefore$ $x = 2$

Example 12 : Prove that $(3+\sqrt{8})^5 + (3-\sqrt{8})^5 = 6726$. Hence deduce that, $6725 < (3+\sqrt{8})^5 < 6726$. Hence obtain $[(3+\sqrt{8})^5]$.

Solution : $(3+\sqrt{8})^5 = \binom{5}{0}(3)^5 + \binom{5}{1}(3)^4(\sqrt{8}) + \binom{5}{2}(3)^3(\sqrt{8})^2 + \binom{5}{3}(3)^2(\sqrt{8})^3$

$+ \binom{5}{4}(3)(\sqrt{8})^4 + \binom{5}{5}(\sqrt{8})^5$ **(i)**

$(3-\sqrt{8})^5 = \binom{5}{0}(3)^5 - \binom{5}{1}(3)^4(\sqrt{8}) + \binom{5}{2}(3)^3(\sqrt{8})^2 - \binom{5}{3}(3)^2(\sqrt{8})^3$

$+ \binom{5}{4}(3)(\sqrt{8})^4 - \binom{5}{5}(\sqrt{8})^5$ **(ii)**

Adding (i) and (ii), we have

$(3+\sqrt{8})^5 + (3-\sqrt{8})^5 = 2\left[\binom{5}{0}(3)^5 + \binom{5}{2}(3)^3(\sqrt{8})^2 + \binom{5}{4}(3)(\sqrt{8})^4\right]$

$= 2\left[1 \cdot 243 + \frac{5 \cdot 4}{1 \cdot 2} \cdot 27 \cdot 8 + 5 \cdot 3 \cdot 64\right]$ $\left(\binom{5}{4} = \binom{5}{1}\right)$

$= 2[243 + 2160 + 960]$

$= 2[3363]$

$= 6726$

Now, $(3 + \sqrt{8})(3 - \sqrt{8}) = 9 - 8 = 1$ and $(3 + \sqrt{8}) > 0$. Hence $3 - \sqrt{8} > 0$.

Also $(3 + \sqrt{8}) > 1$

$\therefore \quad 3 - \sqrt{8} < 1$

$\therefore \quad 0 < 3 - \sqrt{8} < 1$

$\therefore \quad 0 < (3 - \sqrt{8})^5 < 1$

$\therefore \quad (3 + \sqrt{8})^5 < (3 + \sqrt{8})^5 + (3 - \sqrt{8})^5 = 6726 < (3 + \sqrt{8})^5 + 1$

$\therefore \quad (3 + \sqrt{8})^5 < 6726$ and $6726 < (3 + \sqrt{8})^5 + 1$

$\therefore \quad 6725 < (3 + \sqrt{8})^5 < 6726$

$\therefore$ According to definition of integer part, $[(3 + \sqrt{8})^5] = 6725$

Example 13 : The sum of the coefficients of powers of x in the first three terms in the expansion of $\left(x^2 - \frac{2}{x}\right)^n$ $(x \neq 0)$ is 127, find n. $(n \in N)$

Solution : In the expansion of $\left(x^2 - \frac{2}{x}\right)^n$, the first three terms are $\binom{n}{0}(x^2)^n$, $\binom{n}{1}(x^2)^{n-1} \cdot \left(\frac{-2}{x}\right)$ and $\binom{n}{2}(x^2)^{n-2} \cdot \left(\frac{-2}{x}\right)^2$. As the sum of the coefficients of these terms is 127, we have,

$\binom{n}{0} - \binom{n}{1}2 + \binom{n}{2} \cdot 4 = 127$

$\therefore \quad 1 - 2n + \frac{4n(n-1)}{2} = 127$

$\therefore \quad 1 - 2n + 2n(n-1) = 127$

$\therefore \quad 1 - 2n + 2n^2 - 2n - 127 = 0$

$\therefore \quad 2n^2 - 4n - 126 = 0$

$\therefore \quad n^2 - 2n - 63 = 0$

$\therefore \quad (n - 9)(n + 7) = 0$

$\therefore \quad n = 9$ or $n = -7$ But $-7 \notin N$

$\therefore \quad n = 9$

Example 14 : Use the binomial theorem to show that dividing $8^n - 7n$ by 49 leaves the remainder 1.

Solution : $8^n = (1 + 7)^n$

$$= 1 + \binom{n}{1}7 + \binom{n}{2}7^2 + \binom{n}{3}7^3 + ... + \binom{n}{n}7^n$$

$$= 1 + 7n + 7^2\left[\binom{n}{2} + \binom{n}{3}7 + ... + \binom{n}{n}7^{n-2}\right]$$

$\therefore \quad 8^n - 7n = 1 + 49m$, where $m = \left[\binom{n}{2} + \binom{n}{3}7 + ... + \binom{n}{n}7^{n-2}\right] \in N$

$\therefore$ Dividing $8^n - 7n$ by 49 leaves the remainder 1.

Example 15 : Prove that : $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}$, $\forall n \in N$

Solution : [Motivation : See the R.H.S. $= \frac{(2n)!}{(n!)(n!)} = \frac{(2n)!}{(2n-n)!n!} = \binom{2n}{n}$, which is the coefficient of x^n in the expansion of $(1 + x)^{2n}$.]

$$(1 + x)^{2n} = (1 + x)^n (x + 1)^n$$

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right] \times \left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + ... + \binom{n}{n-1}x + \binom{n}{n}\right]$$

Now coefficient of x^n in the expansion of $(1 + x)^{2n}$ is $\binom{2n}{n}$ and

coefficient of x^n in R.H.S. $= \binom{n}{0}^2 + \binom{n}{1}^2 + ... + \binom{n}{n}^2$ **(Taking product term wise)**

$$\therefore \quad \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2 = \binom{2n}{n}$$

$$= \frac{(2n)!}{n! \cdot n!} = \frac{(2n)!}{(n!)^2}$$

Example 16 : Prove that : $\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + ... + \binom{n}{n-1}\binom{n}{n} = \frac{(2n)!}{(n-1)!(n+1)!}$, $\forall n \in N$

Solution : [Motivation : See R.H.S. $= \frac{(2n)!}{[2n-(n-1)]!(n-1)!} = \binom{2n}{n-1}$. It is the coefficient of x^{n-1} in the expansion of $(1 + x)^{2n}$.]

$$(1 + x)^{2n} = (1 + x)^n (x + 1)^n$$

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right] \times \left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + ... + \binom{n}{n}\right]$$

Now coefficient of x^{n-1} in $(1 + x)^{2n}$ is $\binom{2n}{n-1}$ and

the coefficient of x^{n-1} in R.H.S. is $\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + ... + \binom{n}{n-1}\binom{n}{n}$

$$\therefore \quad \binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + ... + \binom{n}{n-1}\binom{n}{n} = \binom{2n}{n-1} = \frac{(2n)!}{(n+1)!(n-1)!}$$

Example 17 : Prove that : $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + ... + (2n + 1)\binom{n}{n} = (n + 1)2^n$, $\forall n \in N$

Solution : Let $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + ... + (2n - 1)\binom{n}{n-1} + (2n + 1)\binom{n}{n} = S$ **(i)**

Using $\binom{n}{r} = \binom{n}{n-r}$ and taking terms in the reverse order, we have

$(2n + 1)\binom{n}{0} + (2n - 1)\binom{n}{1} + (2n - 3)\binom{n}{2} + ... + 5\binom{n}{n-2} + 3\binom{n}{n-1} + \binom{n}{n} = S$ **(ii)**

Adding corresponding terms of (i) and (ii), we have

$$\therefore \quad (1 + (2n + 1))\binom{n}{0} + (3 + (2n - 1))\binom{n}{1} + (5 + (2n - 3))\binom{n}{2} + ... + ((2n - 3) + 5)\binom{n}{n-2} + ((2n - 1) + 3)\binom{n}{n-1} + ((2n + 1) + 1)\binom{n}{n} = 2S$$

$\therefore \quad (2n + 2)\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}\right] = 2S$

$\therefore \quad 2(n + 1) \cdot 2^n = 2S$

$\therefore \quad S = (n + 1)2^n$

So, $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + ... + (2n + 1)\binom{n}{n} = (n + 1)2^n$

Example 18 : If in the expansion of $(x - 2y)^n$, the sum of fifth and sixth term is zero then find the value of $\frac{x}{y}$. If $n = 8$ then find $\frac{x}{y}$.

Solution : Here, $T_5 = \binom{n}{4} \cdot x^{n-4} \cdot (-2y)^4$ and $T_6 = \binom{n}{5} \cdot x^{n-5} \cdot (-2y)^5$

Now, $T_5 + T_6 = 0$. So, $T_5 = -T_6$

$\therefore \quad \binom{n}{4} \cdot (-2)^4 \cdot x^{n-4} \cdot y^4 = -\binom{n}{5} \cdot (-2)^5 \cdot x^{n-5} \cdot y^5$

$\therefore \quad \frac{n!}{4!(n-4)!} \cdot 16 \cdot x^{n-4} \cdot y^4 = -\frac{n!}{5!(n-5)!} \cdot (-32) \cdot x^{n-5} \cdot y^5$

$\therefore \quad \frac{x^{n-4} \cdot y^4}{x^{n-5} \cdot y^5} = \frac{n!}{5!(n-5)!} \times \frac{4!(n-4)!}{n!} \times \frac{32}{16}$

$\therefore \quad \frac{x}{y} = \frac{4!(n-4)(n-5)!}{5 \cdot 4!(n-5)!} \times 2$

$\therefore \quad \frac{x}{y} = \frac{n-4}{5} \times 2$

Taking $n = 8$, we have

$\frac{x}{y} = \frac{8}{5}$

Example 19 : Obtain the sum of the last thirty coefficients in the expansion of $(1 + x)^{59}$.

Solution : There are 60 terms in the expansion of $(1 + x)^{59}$.

$\therefore$ Sum of the coefficients of last thirty terms is,

$S = \binom{59}{30} + \binom{59}{31} + \binom{59}{32} + ... + \binom{59}{58} + \binom{59}{59}$ **(first 30 coefficients $\binom{59}{0}, \binom{59}{1}, ..., \binom{59}{29}$)** **(i)**

i.e. $S = \binom{59}{29} + \binom{59}{28} + \binom{59}{27} + ... + \binom{59}{1} + \binom{59}{0}$ **(Using $\binom{n}{r} = \binom{n}{n-r}$)** **(ii)**

$\therefore \quad 2S = \binom{59}{0} + \binom{59}{1} + ... + \binom{59}{59}$ **(adding respective sides of (i) and (ii))**

$\therefore \quad S = \frac{2^{59}}{2} = 2^{58}$

Exercise 3

1. Obtain the ratio of the coefficients of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$.
2. If the coefficients of $(r - 2)$th and $(2r - 5)$th terms in the expansion of $(1 + x)^{36}$ are equal, find r.
3. Find x, y and n in the expansion of $(x + y)^n$, if the first three terms in the expansion are 64, 960 and 6000.
4. The 2nd, 3rd and 4th terms in the expansion of $(a + b)^n$ are 240, 720 and 1080, find a, b and n.

5. Prove that $(2 + \sqrt{3})^7 + (2 - \sqrt{3})^7 = 10084$.

Hence deduce that, $10083 < (2 + \sqrt{3})^7 < 10084$.

6. Find n, if the ratio of the fourth term to the fourth term from the end in the expansion of $\left(\sqrt[5]{2} + \frac{1}{\sqrt[5]{3}}\right)^n$ is 6 : 1.

7. Find the coefficient of x^4 in the expansion of $(1 - x)^{12} \cdot (1 + 2x)^6$.

8. The sum of the coefficients of the first three terms in the expansion of $\left(x^2 - \frac{3}{x}\right)^n$ $(x \neq 0)$ is 376, find the coefficient of x^8.

9. Using the binomial theorem, show that $3^{2n} - 8n - 1$ is divisible by 64, for $n \in \mathrm{N}$.

10. Prove the following identities : $(\forall n \in \mathrm{N})$

(1) $\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + ... + (n + 1)\binom{n}{n} = (n + 2) \cdot 2^{n-1}$

(2) $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + ... + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

11. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If the coefficients of 5th and 19th terms in the expansion of $(1 + x)^n$ are equal, then n = ... ☐

(a) 18 (b) 24 (c) 22 (d) 20

(2) If the coefficients of $(r - 6)$th and $(2r - 2)$th terms in the expansion of $(1 + x)^{32}$ are equal, then r = ... ☐

(a) -2 (b) 14 (c) 34 (d) 20

(3) The coefficient of x^{21} in the expansion of $(x + x^2)^{20}$ is ...... ☐

(a) $\binom{20}{1}$ (b) $\binom{20}{0}$ (c) $\binom{20}{2}$ (d) $\binom{20}{12}$

(4) The number of terms in the expansion of $(2x + 3y + 4z)^5$ of type $x^a \cdot y^b \cdot z^c$ is ...... ☐

(a) 10 (b) 15 (c) 21 (d) 42

(5) If $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4 = x + y\sqrt{3}$, then y = ... ☐

(a) 0 (b) 56 (c) 112 (d) 97

(6) If T_{r-1} is the middle term of $(a + b)^{10}$, then r = ... ☐

(a) 6 (b) 5 (c) 7 (d) 8

(7) Constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$, $(x \neq 0)$ is ...... ☐

(a) 7920 (b) 495 (c) -7920 (d) -495

(8) $\binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}$ = ... $(n > 1)$ ☐

(a) 2^n (b) 2^{n-1} (c) $2^n - 1$ (d) $2^{n-1} - 1$

(9) Middle term in the expansion of $\left(2x + \frac{1}{2x}\right)^8$ is ...... $(x \neq 0)$ ☐

(a) $\binom{8}{4}$ (b) $\binom{8}{4}(2x)$ (c) $\binom{8}{4}\left(\frac{1}{2x}\right)$ (d) $\binom{8}{4}(2)$

(10) Sum of the coefficients of $x^{13}y^2$ and x^2y^{13} in the expansion of $(x + y)^{15}$ is .... ☐

(a) $\binom{15}{2}$ (b) $2\binom{15}{13}$ (c) $\binom{15}{3}$ (d) $2\binom{15}{3}$

*

Summary

We studied following points in this chapter :

1. The Binomial Expansion for $n \in N$ is given by the binomial theorem as

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1} \cdot b + \binom{n}{2}a^{n-2} \cdot b^2 + ... + \binom{n}{r}a^{n-r} \cdot b^r + ... + \binom{n}{n}b^n, n \in N$$

2. The coefficients of binomial theorem are arranged in an array, known as Pascal's Triangle.

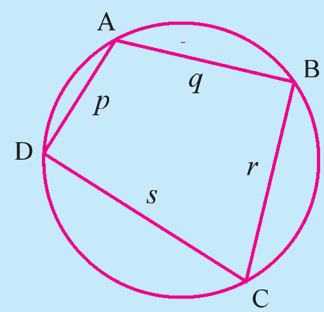
3. The general term of the expansion $(a + b)^n$ is $T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$.

4. The middle term in the expansion of $(a+b)^n$ is $\left(\frac{n}{2}+1\right)$ or $\left(\frac{n+2}{2}\right)^{th}$ term, if n is even and $\left(\frac{n+1}{2}\right)^{th}$ as well as $\left(\frac{n+3}{2}\right)^{th}$ terms are the middle terms, if n is odd.

—❖—

Brahmagupta's formula

Brahmagupta's most famous result in geometry is his formula for cyclic quadrilaterals. Given the lengths of the sides of any cyclic quadrilateral, Brahmagupta gave an approximate and an exact formula for the figure's area.



The approximate area is the product of the halves of the sums of the opposite sides of a quadrilateral. The accurate [area] is the square root the product of the half of the sum of the sides diminished by [each] side of the quadrilateral.

So given the lengths p, q, r and s of sides of a cyclic quadrilateral, the approximate area is $\left(\frac{p+r}{2}\right)\left(\frac{q+s}{2}\right)$ while, letting $t = \frac{p+q+r+s}{2}$, the exact area is

$$\sqrt{(t-p)(t-q)(t-r)(t-s)}$$

Heron's formula is a special case of this formula and it can be derived by setting one of the sides equal to zero.

Chapter 4

ADDITION FORMULAE AND FACTOR FORMULAE

Music is the pleasure the human mind experiences from counting without being aware that it is counting.

– Gottfried Leibnitz

4.1 Introduction

We have studied the fundamental ideas and properties of trigonometric functions. Now, we will see how to express values of trigonometric functions with variables $\alpha + \beta$ and $\alpha - \beta$ in terms of values of trigonometric functions with variables α and β, where α and β are real numbers. These formulae are known as addition formulae. With the help of these formulae, we will derive factor formulae and study their uses.

If $f(x) = ax$, $x \in$ R is a linear function, then

$$f(x - y) = a(x - y) = ax - ay = f(x) - f(y)$$

Thus, $f(x - y) = f(x) - f(y)$

Now, consider the trigonometric function $f(x) = cosx$, $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

For these values of α and β, $\alpha - \beta = \frac{\pi}{3} - \frac{\pi}{6}$. So $cos(\alpha - \beta) = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

But $cos\alpha - cos\beta = cos\frac{\pi}{3} - cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} \neq \frac{\sqrt{3}}{2}$

Thus, $cos(\alpha - \beta) \neq cos\alpha - cos\beta$

Thus, what is true for a linear function may not be true for trigonometric functions. Similarly other results can also be quoted. Now, we will obtain the formula of $cos(\alpha - \beta)$ using $cos\alpha$, $cos\beta$, $sin\alpha$, $sin\beta$.

4.2 The Addition Formulae

We shall first prove a formula for $cos(\alpha - \beta)$ and $cos(\alpha + \beta)$.

Let us see the expression for $cos(\alpha - \beta)$.

Theorem 1 : For $\alpha, \beta \in R$

(1) $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

(2) $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Proof : Case (1) : Let $\alpha, \beta \in [0, 2\pi)$.

We have three possibilities for α and β by law of trichotomy.

They are (i) $\alpha > \beta$ (ii) $\alpha = \beta$ (iii) $\alpha < \beta$

(i) $\alpha > \beta$

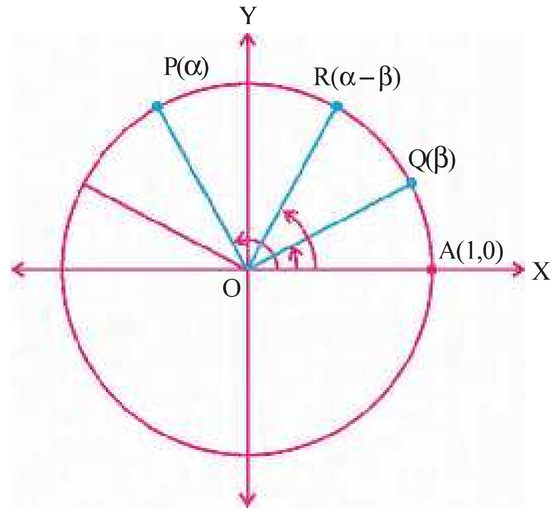


Figure 4.1

Suppose the trigonometric points on the unit circle corresponding to α, β and $\alpha - \beta$ are P, Q and R respectively.

$\therefore$ By definition, $P(\alpha) = (\cos\alpha, \sin\alpha)$,

$Q(\beta) = (\cos\beta, \sin\beta)$ and $R(\alpha - \beta) = (\cos(\alpha - \beta), \sin(\alpha - \beta))$.

Also A is (1, 0).

As shown in figure we have $l(\overset{\frown}{AP}) = \alpha$, $l(\overset{\frown}{AQ}) = \beta$ and $l(\overset{\frown}{AR}) = \alpha - \beta$.

As $\beta < \alpha$ and $Q \in \overset{\frown}{AP}$,

$l(\overset{\frown}{PQ}) = l(\overset{\frown}{AP}) - l(\overset{\frown}{AQ})$

$\therefore$ $l(\overset{\frown}{PQ}) = \alpha - \beta = l(\overset{\frown}{AR})$

$\therefore$ $\overset{\frown}{PQ} \cong \overset{\frown}{AR}$

Chords corresponding to congruent arcs of the same circle are congruent.

$\therefore$ $PQ = AR$

$\therefore$ $PQ^2 = AR^2$

Now using distance formula,

$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= \cos^2\alpha - 2\cos\alpha \cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha \sin\beta + \sin^2\beta$

$= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha \cos\beta - 2\sin\alpha \sin\beta$

$= 2 - 2(\cos\alpha \cos\beta + \sin\alpha \sin\beta)$

$AR^2 = (1 - \cos(\alpha - \beta))^2 + (0 - \sin(\alpha - \beta))^2$

$= 1 - 2\cos(\alpha - \beta) + \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)$

$= 2 - 2\cos(\alpha - \beta)$

But $AR^2 = PQ^2$

$\therefore$ $2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha \cos\beta + \sin\alpha \sin\beta)$

$\therefore$ $-2\cos(\alpha - \beta) = -2(\cos\alpha \cos\beta + \sin\alpha \sin\beta)$

$\therefore$ $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

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(ii) Suppose $\alpha = \beta$

Then, L.H.S. $= cos(\alpha - \beta) = cos(\alpha - \alpha) = cos 0 = 1$

R.H.S. $= cos\alpha\ cos\beta + sin\alpha\ sin\beta$

$= cos\alpha\ cos\alpha + sin\alpha\ sin\alpha$

$= cos^2\alpha + sin^2\alpha = 1$

$\therefore$ L.H.S. = R.H.S.

(iii) Suppose $\alpha < \beta$

Then, $\alpha - \beta = -(\beta - \alpha)$

$\therefore\ cos(\alpha - \beta) = cos(-(\beta - \alpha))$

$= cos(\beta - \alpha)$ **(*cosine* is an even function.)**

$= cos\beta\ cos\alpha + sin\beta\ sin\alpha$ **($\beta > \alpha$)**

$\therefore\ cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

Case (2) : $\alpha, \beta \in R$

For the given $\alpha, \beta \in R$, we can find $\alpha_1, \beta_1 \in [0, 2\pi)$,

such that $\alpha = 2m\pi + \alpha_1$ and $\beta = 2n\pi + \beta_1$, $m, n \in Z$

$\therefore\ \alpha - \beta = 2m\pi + \alpha_1 - (2n\pi + \beta_1)$

$= 2(m - n)\pi + \alpha_1 - \beta_1$

$= 2k\pi + \alpha_1 - \beta_1$, where $k = m - n \in Z$

As *sin* and *cos* are periodic functions whose principal period is 2π

$cos\alpha = cos\alpha_1$, $cos\beta = cos\beta_1$ and $cos(\alpha - \beta) = cos(\alpha_1 - \beta_1)$

Thus, $cos(\alpha - \beta) = cos(\alpha_1 - \beta_1)$

$= cos\alpha_1\ cos\beta_1 + sin\alpha_1\ sin\beta_1$ **(Case (1))**

$= cos\alpha\ cos\beta + sin\alpha\ sin\beta$

$\therefore\ cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

From case (1) and case (2) we see that for all $\alpha, \beta \in R$

$$cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$$

(2) We have, $cos(\alpha + \beta) = cos(\alpha - (-\beta))$

$= cos\alpha\ cos(-\beta) + sin\alpha\ sin(-\beta)$

$= cos\alpha\ cos\beta - sin\alpha\ sin\beta$ **($cos(-\beta) = cos\beta$, $sin(-\beta) = -sin\beta$)**

$$\therefore\ cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$$

Corollay 1 : (1) $cos\left(\frac{\pi}{2} - \theta\right) = sin\theta$ (2) $sin\left(\frac{\pi}{2} - \theta\right) = cos\theta$

Proof : (1) We know that for all $\alpha, \beta \in R$,

$cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

We substitute $\alpha = \frac{\pi}{2}$ and $\beta = \theta$ in the above identity. We get,

$$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= 0 \cdot \cos\theta + 1 \cdot \sin\theta$$
$$= \sin\theta$$

$\therefore$ $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

(2) If we replace θ by $\frac{\pi}{2} - \theta$ in $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, we get

$$\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right] = \sin\left(\frac{\pi}{2}-\theta\right)$$

$\therefore$ $\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$

$\therefore$ $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

Theorem 2 : (1) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(2) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof : (1) $\sin(\alpha - \beta) = \cos\left[\frac{\pi}{2} - (\alpha - \beta)\right]$ $\quad \left(\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right)$

$$= \cos\left[\left(\frac{\pi}{2}-\alpha\right) + \beta\right]$$
$$= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$\therefore$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(2) $\sin(\alpha + \beta) = \sin[\alpha - (-\beta)]$

$$= \sin\alpha \cdot \cos(-\beta) - \cos\alpha \cdot \sin(-\beta)$$
$$= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \qquad (\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta)$$

$\therefore$ $\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$

4.3 Other Formulae for Allied Numbers

We have seen from theorems 1 and 2 that for all real numbers α and β.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \text{(i)}$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{(ii)}$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \text{(iii)}$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{(iv)}$$

We have also seen that for all $\theta \in R$,

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\therefore \quad \tan\left(\frac{\pi}{2}-\theta\right) = \left(\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Putting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$ in (iv) and (ii) respectively, we get

$$sin\left(\frac{\pi}{2}+\theta\right) = sin\frac{\pi}{2}\, cos\theta + cos\frac{\pi}{2}\, sin\theta = 1 \cdot cos\theta + 0 \cdot sin\theta = cos\theta$$

$$\therefore \quad sin\left(\frac{\pi}{2}+\theta\right) = cos\theta$$

$$cos\left(\frac{\pi}{2}+\theta\right) = cos\frac{\pi}{2}\, cos\theta - sin\frac{\pi}{2}\, sin\theta = 0 \cdot cos\theta - 1 \cdot sin\theta = -sin\theta$$

$$\therefore \quad cos\left(\frac{\pi}{2}+\theta\right) = -sin\theta$$

and hence, $tan\left(\frac{\pi}{2}+\theta\right) = -cot\theta$

Similarly putting, $\alpha = \frac{3\pi}{2}$ and $\beta = \theta$ in (i) to (iv), we get

$$sin\left(\frac{3\pi}{2}-\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}-\theta\right) = -sin\theta$$

$$\therefore \quad tan\left(\frac{3\pi}{2}-\theta\right) = cot\theta$$

Similarly, $sin\left(\frac{3\pi}{2}+\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}+\theta\right) = sin\theta$

$$\therefore \quad tan\left(\frac{3\pi}{2}+\theta\right) = -cot\theta$$

Again putting $\alpha = \pi$, $\beta = \theta$ and $\alpha = 2\pi$, $\beta = \theta$ in (i) to (iv), we can prove the following :

$sin(\pi - \theta) = sin\theta,\ cos(\pi - \theta) = -cos\theta,\ tan(\pi - \theta) = -tan\theta$

$sin(\pi + \theta) = -sin\theta,\ cos(\pi + \theta) = -cos\theta,\ tan(\pi + \theta) = tan\theta$

$sin(2\pi - \theta) = -sin\theta,\ cos(2\pi - \theta) = cos\theta,\ tan(2\pi - \theta) = -tan\theta$

$sin(2\pi + \theta) = sin\theta,\ cos(2\pi + \theta) = cos\theta,\ tan(2\pi + \theta) = tan\theta$

We will be using these formulae frequently for solving examples, so it would be very useful to remember them. As an aid to memory, remember the following.

First of all, it is enough to consider values of trigonometric functions $sin\alpha$, $cos\alpha$ etc. where $0 \le \alpha < 2\pi$, because if $\theta \in R$ then $\theta = 2n\pi + \alpha$, $0 \le \alpha < 2\pi$. We let $0 < \beta < \frac{\pi}{2}$. Then typical real numbers $\frac{\pi}{2} - \beta$, $\frac{\pi}{2} + \beta$, $\frac{3\pi}{2} - \beta$ and $\frac{3\pi}{2} + \beta$ correspond to the trigonometric points which lie in the I, II, III, IV quadrants respectively.

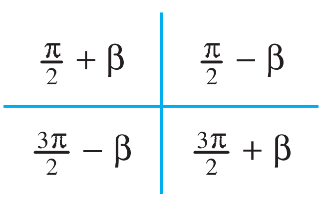

Figure 4.2

From figure 4.2 for any real value, trigonometric function change as under, *sin* → *cos*, *cos* → *sin*, *tan* → *cot*, *cot* → *tan*, *sec* → *cosec*, *cosec* → *sec*.

$P\left(\frac{\pi}{2}+\beta\right)$ is in second quadrant.

In the second quadrant $sin\left(\frac{\pi}{2}+\beta\right) > 0$.

Note : Choice of sign is according to the original function on the left.

$$\therefore \quad sin\left(\frac{\pi}{2}+\beta\right) = cos\beta$$

$P\left(\frac{3\pi}{2}-\beta\right)$ is in the third quadrant and in the third quadrant $cos\left(\frac{3\pi}{2}-\beta\right)$ is $-ve$.

$\therefore \quad cos\left(\frac{3\pi}{2}-\beta\right) = -sin\beta$

Now have a look at the figure 4.3,

$\pi - \beta$	β
$\pi + \beta$	$2\pi - \beta$

Figure 4.3

For any such transformations, the trigonometric functions remain same. *sin* $\rightarrow$ *sin*, *cos* $\rightarrow$ *cos* etc.

Choice of sign is according to the quadrant of the original function. The trigonometric point $P(\pi + \beta)$ is in the third quadrant and $sin(\pi + \beta)$ is $-ve$ in the third quadrant.

Hence, $sin(\pi + \beta) = -sin\beta$,

$tan(\pi + \beta) = tan\beta$ **($tan(\pi + \beta)$ is $+ve$ in third quadrant.)**

Now $P(2\pi - \beta)$ is in the fourth quadrant.

Hence, $sec(2\pi - \beta) = sec\beta$, $cosec(2\pi - \beta) = -cosec\beta$

(as *sec* takes $+ve$ and *cosec* takes $-ve$ values in the fourth quadrant.)

Now, let us find $sin\left(\frac{38\pi}{3}\right)$ and $cos\left(\frac{61\pi}{4}\right)$ using there rules.

$$sin\left(\frac{38\pi}{3}\right) = sin\left(\frac{36\pi + 2\pi}{3}\right)$$

$$= sin\left(12\pi + \frac{2\pi}{3}\right)$$

$$= sin\frac{2\pi}{3} \qquad \textbf{(}12\pi \textbf{ is a period of } sine \textbf{ function.)}$$

$$= sin\left(\frac{3\pi - \pi}{3}\right)$$

$$= sin\left(\pi - \frac{\pi}{3}\right)$$

$$= sin\frac{\pi}{3} \qquad \textbf{(}sine \textbf{ takes } +ve \textbf{ values in the second quadrant.)}$$

$$= \frac{\sqrt{3}}{2}$$

$$cos\left(\frac{61\pi}{4}\right) = cos\left(\frac{60\pi + \pi}{4}\right)$$

$$= cos\left(15\pi + \frac{\pi}{4}\right)$$

$$= cos\left(14\pi + \pi + \frac{\pi}{4}\right)$$

$$= cos\left(\pi + \frac{\pi}{4}\right) \qquad \textbf{(}14\pi \textbf{ is a period of } cosine \textbf{ function.)}$$

$$= -cos\frac{\pi}{4} \qquad \textbf{(}cosine \textbf{ takes } -ve \textbf{ values in the third quadrant.)}$$

$$= -\frac{1}{\sqrt{2}}$$

The Principle Period of *tan* :

We know that $sin(\pi + \theta) = -sin\theta$, $cos(\pi + \theta) = -cos\theta$. So $tan(\pi + \theta) = tan\theta$

Thus, π is a period of *tan*. Now we will prove that π is the principal period of *tan.*

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Suppose the principal period of *tan* is p.

Now, $tan(\theta + p) = tan\theta, \forall \theta, \theta + p \in R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$

In particular, taking $\theta = 0$, we get

$tanp = 0$

$\therefore \quad p = k\pi$

$\therefore$ The least positive value of p is π.

Thus, π is the principal period of *tan*.

Example 1 : Evaluate : (1) $cos120^o$ (2) $sin\left(\frac{-17\pi}{4}\right)$ (3) $tan\left(\frac{13\pi}{4}\right)$ (4) $3sec\left(\frac{-7\pi}{4}\right)$

Solution : (1) $cos120^o = cos(90^o + 30^o) = -sin30^o = \frac{-1}{2}$ $\left(cos\left(\frac{\pi}{2}+\theta\right) = -sin\theta\right)$

$\therefore \quad cos120^o = \frac{-1}{2}$

(2) $sin\left(\frac{-17\pi}{4}\right) = -sin\left(\frac{17\pi}{4}\right)$

$= -sin\left(\frac{16\pi+\pi}{4}\right)$

$= -sin\left(4\pi + \frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ **(4π is a period of *sine*.)**

$\therefore \quad sin\left(\frac{-17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

(3) $tan\left(\frac{13\pi}{4}\right) = tan\left(\frac{12\pi+\pi}{4}\right) = tan\left(3\pi + \frac{\pi}{4}\right)$

$= tan\frac{\pi}{4} = 1$ **(3π is a period of *tan*.)**

$\therefore \quad tan\left(\frac{13\pi}{4}\right) = 1$

(4) $3sec\left(\frac{-7\pi}{4}\right) = 3sec\left(\frac{7\pi}{4}\right)$ **($sec(-\theta) = sec\theta$)**

$= 3sec\left(\frac{8\pi-\pi}{4}\right)$

$= 3sec\left(2\pi - \frac{\pi}{4}\right)$

$= 3sec\left(\frac{-\pi}{4}\right)$ **(2π is a period of *sec*.)**

$= 3sec\frac{\pi}{4} = 3\sqrt{2}$

$\therefore \quad 3sec\left(\frac{-7\pi}{4}\right) = 3\sqrt{2}$

Example 2 : Evaluate : (1) $\frac{sin\left(\theta - \frac{\pi}{2}\right)}{cos(\theta - \pi)} + \frac{tan\left(\frac{\pi}{2}+\theta\right)}{cot(3\pi+\theta)} + \frac{cosec(2\pi+\theta)}{sec\left(\frac{3\pi}{2}-\theta\right)}$

(2) $sin\frac{10\pi}{3} \cdot cos\frac{11\pi}{6} + cos\frac{2\pi}{3} \cdot sin\frac{5\pi}{6}$

(3) $cos^2\frac{\pi}{8} + cos^2\frac{3\pi}{8} + cos^2\frac{5\pi}{8} + cos^2\frac{7\pi}{8}$

Solution : (1) $\dfrac{\sin\left(\theta-\frac{\pi}{2}\right)}{\cos(\theta-\pi)} + \dfrac{\tan\left(\frac{\pi}{2}+\theta\right)}{\cot(3\pi+\theta)} + \dfrac{\operatorname{cosec}(2\pi+\theta)}{\sec\left(\frac{3\pi}{2}-\theta\right)}$

$= \dfrac{-\sin\left(\frac{\pi}{2}-\theta\right)}{\cos(\pi-\theta)} + \dfrac{-\cot\theta}{\cot\theta} + \dfrac{\operatorname{cosec}\theta}{-\operatorname{cosec}\theta}$

$= \dfrac{-\cos\theta}{-\cos\theta} + (-1) + (-1)$

$= 1 - 1 - 1 = -1$

(2) $\sin\frac{10\pi}{3} \cdot \cos\frac{11\pi}{6} + \cos\frac{2\pi}{3} \cdot \sin\frac{5\pi}{6}$

$= \sin\left(\frac{9\pi+\pi}{3}\right) \cdot \cos\left(\frac{12\pi-\pi}{6}\right) + \cos\left(\frac{3\pi-\pi}{3}\right) \cdot \sin\left(\frac{6\pi-\pi}{6}\right)$

$= \sin\left(3\pi+\frac{\pi}{3}\right) \cdot \cos\left(2\pi-\frac{\pi}{6}\right) + \cos\left(\pi-\frac{\pi}{3}\right) \cdot \sin\left(\pi-\frac{\pi}{6}\right)$

$= -\sin\frac{\pi}{3} \cdot \cos\frac{\pi}{6} + \left(-\cos\frac{\pi}{3}\right) \cdot \sin\frac{\pi}{6}$

$= \frac{-\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{-3}{4} - \frac{1}{4} = -1$

(3) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \sin^2\left(\frac{\pi}{2} - \frac{5\pi}{8}\right) + \sin^2\left(\frac{\pi}{2} - \frac{7\pi}{8}\right)$

$= \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \sin^2\left(\frac{-\pi}{8}\right) + \sin^2\left(\frac{-3\pi}{8}\right)$

$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) + \left(\cos^2\frac{3\pi}{8} + \sin^2\frac{3\pi}{8}\right)$

$= 1 + 1 = 2$

Example 3 : Decide whether following numbers are positive or negative.

(1) $\sin 110^\circ + \cos 110^\circ$ (2) $\operatorname{cosec}\frac{17\pi}{12} - \sec\frac{17\pi}{12}$

Solution : (1) $\sin 110^\circ + \cos 110^\circ = \sin(180^\circ - 70^\circ) + \cos(90^\circ + 20^\circ)$

$= \sin 70^\circ - \sin 20^\circ$

Now *sine* is an increasing function in the first quadrant.

$\therefore$ 70 > 20. Hence $\sin 70^\circ > \sin 20^\circ$

$\therefore$ $\sin 70^\circ - \sin 20^\circ > 0$

$\therefore$ $\sin 110^\circ + \cos 110^\circ$ is positive.

(2) $\operatorname{cosec}\frac{17\pi}{12} - \sec\frac{17\pi}{12}$

$= \operatorname{cosec}\left(\frac{12\pi+5\pi}{12}\right) - \sec\left(\frac{18\pi-\pi}{12}\right)$

$= \operatorname{cosec}\left(\pi+\frac{5\pi}{12}\right) - \sec\left(\frac{3\pi}{2}-\frac{\pi}{12}\right)$

$= -\operatorname{cosec}\frac{5\pi}{12} + \operatorname{cosec}\frac{\pi}{12}$

Now, as *sine* is increasing and so *cosec* is a decreasing function in the first quadrant and

$\frac{\pi}{12} < \frac{5\pi}{12}$

$\therefore \quad cosec\frac{\pi}{12} > cosec\frac{5\pi}{12}$

$\therefore \quad \left(cosec\frac{\pi}{12} - cosec\frac{5\pi}{12}\right) > 0$

$\therefore \quad cosec\frac{17\pi}{12} - sec\frac{17\pi}{12}$ is positive.

Exercise 4.1

1. Evaluate :

(1) $cos135^{\circ}$ (2) $tan\left(\frac{-23\pi}{6}\right)$ (3) $cos\left(\frac{-50\pi}{3}\right)$

(4) $sec690^{\circ}$ (5) $cosec\ \frac{15\pi}{4}$ (6) $cot\left(\frac{-7\pi}{3}\right)$

Prove : **(2 to 11)**

2. $cos\left(\frac{\pi}{2}+\theta\right) \cdot sec(-\theta) \cdot tan(\pi - \theta) + sec(2\pi + \theta) \cdot sin(\pi + \theta) \cdot cot\left(\frac{\pi}{2}-\theta\right) = 0$

3. $\frac{sin(\pi-\theta)}{sin(\pi+\theta)} \cdot \frac{cosec(\pi+\theta)}{cosec(-\pi+\theta)} \cdot \frac{cosec(2\pi+\theta)}{sin(3\pi-\theta)} = -cosec^2\theta$

4. $\frac{sin(-\theta) \cdot tan\left(\frac{\pi}{2}-\theta\right) \cdot sin(\pi-\theta) \cdot sec\left(\frac{3\pi}{2}+\theta\right)}{sin(\pi+\theta) \cdot cos\left(\frac{3\pi}{2}-\theta\right) \cdot cosec(\pi-\theta) \cdot cot(2\pi-\theta)} = 1$

5. $sin(n + 1)A \cdot cos(n + 2)A - cos(n + 1)A \cdot sin(n + 2)A = -sinA$

6. $sin^2(40^{\circ} + \theta) + sin^2(50^{\circ} - \theta) = 1$

7. $\frac{cot\ 333^{\circ} - cos\,567^{\circ}}{tan\,297^{\circ} + sin\,477^{\circ}} = 1$

8. $\frac{sec^2 129^{\circ} - cosec^2 31^{\circ}}{cosec39^{\circ} - sec121^{\circ}} = cosec39^{\circ} - sec59^{\circ}$

9. $cos(A + B + C) = cosA\ cosB\ cosC - sinA \cdot sinB \cdot cosC - sinA\ cosB\ sinC - cosA\ sinB\ sinC$

10. $sin\alpha \cdot sin(\beta - \gamma) + sin\beta \cdot sin(\gamma - \alpha) + sin\gamma \cdot sin(\alpha - \beta) = 0$

11. $(sin\alpha - cos\alpha) \cdot (sin\beta + cos\beta) = sin(\alpha - \beta) - cos(\alpha + \beta)$

12. For ΔABC, prove following results :

(1) $sin(B + C) = sinA$ (2) $cos(A + B) = -cosC$

(3) $sin\left(\frac{B+C}{2}\right) = cos\frac{A}{2}$ (4) $tan(A - B - C) = tan2A$

(5) $\frac{sin(B+C) \cdot cos(B+C) \cdot sin\frac{A}{2} \cdot cos\frac{A}{2}}{sin\left(\frac{B+C}{2}\right) \cdot cos\left(\frac{B+C}{2}\right) \cdot sin(\pi+A) \cdot cos(2\pi-A)} = 1$

(6) If $cosA = cosB\ cosC$, then prove that $2cotB\ cotC = 1$.

13. For a convex quadrilateral ABCD, prove that

(1) $sin(A + B) + sin(C + D) = sin(B + C) + sin(A + D)$

(2) $cot(A + B + C) + cotD = 0$

14. For cyclic quadrilateral ABCD, prove that

(1) $\cos A + \cos B + \cos C + \cos D = 0$

(2) $\sin A + \sin B = \sin C + \sin D$

15. If $\alpha - \beta = \frac{\pi}{6}$, then prove that $2\sin\alpha - \cos\beta = \sqrt{3}\sin\beta$.

16. If $\theta = \frac{19\pi}{4}$, then prove that $\cos^2\theta - \sin^2\theta - 2\tan\theta + \sec^2\theta - 4\cot^2\theta = 0$.

17. Evaluate : (1) $\sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\frac{5\pi}{12} + \sin^2\frac{7\pi}{12} + \sin^2\frac{9\pi}{12} + \sin^2\frac{11\pi}{12}$

(2) $\sin x + \sin(\pi + x) + \sin(2\pi + x) + ...\ 2n$ terms.

(3) $\cos x + \cos(\pi - x) + \cos(2\pi - x) + \cos(3\pi - x) + ...\ (2n + 1)$ terms, if $x = \frac{\pi}{3}$

(4) $\cot\frac{\pi}{20} \cdot \cot\frac{3\pi}{20} \cdot \cot\frac{5\pi}{20} \cdot \cot\frac{7\pi}{20} \cdot \cot\frac{9\pi}{20}$

18. Determine whether each of the following is positive or negative :

(1) $\sin 155^\circ + \cos 155^\circ$ (2) $\tan\frac{6\pi}{7} + \cot\left(\frac{-6\pi}{7}\right)$

(3) $\tan 111^\circ - \cot 111^\circ$ (4) $\operatorname{cosec}\frac{7\pi}{12} + \sec\frac{7\pi}{12}$

19. If $\tan\theta = \left(\frac{-3}{4}\right)$ and $\frac{\pi}{2} < \theta < \pi$, then find the value of $\dfrac{\sin(\pi - \theta) + \tan(\pi + \theta) + \tan(4\pi - \theta)}{\sin\left(\frac{3\pi}{2} + \theta\right) + \cos\left(\frac{5\pi}{2} - \theta\right)}$.

20. Prove that $\sin(n\pi + (-1)^n\,\theta) = \sin\theta$, for all $n \in N$.

*

4.4 Some Important Results

(1) We have already obtained values of trigonometric functions for $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. With the help of $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, we will obtain values of $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

Let $\alpha = \frac{\pi}{3}, \beta = \frac{\pi}{4}$ or $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{6}$

$\therefore\ \alpha - \beta = \frac{\pi}{12}$

$$\therefore\ \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\therefore\ \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Similarly, we can show that $\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$

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Also, $sin\frac{5\pi}{12} = sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$cos\frac{5\pi}{12} = cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

(2) **(i)** $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta = cos^2\beta - cos^2\alpha$

(ii) $cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$

(i) $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = (sin\alpha\ cos\beta + cos\alpha\ sin\beta)(sin\alpha\ cos\beta - cos\alpha\ sin\beta)$

$= sin^2\alpha \cdot cos^2\beta - cos^2\alpha \cdot sin^2\beta$

$= sin^2\alpha\ (1 - sin^2\beta) - (1 - sin^2\alpha) \cdot sin^2\beta$

$= sin^2\alpha - sin^2\alpha\ sin^2\beta - sin^2\beta + sin^2\alpha\ sin^2\beta$

$= sin^2\alpha - sin^2\beta$

$\therefore$ $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

Now, $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

$= (1 - cos^2\alpha) - (1 - cos^2\beta)$

$= cos^2\beta - cos^2\alpha$

$\therefore$ $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = cos^2\beta - cos^2\alpha$

Similarly, it can be proved that

$cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$

4.5 The Range of $f(\alpha) = acos\alpha + bsin\alpha,\ \alpha \in R,\ a,\ b \in R,\ a^2 + b^2 \neq 0$

As $a^2 + b^2 \neq 0$, we consider three cases :

(1) $a = 0,\ b \neq 0$ (2) $a \neq 0,\ b = 0$ (3) $a \neq 0,\ b \neq 0$

Case (1) : $a = 0,\ b \neq 0$

Then, $f(\alpha) = bsin\alpha$. Range of $sin\alpha$ is $[-1, 1]$.

$-1 \leq sin\alpha \leq 1$

$\Leftrightarrow$ $-b \leq bsin\alpha \leq b$ $(b > 0)$

$\therefore$ For $b > 0$, the range of $bsin\alpha$ is $[-b, b] = [-|b|, |b|]$. $(|b| = b)$

Now; for $b < 0$, $-1 \leq sin\alpha \leq 1 \Leftrightarrow -b \geq bsin\alpha \geq b$

$\Leftrightarrow b \leq bsin\alpha \leq -b$

$\therefore$ For $b < 0$, the range is $[b, -b] = [-|b|, |b|]$. $(|b| = -b)$

$\therefore$ The range of $f(\alpha) = bsin\alpha$ is $[-|b|, |b|]$.

Case (2) : $a \neq 0,\ b = 0$

Then, $f(\alpha) = acos\alpha$. Its range is $[-|a|, |a|]$ as before.

Case (3) : $a \neq 0,\ b \neq 0$

In this case, we shall express $acos\alpha + bsin\alpha$ in the form $r\ cos(\theta - \alpha)$.

As $r\ cos(\theta - \alpha) = r\ cos\theta\ cos\alpha + r\ sin\theta\ sin\alpha$, we shall find r and θ such that $a = r\ cos\theta$, $b = r\ sin\theta$. $(r > 0)$

These equations imply that $a^2 + b^2 = r^2$ and $tan\theta = \frac{b}{a}$.

$\therefore$ $r = \sqrt{a^2 + b^2}$. As the range of *tan* function is R, corresponding to real number $\frac{b}{a}$ we can find $\theta \in R - \left\{(2n - 1)\frac{\pi}{2} \mid n \in Z\right\}$ such that $tan\theta = \frac{b}{a}$.

Hence, for given *a* and *b* (both non-zero), we can select $r = \sqrt{a^2 + b^2}$ and θ such that $tan\theta = \frac{b}{a}$. We can select θ, so that $rcos\theta = a$, $rsin\theta = b$.

Thus, $f(\alpha) = acos\alpha + bsin\alpha$

$$= rcos\theta\ cos\alpha + r\ sin\theta\ sin\alpha$$

$$= r\,(cos\theta\ cos\alpha + sin\theta\ sin\alpha)$$

$$= r\ cos(\theta - \alpha)$$

$$f(\alpha) = r\ cos(\theta - \alpha)$$

$$-1 \le cos(\theta - \alpha) \le 1 \Leftrightarrow -r \le r\ cos(\theta - \alpha) \le r \qquad (r > 0)$$

$\therefore$ The range of $f(\alpha)$ is $[-r, r]$.

Hence, range of $f(\alpha)$ is $\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right]$.

This means that the maximum value attained by $f(\alpha)$ is $\sqrt{a^2 + b^2}$ and the minimum value is $-\sqrt{a^2 + b^2}$.

4.6 Addition Formulae for *tan* and *cot*

(1) If α, β and $\alpha + \beta \in R - \left\{(2k - 1)\frac{\pi}{2} \mid k \in Z\right\}$, then

$$tan\,(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta}$$

and if α, β and $\alpha - \beta \in R - \left\{(2k - 1)\frac{\pi}{2} \mid k \in Z\right\}$, then

$$tan\,(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$$

Proof : $tan\,(\alpha + \beta) = \frac{sin\,(\alpha + \beta)}{cos\,(\alpha + \beta)} = \frac{sin\,\alpha\ cos\beta + cos\alpha\ sin\,\beta}{cos\alpha\ cos\beta - sin\,\alpha\ sin\,\beta}$ $\left(\alpha + \beta \in R - \left\{(2k - 1)\frac{\pi}{2} \mid k \in Z\right\}\right)$

As, α, $\beta \in R - \left\{(2k - 1)\frac{\pi}{2} \mid k \in Z\right\}$, $cos\alpha \ne 0$, $cos\beta \ne 0$

Hence, dividing both numerator and denominator by $cos\alpha \cdot cos\beta$, we get

$$tan\,(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta}$$

Similarly, we can get, $tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$

(2) If α, β and $\alpha + \beta \in R - \{k\pi \mid k \in Z\}$, then

$$cot\,(\alpha + \beta) = \frac{cot\,\alpha \cdot cot\,\beta - 1}{cot\,\beta + cot\,\alpha}$$

and if α, β and $\alpha - \beta \in R - \{k\pi \mid k \in Z\}$, then

$$cot\,(\alpha - \beta) = \frac{cot\,\alpha \cdot cot\,\beta + 1}{cot\,\beta - cot\,\alpha}$$

Proof : $cot\,(\alpha + \beta) = \frac{cos(\alpha+\beta)}{sin(\alpha+\beta)} = \frac{cos\alpha \cdot cos\beta - sin\,\alpha \cdot sin\,\beta}{sin\,\alpha \cdot cos\beta + cos\alpha \cdot sin\,\beta}$ **$(\alpha + \beta \in R - \{k\pi \mid k \in Z\})$**

As, $\alpha, \beta \in R - \{k\pi \mid k \in Z\}$, $sin\alpha \neq 0$, $sin\beta \neq 0$.

Hence, dividing both numerator and denominator by $sin\alpha \cdot sin\beta$, we get

$$cot\,(\alpha + \beta) = \frac{cot\,\alpha \cdot cot\,\beta - 1}{cot\,\beta + cot\,\alpha}.$$

Similarly, we can prove that, $cot(\alpha - \beta) = \frac{cot\,\alpha \cdot cot\,\beta + 1}{cot\,\beta - cot\,\alpha}$.

4.7 Value of $tan\,\frac{\pi}{12}$ and $cot\,\frac{\pi}{12}$

We have $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ or $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$

(1) $tan\frac{\pi}{12} = tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{tan\,\frac{\pi}{3} - tan\,\frac{\pi}{4}}{1 + tan\,\frac{\pi}{3}\,tan\,\frac{\pi}{4}}$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$\therefore \quad tan\frac{\pi}{12} = 2 - \sqrt{3}$

(2) $cot\frac{\pi}{12} = cot\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{cot\frac{\pi}{3}\,cot\frac{\pi}{4} + 1}{cot\frac{\pi}{4} - cot\frac{\pi}{3}}$

$$= \frac{\frac{1}{\sqrt{3}} \cdot 1 + 1}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{3 + 2\sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$cot\frac{\pi}{12} = 2 + \sqrt{3}$

Also, $tan\frac{5\pi}{12} = tan\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = cot\frac{\pi}{12} = 2 + \sqrt{3}$ and

$$cot\frac{5\pi}{12} = cot\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = tan\frac{\pi}{12} = 2 - \sqrt{3}.$$

Example 4 : If $sin\alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$ and $tan\beta = \frac{-12}{5}$, $-\frac{\pi}{2} < \beta < 0$, then determine the quadrant of $P(\alpha + \beta)$ and $P(\alpha - \beta)$.

Solution : Here $\frac{\pi}{2} < \alpha < \pi$ and $-\frac{\pi}{2} < \beta < 0$. On addition, we get $0 < \alpha + \beta < \pi$.

$\therefore$ $P(\alpha + \beta)$ is in the first or in the second quadrant. As *cosine* takes +*ve* value in the first quadrant and −*ve* value in the second quadrant and *sine* takes +*ve* value in the first and the second both quadrants, so to determine the quadrant of $P(\alpha + \beta)$, we must find $cos(\alpha + \beta)$.

$\therefore$ $cos\alpha = -\sqrt{1 - sin^2\alpha} = -\sqrt{1 - \frac{16}{25}} = \frac{-3}{5}$ **$\left(\frac{\pi}{2} < \alpha < \pi\right)$**

$tan\beta = \frac{-12}{5}$, $-\frac{\pi}{2} < \beta < 0$

$\therefore$ $sec\beta = \sqrt{1 + tan^2\beta} = \sqrt{1 + \frac{144}{25}} = \frac{13}{5}$ **$\left(-\frac{\pi}{2} < \beta < 0\right)$**

$\therefore$ $cos\beta = \frac{5}{13}$, $sin\beta = tan\beta \cdot cos\beta = \frac{-12}{5} \times \frac{5}{13} = \frac{-12}{13}$

$$cos(\alpha + \beta) = cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta$$

$$= \left(\frac{-3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$$

$$= \frac{-15}{65} + \frac{48}{65} = \frac{33}{65}$$

$\therefore$ $cos(\alpha + \beta) > 0$

$\therefore$ $P(\alpha + \beta)$ is in the first quadrant.

Second method for determining quadrant :

To determine the quadrant of $P(\alpha + \beta)$, we can use another method.

$$sin(\alpha + \beta) = sin\alpha\, cos\beta + cos\alpha\, sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) = \frac{20 + 36}{65} = \frac{56}{65} > 0$$

$cos(\alpha + \beta) = cos\alpha\, cos\beta - sin\alpha\, sin\beta = \frac{33}{65}$ **(Method 1)**

As $sin(\alpha + \beta) > 0$ and $cos(\alpha + \beta) > 0$, $P(\alpha + \beta)$ is in the first quadrant.

Now, for $P(\alpha - \beta)$. $\frac{\pi}{2} < \alpha < \pi$ and $-\frac{\pi}{2} < \beta < 0$

$\therefore$ $\frac{\pi}{2} > -\beta > 0$

$\therefore$ $0 < -\beta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \alpha < \pi$ **(i)**

$\frac{\pi}{2} < \alpha - \beta < \frac{3\pi}{2}$ **(adding inequalities in (i))**

$\therefore$ P($\alpha - \beta$) is in the second or in the third quadrant. As *sine* takes +*ve* values in the second quadrant and −*ve* values in the third quadrant and *cosine* takes −*ve* values in the second and in the third both quadrants, so to determine the quadrant of P($\alpha - \beta$), we must find *sin*($\alpha - \beta$).

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) = \frac{20-36}{65} = \frac{-16}{65}$$

$\therefore$ $\sin(\alpha - \beta) < 0$

$\therefore$ P($\alpha - \beta$) is in the third quadrant.

Example 5 : Find the range of $\sin\theta + \cos\left(\theta + \frac{\pi}{3}\right)$.

Solution : Suppose $f(\theta) = \sin\theta + \cos\left(\theta + \frac{\pi}{3}\right)$

$$= \sin\theta + \cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3}$$

$$= \sin\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

$$f(\theta) = \frac{1}{2}\cos\theta + \left(1 - \frac{\sqrt{3}}{2}\right)\sin\theta = a\cos\theta + b\sin\theta$$

Comparing $f(\theta)$ with $a\cos\theta + b\sin\theta$, we get

$$a = \frac{1}{2},\ b = 1 - \frac{\sqrt{3}}{2}$$

Now, $r^2 = a^2 + b^2 = \frac{1}{4} + \left(1 - \frac{\sqrt{3}}{2}\right)^2$

$$= \frac{1}{4} + 1 - \sqrt{3} + \frac{3}{4}$$

$$r^2 = 2 - \sqrt{3}$$

$\therefore$ $r = \sqrt{2-\sqrt{3}} = \sqrt{\frac{4-2\sqrt{3}}{2}} = \sqrt{\frac{3-2\sqrt{3}+1}{2}} = \sqrt{\frac{(\sqrt{3}-1)^2}{2}}$

$\therefore$ $r = \frac{\sqrt{3}-1}{\sqrt{2}} = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}$

$\therefore$ The range of $f(\theta)$ is $[-r, r] = \left[\frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}\right]$.

Example 6 : Determine whether the $\sin 110^\circ + \cos 110^\circ$ is positive or negative.

Solution : Suppose $f(\theta) = \sin 110^\circ + \cos 110^\circ$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 110^\circ + \frac{1}{\sqrt{2}}\cos 110^\circ\right)$$

$$= \sqrt{2}(\cos 45^\circ \sin 110^\circ + \sin 45^\circ \cos 110^\circ)$$

$$= \sqrt{2}\sin(110^\circ + 45^\circ)$$

$$= \sqrt{2}\sin 155^\circ > 0 \qquad (90 < 155 < 180)$$

$\therefore$ $\sin 110^\circ + \cos 110^\circ$ is positive.

Note : The example 3 solved earlier in this chapter can be solved by this alternative method too.

Example 7 : Express $\sqrt{3}\sin\alpha - \cos\alpha$ in the form $r\sin(\alpha - \theta)$ and find r and θ, where, $r > 0$, $0 \le \theta < 2\pi$.

Solution : Let $f(\alpha) = \sqrt{3}\sin\alpha - \cos\alpha$

Multiplying and dividing by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}\sin\alpha - \frac{1}{2}\cos\alpha\right)$$

$$= 2\left(\sin\alpha\cos\frac{\pi}{6} - \cos\alpha\sin\frac{\pi}{6}\right)$$

$$= 2\sin\left(\alpha - \frac{\pi}{6}\right)$$

$$= r\sin(\alpha - \theta)$$

$r = 2$, $\theta = \frac{\pi}{6}$. Here $\theta = \frac{\pi}{6}$ satisfies $0 \le \theta < 2\pi$.

Example 8 : If $\sqrt{3}\cos\alpha - \sin\alpha = r\cos(\alpha - \theta)$, find r and θ. $r > 0$,

where (i) $0 < \theta < 2\pi$ (ii) $\frac{-\pi}{2} < \theta < 0$

Solution : Let $f(\alpha) = \sqrt{3}\cos\alpha - \sin\alpha$

Multiplying and dividing by $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)$$

$$= 2\left(\cos\frac{\pi}{6}\cos\alpha - \sin\frac{\pi}{6}\sin\alpha\right)$$

$$= 2\cos\left(\alpha + \frac{\pi}{6}\right)$$

$$= 2\cos\left(\alpha - \left(-\frac{\pi}{6}\right)\right)$$

Now comparing with $r\cos(\alpha - \theta)$, we get

$r = 2$, $\theta = \frac{-\pi}{6}$ and $\theta = \frac{-\pi}{6}$ satisfies $\frac{-\pi}{2} < \theta < 0$

$$2\cos\left(\alpha + \frac{\pi}{6}\right) = 2\cos\left(\alpha + \frac{\pi}{6} - 2\pi\right) = 2\cos\left(\alpha - \frac{11\pi}{6}\right)$$

$\therefore \quad \theta = \frac{11\pi}{6}$ satisfies $0 < \theta < 2\pi$.

Example 9 : Prove that $\sin^2 A = \cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A\cos B$.

Solution : R.H.S. $= \cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A\cos B$.

$= \cos^2 B + \cos^2(A - B) - 2\cos(A - B)\cos A\cos B$

$= \cos^2 B + \cos(A - B)[\cos(A - B) - 2\cos A\cos B]$

$= \cos^2 B + \cos(A - B)[\cos A\cos B + \sin A\sin B - 2\cos A\cos B]$

$= \cos^2 B + \cos(A - B)(\sin A\sin B - \cos A\cos B)$

$= \cos^2 B - \cos(A - B)\cos(A + B)$

$= \cos^2 B - (\cos^2 A - \sin^2 B)$

$= \cos^2 B + \sin^2 B - \cos^2 A$

$= 1 - \cos^2 A$

$= \sin^2 A =$ L.H.S.

Exercise 4.2

1. Evaluate :

(1) $\sin^2 37\frac{1}{2}^{\circ} - \sin^2 7\frac{1}{2}^{\circ}$ (2) $\sin^2 52\frac{1}{2}^{\circ} - \cos^2 7\frac{1}{2}^{\circ}$ (3) $\cos^2 37\frac{1}{2}^{\circ} - \sin^2 37\frac{1}{2}^{\circ}$

2. Prove that : $\sin^2 A + \sin^2 B + \cos^2(A + B) + 2\sin A \sin B \cos(A + B) = 1$.

3. (1) If $\cos A = \frac{1}{7}$, $\cos B = \frac{13}{14}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A - B = \frac{\pi}{3}$.

(2) If $\sin A = \frac{1}{\sqrt{5}}$, $\cos B = \frac{3}{\sqrt{10}}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A + B = \frac{\pi}{4}$.

4. (1) Find the quadrant of $P(\alpha - \beta)$, if $\cos\alpha = \frac{4}{5}$, $\cos\beta = \frac{12}{13}$, $\frac{3\pi}{2} < \alpha, \beta < 2\pi$.

(2) Find the quadrant of $P(\alpha + \beta)$, if $\cos\alpha = \frac{-5}{13}$, $\frac{\pi}{2} < \alpha < \pi$ and $\tan\beta = \frac{4}{3}$, $\pi < \beta < \frac{3\pi}{2}$.

5. If $\cot\alpha = \frac{1}{2}$, $\sec\beta = \frac{-5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $\tan(\alpha + \beta)$ and find the quadrant of $P(\alpha + \beta)$.

6. Determine the range of (1) $7\sin\theta + 24\cos\theta$ (2) $\cos\theta + \sin\left(\theta - \frac{\pi}{6}\right) + 1$

7. Prove that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 7$ in $[0, 14]$.

8. Express $\sqrt{3}\sin\theta + \cos\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < 2\pi$.

9. $\frac{-\pi}{2} < \theta < 0$ and $\cos\alpha - \sqrt{3}\sin\alpha = r\cos(\alpha - \theta)$, find r and θ.

10. Prove :

(1) $\tan\left(\frac{\pi}{3} - \alpha\right) = \frac{\sqrt{3}\cos\alpha - \sin\alpha}{\cos\alpha + \sqrt{3}\sin\alpha}$ (2) $\tan 39^{\circ} = \frac{\sqrt{3}\cos 21^{\circ} - \sin 21^{\circ}}{\cos 21^{\circ} + \sqrt{3}\sin 21^{\circ}}$

(3) $\tan 3A \cdot \tan 2A \cdot \tan A = \tan 3A - \tan 2A - \tan A$

(4) $\cot A \cdot \cot 2A - \cot 2A \cdot \cot 3A - \cot 3A \cdot \cot A = 1$

(5) $\tan 25^{\circ} \cdot \tan 15^{\circ} + \tan 15^{\circ} \cdot \tan 50^{\circ} + \tan 25^{\circ} \cdot \tan 50^{\circ} = 1$

11. If $A + B = \frac{\pi}{4}$, then prove that

(1) $(1 + \tan A)(1 + \tan B) = 2$

(2) $(\cot A - 1)(\cot B - 1) = 2$

12. (1) Prove that $A + B = \frac{\pi}{2} \Rightarrow \tan A = \tan B + 2\tan(A - B)$

(2) Prove that $\tan 65^{\circ} = \tan 25^{\circ} + 2\tan 40^{\circ}$

13. If $A + B + C = (2k + 1)\frac{\pi}{2}$, $k \in Z$, then prove that

(1) $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

(2) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$

14. If A + B + C = $k\pi$, $k \in$ Z, then prove that

(1) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(2) $\cot B \cdot \cot C + \cot C \cdot \cot A + \cot A \cdot \cot B = 1$

15. If $\tan A = 3$, $\tan B = \frac{1}{2}$, $0 < A, B < \frac{\pi}{2}$, then prove that $A - B = \frac{\pi}{4}$.

16. If $\tan B = 2$ and $\tan C = 3$ in ΔABC, then prove that $\tan A = 1$.

17. If $0 < A, B < \frac{\pi}{2}$, $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, prove that $A + B = \frac{\pi}{4}$.

18. If $\alpha + \beta = \theta$, $\alpha - \beta = \phi$ and $\frac{\tan \alpha}{\tan \beta} = \frac{x}{y}$, then prove that $\frac{\sin \theta}{\sin \phi} = \frac{x+y}{x-y}$.

19. If $\frac{\tan (A - B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, then prove that $\tan A \cdot \tan B = \tan^2 C$.

20. If $\tan(A + B) = 3$ and $\tan(A - B) = 2$, then find $\tan 2A$ and $\tan 2B$.

21. If $\tan\beta = \frac{n \sin\alpha \cos\alpha}{1 - n\sin^2\alpha}$, then prove that $\tan(\alpha - \beta) = (1 - n) \tan\alpha$.

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4.8 Expression of a Product in the Form of a Sum or a Difference

We have studied the following formulae valid for all real $\alpha, \beta \in R$:

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$ **(i)**

$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ **(ii)**

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ **(iii)**

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ **(iv)**

Taking sum and difference of (i) and (ii), we get,

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha \cos\beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha \sin\beta$$

that is,

$2\sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$ **(v)**

$2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$ **(vi)**

In the same way, taking sum and difference of (iii) and (iv), we get

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha \cos\beta$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha \sin\beta$$

that is,

$2\cos\alpha \cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$ **(vii)**

$2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ **(viii)**

In each of (v), (vi), (vii) and (viii), the left side is the product of trigonometric functions where as the right side is the sum or difference of a trigonometric functions with variable $\alpha + \beta$ or $\alpha - \beta$. It would therefore be easy to express product of trigonometric functions in terms of a sum or a difference.

For example, $2\sin 3\theta \cos 5\theta = \sin(3\theta + 5\theta) + \sin(3\theta - 5\theta)$

$= \sin 8\theta + \sin(-2\theta)$

$= \sin 8\theta - \sin 2\theta$ $(\sin(-\theta) = -\sin\theta)$

Now, if we take bigger angle first, then calculation become simpler

$2\cos 3\theta \cdot \sin 5\theta = 2\sin 5\theta \cdot \cos 3\theta = \sin(5\theta + 3\theta) + \sin(5\theta - 3\theta)$

$= \sin 8\theta + \sin 2\theta$

Example 11 : Express each of the following as a sum or a difference.

(1) $2\sin 5\theta \cos\theta$ (2) $2\cos\frac{5\theta}{2} \sin\frac{3\theta}{2}$ (3) $2\sin 3\theta \sin 5\theta$ (4) $\sin^2\theta$ (5) $2\cos 5\theta \cos\frac{\theta}{2}$

Solution : (1) $2\sin 5\theta \cos\theta = \sin(5\theta + \theta) + \sin(5\theta - \theta) = \sin 6\theta + \sin 4\theta$

(2) $2\cos\frac{5\theta}{2} \sin\frac{3\theta}{2} = \sin\left(\frac{5\theta}{2} + \frac{3\theta}{2}\right) - \sin\left(\frac{5\theta}{2} - \frac{3\theta}{2}\right) = \sin 4\theta - \sin\theta$

(3) $2\sin 3\theta \sin 5\theta = \cos(3\theta - 5\theta) - \cos(3\theta + 5\theta) = \cos(-2\theta) - \cos 8\theta$

$= \cos 2\theta - \cos 8\theta$

(4) $\sin^2\theta = \sin\theta \sin\theta = \frac{1}{2}[2\sin\theta \sin\theta] = \frac{1}{2}\left[\cos(\theta - \theta) - \cos(\theta + \theta)\right]$

$= \frac{1}{2}[\cos 0 - \cos 2\theta] = \frac{1}{2}[1 - \cos 2\theta]$

(5) $2\cos 5\theta \cos\frac{\theta}{2} = \cos\left(5\theta + \frac{\theta}{2}\right) + \cos\left(5\theta - \frac{\theta}{2}\right) = \cos\frac{11\theta}{2} + \cos\frac{9\theta}{2}$

Example 12 : Prove that $\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \frac{3}{16}$.

Solution : L.H.S. $= \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ$

$= \sin 60^\circ \cdot (\sin 20^\circ \cdot \sin 40^\circ) \cdot \sin 80^\circ$

$= \frac{\sqrt{3}}{2} \times \frac{1}{2} (2\sin 40^\circ \cdot \sin 20^\circ) \cdot \sin 80^\circ$

$= \frac{\sqrt{3}}{4} [\cos(40^\circ - 20^\circ) - \cos(40^\circ + 20^\circ)] \sin 80^\circ$

$= \frac{\sqrt{3}}{4} [\cos(20^\circ) - \cos 60^\circ] \sin 80^\circ$

$= \frac{\sqrt{3}}{4} \left(\cos 20^\circ - \frac{1}{2}\right) \sin 80^\circ$

$= \frac{\sqrt{3}}{8} (2\sin 80^\circ \cos 20^\circ - \sin 80^\circ)$

$= \frac{\sqrt{3}}{8} [\sin(80^\circ + 20^\circ) + \sin(80^\circ - 20^\circ) - \sin 80^\circ]$

$= \frac{\sqrt{3}}{8} [\sin 100^\circ + \sin 60^\circ - \sin 80^\circ]$

$= \frac{\sqrt{3}}{8} \left[\sin(180^\circ - 80^\circ) + \frac{\sqrt{3}}{2} - \sin 80^\circ\right]$

$= \frac{\sqrt{3}}{8} \left(\sin 80^\circ + \frac{\sqrt{3}}{2} - \sin 80^\circ\right)$

$= \frac{3}{16}$ = R.H.S.

Example 13 : If A + B = 90°, then find the maximum and minimum values of $\sin A \cdot \sin B$.

Solution : Let $y = \sin A \cdot \sin B = \sin A \sin(90^\circ - A) = \sin A \cos A$

Then, $y = \frac{1}{2}(2\sin A \cdot \cos A) = \frac{1}{2}[\sin(A + A) - \sin(A - A)]$

$= \frac{1}{2}\sin 2A$ **($\sin 0 = 0$)**

Now, $-1 \le \sin 2A \le 1 \Leftrightarrow \frac{-1}{2} \le \frac{1}{2}\sin 2A \le \frac{1}{2} \Leftrightarrow \frac{-1}{2} \le y \le \frac{1}{2}$

Hence, $\frac{1}{2}$ and $\frac{-1}{2}$ are respectively the maximum and minimum values of $\sin A \sin B$.

Exercise 4.3

1. Express as a sum or a difference :

(1) $2\sin 7\theta \cdot \cos 3\theta$ (2) $2\sin\frac{\theta}{2} \cdot \cos\frac{5\theta}{2}$ (3) $2\cos 5\theta \cdot \sin 3\theta$

(4) $2\cos\frac{5\theta}{2} \cdot \sin\frac{7\theta}{2}$ (5) $2\cos 11\theta \cdot \cos 3\theta$ (6) $2\cos\frac{5\theta}{2} \cdot \cos\frac{3\theta}{2}$

(7) $\sin 9\theta \cdot \sin 11\theta$ (8) $2\sin\frac{7\theta}{2} \cdot \sin\frac{9\theta}{2}$ (9) $2\sin\theta \cdot \cos\theta$

2. Find the value :

(1) $2\sin\frac{5\pi}{12} \cdot \sin\frac{\pi}{12}$ (2) $2\sin\frac{5\pi}{12} \cdot \cos\frac{7\pi}{12}$ (3) $2\cos\frac{\pi}{12} \cdot \sin\frac{5\pi}{12}$

(4) $2\cos\frac{5\pi}{12} \cdot \cos\frac{7\pi}{12}$ (5) $8\cos 15^\circ \cdot \cos 45^\circ \cdot \cos 75^\circ$ (6) $8\sin 10^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ$

3. Prove :

(1) $\sin\left(\frac{\pi}{4}+\theta\right) \sin\left(\frac{\pi}{4}-\theta\right) = \frac{1}{2}\cos 2\theta$

(2) $\sin\theta \cdot \sin\left(\frac{\pi}{3}-\theta\right) \cdot \sin\left(\frac{\pi}{3}+\theta\right) = \frac{1}{4}\sin 3\theta$

(3) $2\cos\frac{\pi}{13} \cdot \cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

(4) $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ = \frac{1}{16}$

(5) $4\cos 12^\circ \cdot \cos 48^\circ \cdot \cos 72^\circ = \cos 36^\circ$

4. Prove that $4\cos\theta \cdot \cos\left(\frac{\pi}{3}-\theta\right) \cos\left(\frac{\pi}{3}+\theta\right) = \cos 3\theta$ and deduce that

$\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$.

5. Find the value of $\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ$.

6. Prove that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{-1}{2}$.

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4.9 Expressing the Sum or the Difference as a Product

We have seen the formula (v) to (viii) which are reproduced below :

$2sin\alpha \cdot cos\beta = sin(\alpha + \beta) + sin(\alpha - \beta)$ **(v)**

$2cos\alpha \cdot sin\beta = sin(\alpha + \beta) - sin(\alpha - \beta)$ **(vi)**

$2cos\alpha \cdot cos\beta = cos(\alpha + \beta) + cos(\alpha - \beta)$ **(vii)**

$2sin\alpha \cdot sin\beta = cos(\alpha - \beta) - cos(\alpha + \beta)$ **(viii)**

Let us substitute $\alpha + \beta = C$ and $\alpha - \beta = D$ in these formulae.

Then, $\alpha = \frac{C+D}{2}$ and $\beta = \frac{C-D}{2}$. We get

$$sinC + sinD = 2sin\left(\frac{C+D}{2}\right) cos\left(\frac{C-D}{2}\right)$$

$$sinC - sinD = 2cos\left(\frac{C+D}{2}\right) sin\left(\frac{C-D}{2}\right)$$

$$cosC + cosD = 2cos\left(\frac{C+D}{2}\right) cos\left(\frac{C-D}{2}\right)$$

$$cosD - cosC = 2sin\left(\frac{C+D}{2}\right) sin\left(\frac{C-D}{2}\right) \quad \text{or}$$

$$cosC - cosD = -2sin\left(\frac{C+D}{2}\right) sin\left(\frac{C-D}{2}\right).$$

These formulae are useful as they express sums or differences as products.

Example 14 : Express the following as products :

(1) $sin6\theta + sin4\theta$ (2) $sin6\theta - sin2\theta$ (3) $cos5\theta + cos2\theta$

(4) $cos6\theta - cos10\theta$ (5) $sin\theta - 1$ (6) $cos\theta + 1$

Solution : (1) $sin6\theta + sin4\theta = 2sin\left(\frac{6\theta+4\theta}{2}\right) cos\left(\frac{6\theta-4\theta}{2}\right) = 2sin5\theta\ cos\theta$

(2) $sin6\theta - sin2\theta = 2cos\left(\frac{6\theta+2\theta}{2}\right) sin\left(\frac{6\theta-2\theta}{2}\right) = 2cos4\theta\ sin2\theta$

(3) $cos5\theta + cos2\theta = 2cos\left(\frac{5\theta+2\theta}{2}\right) cos\left(\frac{5\theta-2\theta}{2}\right) = 2cos\frac{7\theta}{2}\ cos\frac{3\theta}{2}$

(4) $cos6\theta - cos10\theta = -2sin\left(\frac{6\theta+10\theta}{2}\right) sin\left(\frac{6\theta-10\theta}{2}\right)$

$$= -2sin8\theta\ sin(-2\theta) = 2sin8\theta\ sin2\theta$$

(5) $sin\theta - 1 = sin\theta - sin\frac{\pi}{2} = 2cos\left(\frac{\theta+\frac{\pi}{2}}{2}\right) sin\left(\frac{\theta-\frac{\pi}{2}}{2}\right)$

$$= 2cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right) sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$$

(6) $\cos\theta + 1 = \cos\theta + \cos 0 = 2\cos\left(\frac{\theta + 0}{2}\right) \cos\left(\frac{\theta - 0}{2}\right) = 2\cos\frac{\theta}{2} \cos\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2}$

Example 15 : Prove that

(1) $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ = \frac{1}{2}$

(2) $1 + \cos 2A + \cos 4A + \cos 6A = 4\cos A \cdot \cos 2A \cdot \cos 3A$

(3) $\sqrt{3}\sin 10^\circ + \sqrt{2}\sin 55^\circ = \cos 80^\circ + 2\cos 50^\circ$

Solution : (1) L.H.S. $= \cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ$

$= \cos 20^\circ + \frac{1}{2} + 2\cos\left(\frac{100^\circ + 140^\circ}{2}\right) \cos\left(\frac{100^\circ - 140^\circ}{2}\right)$

$= \cos 20 + \frac{1}{2} + 2\cos 120^\circ \cos(20^\circ)$ **($\cos(-20^\circ) = \cos 20^\circ$)**

$= \cos 20^\circ + \frac{1}{2} + 2\cos(180^\circ - 60^\circ) \cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - 2\cos 60^\circ \cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - 2 \cdot \frac{1}{2}\cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - \cos 20^\circ$

$= \frac{1}{2}$ = R.H.S.

(2) L.H.S. $= 1 + \cos 2A + \cos 4A + \cos 6A$

$= (\cos 0 + \cos 2A) + (\cos 4A + \cos 6A)$

$= 2\cos A \cdot \cos A + 2\cos 5A \cdot \cos A$

$= 2\cos A(\cos A + \cos 5A)$

$= 2\cos A(2\cos 3A \cdot \cos 2A)$

$= 4\cos A \cdot \cos 2A \cdot \cos 3A$ = R.H.S.

(3) L.H.S. $= \sqrt{3}\sin 10^\circ + \sqrt{2}\sin 55^\circ$

$= 2 \cdot \frac{\sqrt{3}}{2}\sin 10^\circ + 2 \cdot \frac{1}{\sqrt{2}}\sin 55^\circ$

$= 2\sin 60^\circ \sin 10^\circ + 2\sin 45^\circ \sin 55^\circ$

$= \cos 50^\circ - \cos 70^\circ + \cos 10^\circ - \cos 100^\circ$

$= \cos 50^\circ - \cos(180^\circ - 80^\circ) - (\cos 70^\circ - \cos 10^\circ)$ **(rearranging)**

$= \cos 50^\circ + \cos 80^\circ + 2\sin 40^\circ \sin 30^\circ$

$= \cos 50^\circ + \cos 80^\circ + 2\sin(90^\circ - 50^\circ) \cdot \frac{1}{2}$

$= \cos 50^\circ + \cos 80^\circ + \cos 50^\circ$

$= \cos 80^\circ + 2\cos 50^\circ$ = R.H.S.

Exercise 4.4

1. Convert into a form of product :

(1) $\sin 7\theta + \sin 3\theta$ (2) $\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}$ (3) $\sin 3\theta - \sin 5\theta$

(4) $\sin\frac{7\theta}{2} - \sin\frac{3\theta}{2}$ (5) $\cos 11\theta + \cos 9\theta$ (6) $\cos\frac{5\theta}{2} + \cos\frac{11\theta}{2}$

(7) $\cos 5\theta - \cos 11\theta$ (8) $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}$ (9) $\cos\theta - 1$

(10) $\sin\theta + 1$ (11) $\cos\theta + \sin\theta$ (12) $\sin\theta - \cos\theta$

Prove : **(2 to 7)**

2. (1) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$ (2) $\cos\frac{5\pi}{12} - \cos\frac{\pi}{12} = \frac{-1}{\sqrt{2}}$

(3) $\sin 65^\circ + \cos 65^\circ = \sqrt{2}\cos 20^\circ$ (4) $\dfrac{\sin\frac{5\pi}{12} - \cos\frac{5\pi}{12}}{\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}} = \frac{1}{\sqrt{3}}$

(5) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

(6) $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$

(7) $\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

3. (1) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

(2) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$

4. (1) $\sin A + \sin B + \sin C - \sin(A + B + C) = 4\sin\frac{A+B}{2} \sin\frac{B+C}{2} \sin\frac{C+A}{2}$

(2) $\cos A + \cos B + \cos C + \cos(A + B + C) = 4\cos\frac{A+B}{2} \cos\frac{B+C}{2} \cos\frac{C+A}{2}$

5. (1) $\dfrac{\sin(A+B) - 2\sin A + \sin(A-B)}{\cos(A+B) - 2\cos A + \cos(A-B)} = \tan A$

(2) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \cos 2A - \sin 2A \tan 3A$

6. (1) $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$ (2) $\sqrt{2}\sin 10^\circ + \sqrt{3}\cos 35^\circ = \sin 55^\circ + 2\cos 65^\circ$

7. (1) $\sin\theta = n\sin(\theta + 2\alpha) \Leftrightarrow \tan(\theta + \alpha) = \frac{1+n}{1-n}\tan\alpha$

(2) $\sin(2A + 3B) = 5\sin B \Rightarrow 2\tan(A + 2B) = 3\tan(A + B)$

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Miscellaneous Problems :

Example 16 : Prove that $0 < \alpha, \beta < \frac{\pi}{2} \Rightarrow \sin(\alpha + \beta) < \sin\alpha + \sin\beta$ and deduce from this that $\sin 49^\circ + \sin 41^\circ > 1$.

Solution : $sin(\alpha + \beta) - sin\alpha - sin\beta$

$= sin\alpha\ cos\beta + cos\alpha\ sin\beta - sin\alpha - sin\beta$

$= sin\alpha\ (cos\beta - 1) + sin\beta\ (cos\alpha - 1)$ **(i)**

Now, as $0 < \alpha, \beta < \frac{\pi}{2}$, so $0 < sin\alpha < 1$, $0 < sin\beta < 1$ and

$0 < cos\alpha < 1$, $0 < cos\beta < 1$

$\therefore$ $cos\alpha - 1 < 0$, $cos\beta - 1 < 0$

$\therefore$ $sin\alpha(cos\beta - 1) < 0$ and $sin\beta(cos\alpha - 1) < 0$

$\therefore$ $sin\alpha(cos\beta - 1) + sin\beta(cos\alpha - 1) < 0$

$\therefore$ $sin(\alpha + \beta) - sin\alpha - sin\beta < 0$

$\therefore$ $sin(\alpha + \beta) < sin\alpha + sin\beta$

Now, taking $\alpha = 49^o$, $\beta = 41^o$

As $0 < 49 < 90$ and $0 < 41 < 90$

$sin(49^o + 41^o) < sin49^o + sin41^o$

$\therefore$ $sin90^o < sin49^o + sin41^o$

$\therefore$ $sin49^o + sin41^o > 1$

Example 17 : If $cos(\alpha + \beta) = \frac{4}{5}$, $sin(\alpha - \beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then prove that $tan2\alpha = \frac{56}{33}$.

Solution : We have, $0 < \alpha < \frac{\pi}{4}$, $0 < \beta < \frac{\pi}{4}$

$\therefore$ $0 < \alpha + \beta < \frac{\pi}{2}$ and $\frac{-\pi}{4} < \alpha - \beta < \frac{\pi}{4}$.

$\therefore$ $cos(\alpha - \beta)$ and $sin(\alpha + \beta)$ are positive.

Now, $sin(\alpha + \beta) = \sqrt{1 - cos^2(\alpha + \beta)} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$ $\left(0 < \alpha + \beta < \frac{\pi}{2}\right)$

$cos(\alpha - \beta) = \sqrt{1 - sin^2(\alpha - \beta)} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$ $\left(-\frac{\pi}{2} < -\frac{\pi}{4} < \alpha - \beta < \frac{\pi}{4} < \frac{\pi}{2}\right)$

$\therefore$ $sin(\alpha + \beta) = \frac{3}{5}$ and $cos(\alpha - \beta) = \frac{12}{13}$

Now, $tan(\alpha + \beta) = \frac{sin(\alpha + \beta)}{cos(\alpha + \beta)} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

and $tan(\alpha - \beta) = \frac{sin(\alpha - \beta)}{cos(\alpha - \beta)} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$

$tan2\alpha = tan[(\alpha + \beta) + (\alpha - \beta)]$

$= \frac{tan(\alpha + \beta) + tan(\alpha - \beta)}{1 - tan(\alpha + \beta)\ tan(\alpha - \beta)}$

$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{56}{33}$

$\therefore$ $tan2\alpha = \frac{56}{33}$

Example 18 : If α and β are roots of $acos\theta + bsin\theta = c$, then show that,

(1) $cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$ (2) $cos(\alpha - \beta) = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$

Solution : We have, $acos\theta + bsin\theta = c$ **(i)**

$\therefore \quad acos\theta = c - bsin\theta$

$\therefore \quad a^2cos^2\theta = (c - bsin\theta)^2$

$\therefore \quad a^2(1 - sin^2\theta) = c^2 - 2bcsin\theta + b^2 sin^2\theta$

$\therefore \quad (a^2 + b^2) sin^2\theta - 2bcsin\theta + (c^2 - a^2) = 0$ **(ii)**

Since α and β are roots of equation (i), $sin\alpha$ and $sin\beta$ are the roots of the equation (ii).

$\therefore \quad sin\alpha \; sin\beta = \frac{c^2 - a^2}{a^2 + b^2}$ **(iii)**

Again, $acos\theta + bsin\theta = c$

$\therefore \quad bsin\theta = c - acos\theta$

$\therefore \quad b^2(1 - cos^2\theta) = c^2 - 2accos\theta + a^2 cos^2\theta$

$\therefore \quad b^2 - b^2cos^2\theta = a^2 cos^2\theta - 2accos\theta + c^2$

$\therefore \quad (a^2 + b^2) cos^2\theta - 2ac\, cos\theta + (c^2 - b^2) = 0$ **(iv)**

Since α and β are roots of equation (i), $cos\alpha$, $cos\beta$ are the roots the equation (iv).

$\therefore \quad cos\alpha \; cos\beta = \frac{c^2 - b^2}{a^2 + b^2}$ **(v)**

Now, $cos(\alpha + \beta) = cos\alpha \; cos\beta - sin\alpha \; sin\beta$

$$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2}$$

$\therefore \quad cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$

and $cos(\alpha - \beta) = cos\alpha \; cos\beta + sin\alpha \; sin\beta$

$$= \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2} = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$$ **(from (iii) and (v))**

$\therefore \quad cos(\alpha - \beta) = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$

Example 19 : If $asin\theta = bsin\left(\theta + \frac{2\pi}{3}\right) = csin\left(\theta + \frac{4\pi}{3}\right)$, then prove that $ab + bc + ca = 0$. $(abc \neq 0)$

Solution : Let $asin\theta = bsin\left(\theta + \frac{2\pi}{3}\right) = csin\left(\theta + \frac{4\pi}{3}\right) = k$

It is clear that $k \neq 0$ (why ?)

$$\therefore \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = sin\theta + sin\left(\theta + \frac{2\pi}{3}\right) + sin\left(\theta + \frac{4\pi}{3}\right)$$

$$= sin\theta + sin\left(\theta + \frac{4\pi}{3}\right) + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= 2sin\left(\theta + \frac{2\pi}{3}\right) cos\frac{2\pi}{3} + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= 2sin\left(\theta + \frac{2\pi}{3}\right) \times \left(-\frac{1}{2}\right) + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= -sin\left(\theta + \frac{2\pi}{3}\right) + sin\left(\theta + \frac{2\pi}{3}\right) = 0$$

$$\therefore \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = 0$$

$$\therefore \quad k\left(\frac{bc + ca + ab}{abc}\right) = 0$$

$\therefore \quad ab + bc + ca = 0$ **($k \neq 0$)**

Exercise 4

1. Prove that :

(1) $\dfrac{cos^2 33^\circ - cos^2 57^\circ}{sin^2 \frac{21^\circ}{2} - sin^2 \frac{69^\circ}{2}} = -\sqrt{2}$ (2) $\dfrac{\sqrt{3}}{sin\,20^\circ} - \dfrac{1}{cos\,20^\circ} = 4$

2. Prove that $0 < \alpha, \beta < \frac{\pi}{4} \Rightarrow tan(\alpha + \beta) > tan\alpha + tan\beta$ and deduce that $tan35^\circ + tan25^\circ < \sqrt{3}$.

3. Prove that $2tan\beta + cot\beta = tan\alpha \Rightarrow 2tan(\alpha - \beta) = cot\beta$.

4. If $\theta + \beta = \alpha$ and $sin\theta = ksin\beta$, prove that $tan\theta = \dfrac{k\,sin\alpha}{1 + k\,cos\alpha}$ and $tan\beta = \dfrac{sin\,\alpha}{k + cos\alpha}$.

5. If $sinA + cosB = 0$ in ΔABC, prove that ΔABC is an obtuse angled triangle and that $0 < sinA < \frac{1}{\sqrt{2}}$.

6. If $cos(\beta - \gamma) + cos(\gamma - \alpha) + cos(\alpha - \beta) = \frac{-3}{2}$, prove that $sin\alpha + sin\beta + sin\gamma = 0$ and $cos\alpha + cos\beta + cos\gamma = 0$.

7. If $tan(\alpha + \theta) = ntan(\alpha - \theta)$, then prove that $(n + 1)\, sin2\theta = (n - 1)\, sin2\alpha$.

8. If α and β are the roots of the equation $atan\theta + bsec\theta = c$, then prove that $tan(\alpha + \beta) = \dfrac{2ac}{a^2 - c^2}$.

9. Find the maximum and minimum values of $3cos\theta + 5sin\left(\theta - \frac{\pi}{6}\right)$.

10. Prove that $sin10^\circ \cdot sin30^\circ \cdot sin50^\circ \cdot sin70^\circ = \frac{1}{16}$.

11. Prove : $cos\frac{\pi}{11} + cos\frac{3\pi}{11} + cos\frac{5\pi}{11} + cos\frac{7\pi}{11} + cos\frac{9\pi}{11} = \frac{1}{2}$

12. Prove that $\dfrac{\cos 8\theta \cos 5\theta - \cos 12\theta \cos 9\theta}{\sin 8\theta \cos 5\theta + \cos 12\theta \sin 9\theta} = \tan 4\theta$.

13. Prove that $m\tan\left(\theta - \frac{\pi}{6}\right) = n\tan\left(\theta + \frac{2\pi}{3}\right) \Rightarrow \cos 2\theta = \dfrac{m+n}{2(m-n)}$.

14. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The value of $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$ is ... ☐

(a) $\tan 25^\circ$ (b) $\tan 35^\circ$ (c) $\tan 55^\circ$ (d) $\tan 80^\circ$

(2) The value of $\cos 245^\circ + \sin 155^\circ$ is ... ☐

(a) 0 (b) $\dfrac{\sqrt{2}+1}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(3) The value of $\cos(270^\circ + \alpha)\cos(90^\circ - \alpha) - \sin(270^\circ - \alpha)\cos\alpha$ is... ☐

(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

(4) The value of $2\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right)$ is ... ☐

(a) $\frac{-1}{4}$ (b) 1 (c) $\frac{-\sqrt{3}}{2}$ (d) $\frac{1}{2}$

(5) If A = 125 and $x = \sin A^\circ + \cos A^\circ$, then ☐

(a) $x < 0$ (b) $x = 0$ (c) $x > 0$ (d) $x \geq 0$

(6) If $\tan\alpha = \dfrac{n}{n+1}$ and $\tan\beta = \dfrac{1}{2n+1}$, $(0 < \alpha, \beta < \frac{\pi}{4})$, then $\alpha + \beta$ is ... ☐

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(7) The value of $\dfrac{\tan 50^\circ - \tan 40^\circ}{\tan 10^\circ}$ is ... ☐

(a) 0 (b) 1 (c) 2 (d) 3

(8) $\sin 190^\circ + \cos 190^\circ$ ... ☐

(a) is negative (b) is zero

(c) is positive (d) is not defined.

(9) If $\cot 15^\circ = m$, then $\dfrac{\tan 225^\circ + \tan 345^\circ}{\tan 195^\circ - \tan 105^\circ}$ is ... ☐

(a) $\dfrac{m-1}{m^2+1}$ (b) $\dfrac{2m}{m^2+1}$ (c) $\dfrac{m^2-1}{m^2+1}$ (d) $\dfrac{m+1}{m^2+1}$

(10) The value of $\log \tan 1^\circ + \log \tan 2^\circ + ... + \log \tan 89^\circ$ is ... ☐

(a) 0 (b) 1 (c) 2 (d) 3

(11) The value of $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$ is ... ☐

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) 2 (d) $\sqrt{3}$

(12) The value of $cos480^o \ sin150^o + sin600^o \ cos390^o$ is ...

(a) $\frac{-1}{2}$ (b) 0 (c) -1 (d) $\frac{1}{2}$

(13) $tan25^o + tan20^o + tan25^o \ tan20^o$ is equal to ...

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

(14) In ΔABC, if $tanA = \frac{1}{2}$, $tanB = \frac{1}{3}$, then the measure of angle C is ...

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

(15) The value of $\sqrt{3}cosec20^o - sec20^o$ is ...

(a) -4 (b) 1 (c) 2 (d) 4

(16) The value of $\sqrt{3}sin75^o - cos75^o$ is ...

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

(17) $cos^2\frac{\pi}{12} + cos^2\frac{3\pi}{12} + cos^2\frac{5\pi}{12} = ...$

(a) $\frac{-1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

(18) The value of $cos15^o - sin15^o$ is ...

(a) $\frac{-1}{\sqrt{2}}$ (b) 0 (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

(19) $cos^2 7\frac{1}{2}^o - cos^2 37\frac{1}{2}^o$ is equal to ...

(a) $\frac{3}{4}$ (b) $\frac{2}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2\sqrt{2}}$

*

Summary

We studied following points in this chapter :

1. $cos(\alpha - \beta) = cos\alpha \ cos\beta + sin\alpha \ sin\beta$
2. $cos(\alpha + \beta) = cos\alpha \ cos\beta - sin\alpha \ sin\beta$
3. $cos\left(\frac{\pi}{2} - \theta\right) = sin\theta$, $sin\left(\frac{\pi}{2} - \theta\right) = cos\theta$
4. $sin(\alpha + \beta) = sin\alpha \ cos\beta + cos\alpha \ sin\beta$
5. $sin(\alpha - \beta) = sin\alpha \ cos\beta - cos\alpha \ sin\beta$
6. $sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$, $cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$
7. $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

   $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = cos^2\beta - cos^2\alpha$

8. $cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\alpha - sin^2\beta$

$cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\beta - sin^2\alpha$

9. The range of $f(\alpha) = acos\alpha + bsin\beta$, $\alpha, \beta \in$ R, $a, b, \in$ R, $a^2 + b^2 \neq 0$

is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.

In proper domain,

10. $tan(\alpha + \beta) = \dfrac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta}$

11. $tan(\alpha - \beta) = \dfrac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$

12. $cot(\alpha + \beta) = \dfrac{cot\alpha \cdot cot\beta - 1}{cot\beta + cot\alpha}$

13. $cot(\alpha - \beta) = \dfrac{cot\alpha \cdot cot\beta + 1}{cot\beta - cot\alpha}$

14. $tan\frac{\pi}{12} = 2 - \sqrt{3}$, $cot\frac{\pi}{12} = 2 + \sqrt{3}$

15. $2sin\alpha\ cos\beta = sin(\alpha + \beta) + sin(\alpha - \beta)$

16. $2cos\alpha\ sin\beta = sin(\alpha + \beta) - sin(\alpha - \beta)$

17. $2cos\alpha\ cos\beta = cos(\alpha + \beta) + cos(\alpha - \beta)$

18. $2sin\alpha\ sin\beta = cos(\alpha - \beta) - cos(\alpha + \beta)$

19. $sinC + sinD = 2sin\left(\dfrac{C+D}{2}\right) cos\left(\dfrac{C-D}{2}\right)$

20. $sinC - sinD = 2cos\left(\dfrac{C+D}{2}\right) sin\left(\dfrac{C-D}{2}\right)$

21. $cosC + cosD = 2cos\left(\dfrac{C+D}{2}\right) cos\left(\dfrac{C-D}{2}\right)$

22. $cosC - cosD = -2sin\left(\dfrac{C+D}{2}\right) sin\left(\dfrac{C-D}{2}\right)$.

Aryabhata is also known as **Aryabhata I** to distinguish him from the later mathematician of the same name who lived about 400 years later.

The surviving text is Aryabhata's masterpiece the *Aryabhatiya* which is a small astronomical treatise written in 118 verses giving a summary of Hindu mathematics up to that time. Its mathematical section contains 33 verses giving 66 mathematical rules without proof.

The mathematical part of the *Aryabhatiya* covers arithmetic, algebra, plane trigonometry and spherical trigonometry. It also contains continued fractions, quadratic equations, sums of power series and a table of *sines*.

Chapter 5

VALUES OF TRIGONOMETRIC FUNCTIONS FOR MULTIPLES AND SUBMULTIPLES

Geometry is not true, it is advantageous.

– Henri Poincare

Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore.

– Albert Einstein

5.1 Introduction

In this chapter we shall use addition formulae to obtain values of trigonometric functions for multiples like 2α, 3α etc. of α and for sub-multiples like $\frac{\alpha}{2}$ of α. Then we will obtain the values of trigonometric functions for some standard particular numbers and finally, we will use them for proving some conditional identities.

5.2 Trigonometric Functions of 2α

(1) Formula for $\sin 2\alpha$: For $\alpha, \beta \in R$,

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Substituting $\beta = \alpha$ in this formula,

$\sin(\alpha + \alpha) = \sin\alpha \cos\alpha + \cos\alpha \sin\alpha$

$\therefore \quad \sin 2\alpha = 2\sin\alpha \cos\alpha$ **(i)**

(2) Formulae for $\cos 2\alpha$: For $\alpha, \beta \in R$,

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Putting $\beta = \alpha$ in this we see that,

$\cos(\alpha + \alpha) = \cos\alpha \cos\alpha - \sin\alpha \sin\alpha$

$\therefore \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ **(ii)**

$\therefore \quad \cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$

$\therefore \quad \cos 2\alpha = 2\cos^2\alpha - 1$ **(iii)**

Again, $cos2\alpha = cos^2\alpha - sin^2\alpha$

$= 1 - sin^2\alpha - sin^2\alpha$

$\therefore \quad cos2\alpha = 1 - 2sin^2\alpha$ **(iv)**

So we have, $cos2\alpha = cos^2\alpha - sin^2\alpha = 1 - 2sin^2\alpha = 2cos^2\alpha - 1$

Thus, once $sin\alpha$ and $cos\alpha$ for $\alpha \in R$ are known, we can obtain the values of $sin2\alpha$ and $cos2\alpha$ using above formulae. Also values of *sine* and *cosine* functions for numbers that are twice of the given numbers can be obtained.

From (iii) and (iv) we have,

$1 + cos2\alpha = 2cos^2\alpha, \ 1 - cos2\alpha = 2sin^2\alpha$

These are quite useful forms.

If we replace 2α by α (and so α by $\frac{\alpha}{2}$), we get

$$sin\alpha = 2sin\frac{\alpha}{2} \cdot cos\frac{\alpha}{2}$$

$$cos\alpha = cos^2\frac{\alpha}{2} - sin^2\frac{\alpha}{2}$$

Also we have $1 + cos\alpha = 2cos^2\frac{\alpha}{2}$ and $1 - cos\alpha = 2sin^2\frac{\alpha}{2}$

(3) $sin2\alpha$, $cos2\alpha$ and $tan2\alpha$ in terms of $tan\alpha$.

$$sin2\alpha = 2sin\alpha \cdot cos\alpha$$

$$= \frac{2sin\alpha \cdot cos\alpha}{cos^2\alpha + sin^2\alpha} \qquad (cos^2\alpha + sin^2\alpha = 1)$$

If $\alpha \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, then $cos\alpha \neq 0$. So let us divide both numerator and denominator by $cos^2\alpha$. Then,

$$sin2\alpha = \frac{2tan\alpha}{1 + tan^2\alpha} \qquad \textbf{(v)}$$

$$cos2\alpha = cos^2\alpha - sin^2\alpha$$

$$= \frac{cos^2\alpha - sin^2\alpha}{cos^2\alpha + sin^2\alpha}$$

Again, taking $\alpha \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, $cos\alpha \neq 0$, we divide both numerator and denominator by $cos^2\alpha$, to get

$$cos2\alpha = \frac{1 - tan^2\alpha}{1 + tan^2\alpha} \qquad \textbf{(vi)}$$

Now suppose α and 2α both are in the domain of *tan*. Then

$$tan2\alpha = tan(\alpha + \alpha)$$

$$= \frac{tan\alpha + tan\alpha}{1 - tan\alpha\, tan\alpha} \quad \left(\alpha \in R - \left[\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\} \cup \left\{(2k-1)\frac{\pi}{4} \mid k \in Z\right\}\right]\right)$$

That is $tan2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha}$ **(vii)**

Finally, assuming that α and 2α are in the domain of *cot*, we can similarly prove that

$$cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha} \left(\alpha \in R - \left\{\frac{k\pi}{2} \mid k \in Z\right\}\right) \qquad \textbf{(viii)}$$

Note that if $\alpha \neq \frac{k\pi}{2}$ for all $k \in Z$, then certainly $\alpha \neq k\pi$ for all $k \in Z$, because $k\pi = \frac{2k\pi}{2}$, $2k \in Z$.

Thus, if $\alpha \neq \frac{k\pi}{2}$ for all $k \in Z$, then $cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha}$

In the results (v), (vi) and (vii), if we replace 2α by α (and so replace α by $\frac{\alpha}{2}$), we get

$$sin\alpha = \frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}}, \quad cos\alpha = \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \text{ and } tan\alpha = \frac{2tan\frac{\alpha}{2}}{1-tan^2\frac{\alpha}{2}}.$$

If we put $tan\frac{\alpha}{2} = t$, then above formulae become

$$sin\alpha = \frac{2t}{1+t^2}, \ cos\alpha = \frac{1-t^2}{1+t^2} \text{ and } tan\alpha = \frac{2t}{1-t^2}.$$

5.3 Trigonometric Functions of 3α

(1) $sin3\alpha = sin(2\alpha + \alpha)$

$= sin2\alpha \cdot cos\alpha + cos2\alpha \cdot sin\alpha$

$= (2sin\alpha \cdot cos\alpha) \cdot cos\alpha + (1 - 2sin^2\alpha) \cdot sin\alpha$

$= 2sin\alpha \cdot cos^2\alpha + sin\alpha - 2sin^3\alpha$

$= 2sin\alpha\,(1 - sin^2\alpha) + sin\alpha - 2sin^3\alpha$

$= 2sin\alpha - 2sin^3\alpha + sin\alpha - 2sin^3\alpha$

$= 3sin\alpha - 4sin^3\alpha$

$\therefore \ sin3\alpha = 3sin\alpha - 4sin^3\alpha$ **(ix)**

(2) $cos3\alpha = cos(\alpha + 2\alpha)$

$= cos\alpha \cdot cos2\alpha - sin\alpha \cdot sin2\alpha$

$= cos\alpha \cdot (2cos^2\alpha - 1) - sin\alpha\,(2sin\alpha\, cos\alpha)$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha \cdot sin^2\alpha$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha\,(1 - cos^2\alpha)$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha + 2cos^3\alpha$

$= 4cos^3\alpha - 3cos\alpha$

$\therefore \ cos3\alpha = 4cos^3\alpha - 3cos\alpha$ **(x)**

(3) Taking α, 2α, 3α in the domain of *tan*,

that is $\alpha \neq (2k-1)\frac{\pi}{2}$, $\alpha \neq (2k-1)\frac{\pi}{4}$ and $\alpha \neq (2k-1)\frac{\pi}{6}$, $k \in Z$

(Remember that every odd multiple of $\frac{\pi}{2}$ is an odd multiple of $\frac{\pi}{6}$, for example $\frac{3\pi}{2} = \frac{9\pi}{6}$.

i.e. $\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\} \subset \left\{(2k-1)\frac{\pi}{6} \mid k \in Z\right\}$)

$tan3\alpha = tan(2\alpha + \alpha)$

$$= \frac{tan2\alpha + tan\alpha}{1 - tan2\alpha\, tan\alpha}$$

$$= \frac{\frac{2tan\alpha}{1-tan^2\alpha} + tan\alpha}{1 - \frac{2tan\alpha}{1-tan^2\alpha} \cdot tan\alpha}$$

$$= \frac{2tan\alpha + tan\alpha - tan^3\alpha}{1 - tan^2\alpha - 2tan^2\alpha}$$

$$= \frac{3tan\,\alpha - tan^3\alpha}{1 - 3tan^2\alpha},\ \alpha \in R - \left\{(2k-1)\frac{\pi}{6},\ k \in Z\right\}$$

$$\therefore\quad tan3\alpha = \frac{3tan\,\alpha - tan^3\alpha}{1 - 3tan^2\alpha},\ \alpha \in R - \left\{(2k-1)\frac{\pi}{6},\ k \in Z\right\} \qquad \textbf{(xi)}$$

This formula remains true even if 2α is in the domain of *tan*. ($2\alpha \in D_{tan}$)

Similarly, we can prove that if α, 2α and $3\alpha \in D_{cot}$, then

$$\alpha \neq k\pi,\ \alpha \neq \frac{k\pi}{2},\ \alpha \neq \frac{k\pi}{3},\ k \in Z$$

$$\{k\pi \mid k \in Z\} \subset \left\{\frac{k\pi}{3} \mid k \in Z\right\}$$

$$cot3\alpha = \frac{cot^3\alpha - 3cot\,\alpha}{3cot^2\alpha - 1},\ \alpha \in R - \left\{\frac{k\pi}{3} \mid k \in Z\right\} \qquad \textbf{(xii)}$$

Indeed, this is true for all $\alpha = \frac{k\pi}{2}$, $k \in Z$.

Thus, for any $\alpha \in R$, we can calculate $sin3\alpha$, $cos3\alpha$ and $tan3\alpha$, if $sin\alpha$, $cos\alpha$ and $tan\alpha$ are given. Also values of trigonometric functions of 4α, 5α, ... etc. can be expressed in terms of trigonometric functions of α.

Example 1 : Prove : (1) $\frac{sin2\theta}{1+cos2\theta} = tan\theta$ (2) $\frac{sin\,\theta + cos\,\frac{\theta}{2}}{1 + sin\,\frac{\theta}{2} - cos\,\theta} = cot\frac{\theta}{2}$

(3) $\frac{cos2\theta}{1+sin2\theta} = tan\left(\frac{\pi}{4} - \theta\right)$ (4) $sec\theta + tan\theta = tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Solution : (1) L.H.S. $= \frac{sin2\theta}{1+cos2\theta} = \frac{2sin\theta\,cos\theta}{2cos^2\theta} = tan\theta$ = R.H.S.

(2) L.H.S. $= \frac{sin\,\theta + cos\,\frac{\theta}{2}}{1 + sin\,\frac{\theta}{2} - cos\,\theta}$

$$= \frac{2sin\,\frac{\theta}{2}cos\,\frac{\theta}{2} + cos\,\frac{\theta}{2}}{sin\,\frac{\theta}{2} + (1 - cos\,\theta)}$$

$$= \frac{cos\frac{\theta}{2}\left(2\,sin\,\frac{\theta}{2} + 1\right)}{sin\frac{\theta}{2} + 2sin^2\frac{\theta}{2}}$$

$$= \frac{cos\frac{\theta}{2}\left(2\,sin\,\frac{\theta}{2} + 1\right)}{sin\frac{\theta}{2}\left(1 + 2sin\frac{\theta}{2}\right)}$$

$$= \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}} = cot\frac{\theta}{2} = \text{R.H.S.}$$

(3) L.H.S. $= \frac{cos2\theta}{1+sin2\theta} = \frac{sin\left(\frac{\pi}{2} - 2\theta\right)}{1 + cos\left(\frac{\pi}{2} - 2\theta\right)}$ $\left(cosA = sin\left(\frac{\pi}{2} - A\right),\ sinA = cos\left(\frac{\pi}{2} - A\right)\right)$

$$= \frac{2sin\left(\frac{\pi}{4} - \theta\right)cos\left(\frac{\pi}{4} - \theta\right)}{2cos^2\left(\frac{\pi}{4} - \theta\right)} = tan\left(\frac{\pi}{4} - \theta\right) = \text{R.H.S.}$$

(4) L.H.S. $= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$

$= \frac{1+\sin\theta}{\cos\theta}$

$= \frac{1-\cos\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)}$

$= \frac{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$

$= \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ = R.H.S.

Example 2 : Express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin 5\theta$ in terms of $\sin\theta$.

Solution : $\cos 4\theta = \cos 2(2\theta)$

$= 2\cos^2 2\theta - 1$

$= 2(2\cos^2\theta - 1)^2 - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

$\sin 5\theta = (\sin 5\theta + \sin\theta) - \sin\theta$

$= 2\sin 3\theta \cos 2\theta - \sin\theta$

$= 2(3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta) - \sin\theta$

$= 6\sin\theta - 12\sin^3\theta - 8\sin^3\theta + 16\sin^5\theta - \sin\theta$

$\therefore \quad \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Example 3 : Prove that $\cos A \cdot \cos(60^o - A)\cos(60^o + A) = \frac{1}{4}\cos 3A$ and use it to find the value of $\cos 20^o \cdot \cos 40^o \cdot \cos 60^o \cdot \cos 80^o$.

Solution : L.H.S. $= \cos A \cdot \cos(60^o - A)\cos(60^o + A)$

$= \cos A(\cos^2 60^o - \sin^2 A)$

$= \cos A\left(\frac{1}{4} - \sin^2 A\right)$

$= \cos A\left(\frac{1}{4} - (1 - \cos^2 A)\right)$

$= \cos A\left(-\frac{3}{4} + \cos^2 A\right)$

$= \frac{1}{4}\left(4\cos^3 A - 3\cos A\right)$

$= \frac{1}{4}\cos 3A$ = R.H.S.

Now, $\cos 20^o \cdot \cos 40^o \cdot \cos 60^o \cdot \cos 80^o = \frac{1}{2}(\cos 20^o \cdot \cos(60^o + 20^o)\cos(60^o - 20^o))$

$= \frac{1}{2}\left[\frac{1}{4}\cos 3(20^o)\right]$ **(A = 20°)**

$= \frac{1}{8}\cos 60^o = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

Example 4 : Prove that $\cos^3\theta + \cos^3\left(\frac{2\pi}{3}+\theta\right) + \cos^3\left(\frac{4\pi}{3}+\theta\right) = \frac{3}{4}\cos 3\theta$.

Solution : We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. So, $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$$\text{L.H.S.} = \cos^3\theta + \cos^3\left(\frac{2\pi}{3}+\theta\right) + \cos^3\left(\frac{4\pi}{3}+\theta\right)$$

$$= \frac{1}{4}[\cos 3\theta + 3\cos\theta] + \frac{1}{4}\left[\cos(2\pi + 3\theta) + 3\cos\left(\frac{2\pi}{3}+\theta\right)\right]$$

$$+ \frac{1}{4}\left[\cos(4\pi + 3\theta) + 3\cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{1}{4}[\cos 3\theta + 3\cos\theta] + \frac{1}{4}\left[\cos 3\theta + 3\cos\left(\frac{2\pi}{3}+\theta\right)\right]$$

$$+ \frac{1}{4}\left[\cos 3\theta + 3\cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\left[\cos\theta + \cos\left(\frac{2\pi}{3}+\theta\right) + \cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\left[\cos\theta + 2\cos(\pi + \theta)\cos\frac{\pi}{3}\right]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\left[\cos\theta - 2\cos\theta \times \frac{1}{2}\right]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}(\cos\theta - \cos\theta) = \frac{3}{4}\cos 3\theta = \text{R.H.S.}$$

Example 5 : Prove that $\cos A \cdot \cos 2A \cdot \cos 2^2A \cdot \cos 2^3A \cdot \ldots \cdot \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \cdot \sin A}$ and use it to find the value of $\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15}$.

Solution : $\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore \quad \cos\theta = \frac{\sin 2\theta}{2\sin\theta}$$

$$\text{L.H.S.} = \cos A \cdot \cos 2A \cdot \cos 2^2A \cdot \cos 2^3A \cdot \ldots \cdot \cos 2^{n-1}A$$

$$= \frac{\sin 2A}{2\sin A} \cdot \frac{\sin 2(2A)}{2\sin 2A} \cdot \frac{\sin 2(2^2A)}{2\sin 2^2A} \cdot \frac{\sin 2(2^3A)}{2\sin 2^3A} \cdot \ldots \cdot \frac{\sin 2(2^{n-1}A)}{2\sin 2^{n-1}A}$$

$$= \frac{\sin 2(2^{n-1}A)}{2^n \cdot \sin A} = \frac{\sin 2^n A}{2^n \cdot \sin A} = \text{R.H.S.}$$

$$\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15} = -\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{\pi}{15}$$

$$\left(\cos\frac{14\pi}{15} = \cos\left(\pi - \frac{\pi}{15}\right) = -\cos\frac{\pi}{15}\right)$$

$$= -\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}$$

$$= -\frac{\sin\left(\pi + \frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}$$

$$= \frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}} = \frac{1}{16}$$

Exercise 5.1

Prove **(1 to 19) :**

1. $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

2. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \cot\left(\frac{\pi}{4}+\theta\right)$

3. $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = 2\operatorname{cosec}\theta$

4. $\dfrac{\cos\theta}{1+\sin\theta} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

5. $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta \quad \cos 2\theta} = \cot\theta$

6. $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = 2\sec\theta$

7. $\dfrac{\cot\theta - \tan\theta}{1-2\sin^2\theta} = \sec\theta \cdot \operatorname{cosec}\theta = 2\operatorname{cosec}2\theta$

8. $\sec 2\theta - \tan 2\theta = \tan\left(\frac{\pi}{4}-\theta\right)$

9. $\dfrac{\sin 5\theta - 2\sin 3\theta + \sin\theta}{\cos 5\theta - \cos\theta} = \tan\theta$

10. $\dfrac{\sin\theta - \sin 3\theta}{\sin^2\theta - \cos^2\theta} = 2\sin\theta$

11. $\sqrt{3}\operatorname{cosec}20^\circ - \sec 20^\circ = 4$

12. $2(\cos^8\theta - \sin^8\theta) = \cos 2\theta + \cos^3 2\theta$

13. If $\tan\alpha = \frac{1}{3}$ and $\tan\frac{\beta}{2} = \frac{1}{2}$, then $\tan(\alpha + \beta) = 3$.

14. If $\cos\theta = \frac{1}{2}\left(x+\frac{1}{x}\right)$, then $\cos 2\theta = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$ and $\cos 3\theta = \frac{1}{2}\left(x^3+\frac{1}{x^3}\right)$.

15. $\dfrac{\sin^2 A - \sin^2 B}{\sin 2A - \sin 2B} = \frac{1}{2}\tan(A + B)$

16. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

17. $\dfrac{\cos 3\theta}{\cos\theta} + \dfrac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$

18. $\cos^3\theta \sin 3\theta + \sin^3\theta \cos 3\theta = \frac{3}{4}\sin 4\theta$

19. $\cos^3\theta \cos 3\theta + \sin^3\theta \sin 3\theta = \cos^3 2\theta$

20. If $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ then find the value of $\sin 2A$, $\cos 2A$, $\tan 2A$ and $\sin 4A$.

21. If $15\theta = \pi$, then prove that $\cos\theta \cdot \cos 2\theta \cdot \cos 3\theta \cdot \cos 4\theta \cdot \cos 5\theta \cdot \cos 6\theta \cdot \cos 7\theta = \frac{1}{128}$.

22. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$, where $0 < \theta < \frac{\pi}{8}$.

23. Prove that $\tan\theta + \tan\left(\frac{\pi}{3}+\theta\right) + \tan\left(\frac{2\pi}{3}+\theta\right) = 3\tan 3\theta$ and deduce that

$\tan 20^\circ + \tan 80^\circ + \tan 140^\circ = 3\sqrt{3}$.

24. Prove that $\tan\theta \cdot \tan\left(\frac{\pi}{3}+\theta\right) \cdot \tan\left(\frac{\pi}{3}-\theta\right) = \tan 3\theta$ and deduce that

$\tan 6^\circ \cdot \tan 42^\circ \cdot \tan 66^\circ \cdot \tan 78^\circ = 1$.

25. Prove : $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

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5.4 Trigonometric Functions of $\frac{\alpha}{2}$ in Terms of $\cos\alpha$

(1) We know that $\cos 2\alpha = 1 - 2\sin^2\alpha$. If we put α in place of 2α (and $\frac{\alpha}{2}$ in place of α), we get

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$.

$\therefore \quad 2sin^2\frac{\alpha}{2} = 1 - cos\alpha$

$\therefore \quad sin^2\frac{\alpha}{2} = \frac{1-cos\alpha}{2}$

(2) Similarly, substituting α in place of 2α (and $\frac{\alpha}{2}$ in place of α) in $cos2\alpha = 2cos^2\alpha - 1$

$\therefore \quad 2cos^2\frac{\alpha}{2} = 1 + cos\alpha$

$\therefore \quad cos^2\frac{\alpha}{2} = \frac{1+cos\alpha}{2}$

(3) $tan^2\frac{\alpha}{2} = \frac{sin^2\frac{\alpha}{2}}{cos^2\frac{\alpha}{2}} = \frac{\frac{1-cos\alpha}{2}}{\frac{1+cos\alpha}{2}}$ **($\alpha \neq (2k-1)\pi;\ k \in Z$)**

$\therefore \quad tan^2\frac{\alpha}{2} = \frac{1-cos\alpha}{1+cos\alpha}$

5.5 Values of Trigonometric Functions for Some Special Numbers

(1) $sin18^\circ$:

Suppose $\theta = 18^\circ$

$\therefore \quad 5\theta = 90^\circ$

$\therefore \quad 3\theta + 2\theta = 90^\circ$

$\therefore \quad 2\theta = 90^\circ - 3\theta$

$\therefore \quad sin2\theta = sin(90^\circ - 3\theta)$

$\therefore \quad sin2\theta = cos3\theta$

$\therefore \quad 2sin\theta\, cos\theta = 4cos^3\theta - 3cos\theta$

$\therefore \quad 2sin\theta = 4cos^2\theta - 3$ **($cos18^\circ \neq 0$)**

$\therefore \quad 2sin\theta = 4(1 - sin^2\theta) - 3$

$\therefore \quad 2sin\theta = 4 - 4sin^2\theta - 3$

$\therefore \quad 4sin^2\theta + 2sin\theta - 1 = 0$

$\therefore \quad sin\theta = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$

$= \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

Here $\theta = 18^\circ$. Hence, $P(\theta)$ is in the first quadrant.

$\therefore \quad sin\theta > 0$

$\therefore \quad sin18^\circ = \frac{\sqrt{5}-1}{4}$

(2) $cos18^\circ$:

Substituting $\theta = 18^\circ$ in $cos^2\theta = 1 - sin^2\theta$, we get

$cos^2 18^\circ = 1 - sin^2 18^\circ$

$= 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{16 - 5 + 2\sqrt{5} - 1}{16}$

$\therefore \quad cos^2 18^o = \dfrac{10 + 2\sqrt{5}}{16}$

$\therefore \quad cos 18^o = \sqrt{\dfrac{10 + 2\sqrt{5}}{16}}$ **(0 < 18 < 90. So $cos 18^o > 0$)**

(3) $cos 36^o$:

Substituting $\theta = 18^o$ in $cos 2\theta = 1 - 2sin^2\theta$, we get

$$cos 36^o = 1 - 2sin^2 18^o$$

$$= 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$= 1 - 2\left(\frac{5 - 2\sqrt{5} + 1}{16}\right)$$

$$= \frac{8 - 5 + 2\sqrt{5} - 1}{8} = \frac{2 + 2\sqrt{5}}{8} = \frac{\sqrt{5} + 1}{4}$$

$\therefore \quad cos 36^o = \dfrac{\sqrt{5} + 1}{4}$

(4) $sin 36^o$:

Substituting $\theta = 36^o$ in $sin^2\theta = 1 - cos^2\theta$, we get

$$sin^2 36^o = 1 - cos^2 36^o$$

$$= 1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2$$

$$= 1 - \left(\frac{5 + 2\sqrt{5} + 1}{16}\right) = \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}$$

$\therefore \quad sin 36^o = \sqrt{\dfrac{10 - 2\sqrt{5}}{16}}$ **(0 < 36 < 90. So $sin 36^o > 0$)**

We can similarly get *sines* and *cosines* of multiples of 18 like 54, 72, 144 etc. In fact,

$$sin 72^o = sin(90^o - 18^o) = cos 18^o = \sqrt{\frac{10 + 2\sqrt{5}}{16}}$$

and $\quad sin 54^o = sin(90^o - 36^o) = cos 36^o = \dfrac{\sqrt{5} + 1}{4}$

Similarly, $cos 72^o = sin 18^o$ and $cos 54^o = sin 36^o$

(5) $sin 22\frac{1}{2}^o$ or $sin\frac{\pi}{8}$:

Putting $\theta = \dfrac{45}{2}^o$ in $sin^2\theta = \dfrac{1 - cos 2\theta}{2}$, we get

$$sin^2 \frac{45}{2}^o = \frac{1 - cos 45^o}{2}$$

$$= \frac{1 - \frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\sqrt{2} - 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{2} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2 - \sqrt{2}}{4}$$

$\therefore \quad sin \dfrac{45}{2}^o = \dfrac{\sqrt{2 - \sqrt{2}}}{2}$ **(0 < $22\frac{1}{2}$ < 90. So $sin 22\frac{1}{2}^o > 0$)**

(6) In the same way, we get $cos22\frac{1}{2}^{o} = \frac{\sqrt{2+\sqrt{2}}}{2}$

(7) ***$tan22\frac{1}{2}^{o}$:***

$$tan^2 22\frac{1}{2}^{o} = tan^2\frac{45}{2}^{o} = \frac{1-cos45}{1+cos45}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{(\sqrt{2}-1)^2}{2-1}$$

Now, $0 < 22\frac{1}{2} < 90$. Hence, $tan22\frac{1}{2}^{o} > 0$

$\therefore \quad tan22\frac{1}{2}^{o} = \sqrt{2} - 1$

Similarly, we can show that $cot22\frac{1}{2}^{o} = \sqrt{2} + 1$. We can also get value of *sines* and *cosines* of $67\frac{1}{2}^{o}$ etc.

$cos67\frac{1}{2}^{o} = sin22\frac{1}{2}^{o}$, $sin67\frac{1}{2}^{o} = cos22\frac{1}{2}^{o}$ and $tan67\frac{1}{2}^{o} = cot22\frac{1}{2}^{o}$

Example 6 : If $cot\theta = \frac{-5}{12}$, $\frac{\pi}{2} < \theta < \pi$, then find the value of $sin\frac{\theta}{2} + cos\frac{\theta}{2}$.

Solution : Since $cot\theta = \frac{-5}{12}$, $tan\theta = \frac{-12}{5}$

$\therefore \quad sec^2\theta = 1 + tan^2\theta = 1 + \frac{144}{25} = \frac{169}{25}$

$\therefore \quad sec\theta = \pm\frac{13}{5}$. Since $\frac{\pi}{2} < \theta < \pi$, $sec\theta < 0$

$\therefore \quad sec\theta = -\frac{13}{5}$. Hence, $cos\theta = \frac{-5}{13}$

Now, $sin^2\frac{\theta}{2} = \frac{1-cos\theta}{2} = \frac{1+\frac{5}{13}}{2} = \frac{18}{26} = \frac{9}{13}$

Since $\frac{\pi}{2} < \theta < \pi$, $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$. So $sin\frac{\theta}{2} > 0$

$\therefore \quad sin\frac{\theta}{2} = \frac{3}{\sqrt{13}}$

$cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2} = \frac{1-\frac{5}{13}}{2} = \frac{8}{26} = \frac{4}{13}$

$\therefore \quad cos\frac{\theta}{2} = \frac{2}{\sqrt{13}}$ $\quad \left(\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}\right)$

$\therefore \quad sin\frac{\theta}{2} + cos\frac{\theta}{2} = \frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}} = \frac{5}{\sqrt{13}}$

Example 7 : Prove that $sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\frac{5\pi}{8} + sin^4\frac{7\pi}{8} = \frac{3}{2}$.

Solution : L.H.S. $= sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\frac{5\pi}{8} + sin^4\frac{7\pi}{8}$

$$= sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\left(\pi - \frac{3\pi}{8}\right) + sin^4\left(\pi - \frac{\pi}{8}\right)$$

$$= 2\left(sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8}\right)$$

$$= 2\left[\left(sin^2\frac{\pi}{8}\right)^2 + \left(sin^2\frac{3\pi}{8}\right)^2\right]$$

$$= 2\left[\left(\frac{1-cos\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1-cos\frac{3\pi}{4}}{2}\right)^2\right] \quad \left(sin^2\theta = \frac{1-cos2\theta}{2}\right)$$

$$= \frac{2}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)^2 + \left(1 - \cos\frac{3\pi}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \sqrt{2} + 1 + \frac{1}{2} + \sqrt{2}\right]$$

$$= \frac{3}{2} = \text{R.H.S.}$$

Example 8 : If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, prove that

(1) $\cos(\alpha - \beta) = \dfrac{a^2+b^2-2}{2}$ (2) $\tan\left(\dfrac{\alpha-\beta}{2}\right) = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

Solution : (1) We have $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$

Squaring and adding,

$$(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = a^2 + b^2$$

$\therefore$ $\sin^2\alpha + 2\sin\alpha\,\sin\beta + \sin^2\beta + \cos^2\alpha + 2\cos\alpha\,\cos\beta + \cos^2\beta = a^2 + b^2$

$\therefore$ $2 + 2(\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta) = a^2 + b^2$

$\therefore$ $2 + 2\cos(\alpha - \beta) = a^2 + b^2$

$\therefore$ $\cos(\alpha - \beta) = \dfrac{a^2+b^2-2}{2}$

(2) Now, $\tan^2\left(\dfrac{\alpha-\beta}{2}\right) = \dfrac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}$

$$\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}$$

$$\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$$

$\therefore$ $\tan\left(\dfrac{\alpha-\beta}{2}\right) = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

Example 9 : Prove $\sin^4\theta \cdot \cos^4\theta = \frac{1}{128}$: $(3 - 4\cos4\theta + \cos8\theta)$

Solution : $\sin^4\theta \cdot \cos^4\theta = (\sin\theta\,\cos\theta)^4$

$$= \frac{1}{16}(2\sin\theta\,\cos\theta)^4$$

$$= \frac{1}{16}(\sin2\theta)^4$$

$$= \frac{1}{16}(\sin^2 2\theta)^2$$

$$= \frac{1}{16}\left(\frac{1-\cos4\theta}{2}\right)^2$$

$$= \frac{1}{64}(1 - 2\cos4\theta + \cos^2 4\theta)$$

$$= \frac{1}{64}\left(1 - 2\cos4\theta + \frac{1+\cos8\theta}{2}\right)$$

$$= \frac{1}{128}(2 - 4\cos4\theta + 1 + \cos8\theta)$$

$$= \frac{1}{128}(3 - 4\cos4\theta + \cos8\theta)$$

Exercise 5.2

1. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, find the values of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

2. If $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{5}{13}$, $0 < \alpha, \beta < \frac{\pi}{2}$, then find the values of $\sin^2\left(\frac{\alpha-\beta}{2}\right)$ and $\cos^2\left(\frac{\alpha-\beta}{2}\right)$.

Prove : **(3 to 12)**

3. $\cos^6\theta - \sin^6\theta = \frac{1}{4}(\cos^3 2\theta + 3\cos 2\theta)$

4. $\cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right) = \frac{3}{2}$

5. $\sin^2 A + \sin^2\left(A + \frac{2\pi}{3}\right) + \sin^2\left(A + \frac{4\pi}{3}\right) = \frac{3}{2}$. Deduce this from example 4.

6. $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$

7. $\sin^4\theta \cdot \cos^2\theta = \frac{1}{32}[2 - \cos 2\theta - 2\cos 4\theta + \cos 6\theta]$

8. $\sin^6\theta = \frac{1}{32}[10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta]$

9. $\sin 6^\circ \cdot \sin 42^\circ \cdot \sin 66^\circ \cdot \sin 78^\circ = \frac{1}{16}$

10. $\cos 6^\circ \cdot \cos 42^\circ \cdot \cos 66^\circ \cdot \cos 78^\circ = \frac{1}{16}$

11. $16\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15} = 1$

12. $\left(1 + \cos\frac{\pi}{10}\right)\left(1 + \cos\frac{3\pi}{10}\right)\left(1 + \cos\frac{7\pi}{10}\right)\left(1 + \cos\frac{9\pi}{10}\right) = \frac{1}{16}$

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5.6 Conditional Identities

Now we shall discuss some identities satisfying certain conditions.

e.g. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$ for $A + B + C = \pi$. This identity is true for all A, B, C satisfying the condition $A + B + C = \pi$. Therefore, this identity is called a **conditional identity**. On the other hand $\sin^2 A + \cos^2 A = 1$ is true for every A without any condition. This is an example of an unconditional identity.

Most of the relations, relating to the angles of a triangle are of the type of conditional identities. They are useful in understanding the properties of a triangle. Here we need to keep the following in mind.

$A + B + C = \pi$

$\therefore \quad A + B = \pi - C$ and $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\therefore \quad \sin(A + B) = \sin(\pi - C)$ and $\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\therefore \quad \sin(A + B) = \sin C$ and $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

In the same way,

$\cos(A + B) = -\cos C$ and $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

Example 10 : If $A + B + C = \pi$, then prove that

$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

Solution : L.H.S. $= \sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A + B)\cos(A - B) + 2\sin C \cdot \cos C$

$= 2\sin(\pi - C)\cos(A - B) + 2\sin C \cdot \cos C$ **$(A + B + C = \pi)$**

$= 2\sin C \cdot \cos(A - B) + 2\sin C \cdot \cos C$

$= 2\sin C\,[\cos(A - B) + \cos C]$

$= 2\sin C\,[\cos(A - B) - \cos(A + B)]$ **$(A + B + C = \pi)$**

$= 2\sin C\,[-2\sin A \cdot \sin(-B)]$

$= 4\sin A \sin B \sin C$ **$(\sin(-B) = -\sin B)$**

$=$ R.H.S.

Example 11 : If $A + B + C = \frac{\pi}{2}$, then prove that

$\cos^2 A + \cos^2 B + \cos^2 C = 2[1 + \sin A \sin B \sin C]$.

Solution : L.H.S. $= \cos^2 A + \cos^2 B + \cos^2 C$

$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$= \frac{1}{2}[3 + \cos 2A + \cos 2B + \cos 2C]$

$= \frac{1}{2}[3 + 2\cos(A + B)\cos(A - B) + 1 - 2\sin^2 C]$

$= \frac{1}{2}[4 + 2\cos(A + B) \cdot \cos(A - B) - 2\sin^2 C]$

$= 2 + \cos\left(\frac{\pi}{2} - C\right) \cdot \cos(A - B) - \sin^2 C$ **$\left(A + B = \frac{\pi}{2} - C\right)$**

$= 2 + \sin C\,[\cos(A - B) - \sin C]$

$= 2 + \sin C\,[\cos(A - B) - \cos(A + B)]$ **$\left(A + B = \frac{\pi}{2} - C\right)$**

$= 2 + \sin C\,[-2\sin A \cdot \sin(-B)]$

$= 2 + 2\sin A \sin B \sin C$

$= 2\,[1 + \sin A \sin B \sin C]$

or second method :

L.H.S. $= \cos^2 A + \cos^2 B + \cos^2 C$

$= \cos^2 A + 1 - \sin^2 B + 1 - \sin^2 C$

$= 2 + (\cos^2 A - \sin^2 B) - \sin^2 C$

$= 2 + \cos(A + B)\cos(A - B) - \sin^2 C$

$= 2 + \cos\left(\frac{\pi}{2} - C\right) \cdot \cos(A - B) - \sin^2 C$

$= 2 + \sin C \cdot \cos(A - B) - \sin^2 C$

$= 2 + \sin C\,[\cos(A - B) - \sin C]$

$= 2 + \sin C\,[\cos(A - B) - \cos(A + B)]$

$= 2 + \sin C\,[-2\sin A \cdot \sin(-B)]$

$= 2\,[1 + \sin A \sin B \sin C] =$ R.H.S.

Exercise 5.3

1. If A + B + C = π, prove that

(1) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(2) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(3) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(4) $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C)$

(5) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

(6) $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(7) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right)$

(8) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

2. If A + B + C = $\frac{\pi}{2}$, prove that

(1) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C$

(2) $\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C$

(3) $\sin^2 A - \sin^2 B + \sin^2 C = 1 - 2\cos A \sin B \cos C$

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Miscellaneous Problems :

Example 12 : Prove that $\tan 142\frac{1}{2}^{\circ} = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

Solution : $\tan 142\frac{1}{2}^{\circ} = \tan\left(90^{\circ} + 52\frac{1}{2}^{\circ}\right)$

$$= -\cot 52\frac{1}{2}^{\circ}$$

$$= -\cot\left(45^{\circ} + 7\frac{1}{2}^{\circ}\right)$$

$$= -\frac{\cot 7\frac{1}{2}^{\circ} - 1}{\cot 7\frac{1}{2}^{\circ} + 1}$$

$$= -\frac{\cos 7\frac{1}{2}^{\circ} - \sin 7\frac{1}{2}^{\circ}}{\cos 7\frac{1}{2}^{\circ} + \sin 7\frac{1}{2}^{\circ}}$$

$$= -\frac{\cos 7\frac{1}{2}^{\circ} - \sin 7\frac{1}{2}^{\circ}}{\cos 7\frac{1}{2}^{\circ} + \sin 7\frac{1}{2}^{\circ}} \times \frac{\cos 7\frac{1}{2}^{\circ} - \sin 7\frac{1}{2}^{\circ}}{\cos 7\frac{1}{2}^{\circ} - \sin 7\frac{1}{2}^{\circ}}$$

$$= -\frac{\left(\cos 7\frac{1}{2}^{\circ} - \sin 7\frac{1}{2}^{\circ}\right)^2}{\cos^2 7\frac{1}{2}^{\circ} - \sin^2 7\frac{1}{2}^{\circ}}$$

$$= -\frac{1 - 2\sin 7\frac{1}{2}^{\circ} \times \cos 7\frac{1}{2}^{\circ}}{\cos^2 7\frac{1}{2}^{\circ} - \sin^2 7\frac{1}{2}^{\circ}}$$

$$= -\frac{1 - \sin 15^\circ}{\cos 15^\circ}$$

$$= -\frac{1 - \sin(45^\circ - 30^\circ)}{\cos(45^\circ - 30^\circ)}$$

$$= -\frac{1 - \frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$= -\frac{2\sqrt{2} - \sqrt{3} + 1}{\sqrt{3}+1}$$

$$= -\frac{(2\sqrt{2} - \sqrt{3} + 1)}{(\sqrt{3}+1)} \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$$

$$= -\frac{(2\sqrt{6} - 2\sqrt{2} - 3 + \sqrt{3} + \sqrt{3} - 1)}{2}$$

$$= -\sqrt{6} + \sqrt{2} + 2 - \sqrt{3}$$

$$= 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$$

Example 13 : If A + B + C = π, then prove that

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi - A}{4}\right) \sin\left(\frac{\pi - B}{4}\right) \sin\left(\frac{\pi - C}{4}\right).$$

Solution : R.H.S. $= 1 + 4\sin\left(\frac{\pi - A}{4}\right) \sin\left(\frac{\pi - B}{4}\right) \sin\left(\frac{\pi - C}{4}\right)$

$$= 1 + 4\sin\left(\frac{B+C}{4}\right) \sin\left(\frac{A+C}{4}\right) \sin\left(\frac{A+B}{4}\right) \qquad \mathbf{(A + B + C = \pi)}$$

$$= 1 + 2\left(2\sin\left(\frac{B+C}{4}\right) \sin\left(\frac{A+C}{4}\right)\right) \sin\left(\frac{A+B}{4}\right)$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right) \left[\cos\left(\frac{B-A}{4}\right) - \cos\left(\frac{A+B+2C}{4}\right)\right]$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right) \cos\left(\frac{B-A}{4}\right) - 2\sin\left(\frac{\pi - C}{4}\right) \cos\left(\frac{\pi + C}{4}\right)$$

$$= 1 + \left(\sin\frac{B}{2} + \sin\frac{A}{2}\right) - \left(\sin\frac{\pi}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + \sin\frac{B}{2} + \sin\frac{A}{2} - \sin\frac{\pi}{2} + \sin\frac{C}{2}$$

$$= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = \text{L.H.S.}$$

Example 14 : If α and β be the roots of the equation $a\cos\theta + b\sin\theta = c$, prove that $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{2b}{a+c}$. Hence, deduce that $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}$.

Solution : $a\cos\theta + b\sin\theta = c$

$$\therefore \quad a\left(\frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}\right) + b\left(\frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}\right) = c$$

$$\therefore \quad a - a\tan^2\frac{\theta}{2} + 2b\tan\frac{\theta}{2} = c + c\tan^2\frac{\theta}{2}$$

$$\therefore \quad (a + c)\tan^2\frac{\theta}{2} - 2b\tan\frac{\theta}{2} + (c - a) = 0$$

This is a quadratic equation in $tan\frac{\theta}{2}$ and its roots are $tan\frac{\alpha}{2}$ and $tan\frac{\beta}{2}$.

$$\therefore \quad tan\frac{\alpha}{2} + tan\frac{\beta}{2} = -\left(\frac{-2b}{a+c}\right) = \frac{2b}{a+c} \text{ and } tan\frac{\alpha}{2} \cdot tan\frac{\beta}{2} = \frac{c-a}{c+a}$$

Now, $$tan\left(\frac{\alpha+\beta}{2}\right) = \frac{tan\frac{\alpha}{2} + tan\frac{\beta}{2}}{1 - tan\frac{\alpha}{2}\, tan\frac{\beta}{2}}$$

$$= \frac{\frac{2b}{a+c}}{1 - \frac{c-a}{c+a}} = \frac{2b}{a+c-c+a} = \frac{2b}{2a} = \frac{b}{a}$$

Example 15 : Prove using principle of mathematical induction,

$$cos\theta + cos2\theta + cos3\theta + ... + cosn\theta = sin\,\frac{(n+1)\theta}{2} \cdot cos\,\frac{n\theta}{2} \cdot cosec\frac{\theta}{2} - 1.$$

Solution :

Let, P(n) : $cos\theta + cos2\theta + cos3\theta + ... + cosn\theta = sin\,\frac{(n+1)\theta}{2} \cdot cos\frac{n\theta}{2} \cdot cosec\frac{\theta}{2} - 1$

For $n = 1$, L.H.S. $= cos\theta$, R.H.S. $= sin\theta \cdot cos\frac{\theta}{2} \cdot cosec\frac{\theta}{2} - 1$

$$= \frac{sin\theta \cdot cos\frac{\theta}{2}}{sin\frac{\theta}{2}} - 1$$

$$= \frac{2sin\frac{\theta}{2} \cdot cos\frac{\theta}{2} \cdot cos\frac{\theta}{2}}{sin\frac{\theta}{2}} - 1$$

$$= 2cos^2\frac{\theta}{2} - 1$$

$$= cos\theta = \text{R.H.S.} \qquad (cos2\theta = 2cos^2\theta - 1)$$

$\therefore$ P(1) is true.

Let P(k) is true.

$$\therefore \quad cos\theta + cos2\theta + cos3\theta + ... + cosk\theta = sin(k+1)\frac{\theta}{2} \cdot cos\frac{k\theta}{2} \cdot cosec\frac{\theta}{2} - 1$$

Let, $n = k + 1$

L.H.S. $= cos\theta + cos2\theta + cos3\theta + ... + cosk\theta + cos(k+1)\theta$

$$= \frac{sin\left(\frac{k+1}{2}\right)\theta \cdot cos\frac{k\theta}{2}}{sin\frac{\theta}{2}} - 1 + cos(k+1)\theta$$

$$= \frac{1}{2sin\frac{\theta}{2}}\left(2sin\left(\frac{k+1}{2}\right)\theta\ cos\,\frac{k\theta}{2} + 2sin\frac{\theta}{2} \cdot cos(k+1)\theta\right) - 1$$

$$= \frac{1}{2sin\frac{\theta}{2}}\left[sin\frac{(2k+1)\theta}{2} + sin\frac{\theta}{2} + sin\frac{(2k+3)\theta}{2} - sin\frac{(2k+1)\theta}{2}\right] - 1$$

$$= \frac{1}{sin\frac{\theta}{2}}\left[\frac{1}{2}\left(sin\,\frac{(2k+3)\theta}{2} + sin\frac{\theta}{2}\right)\right] - 1$$

$$= \frac{1}{sin\frac{\theta}{2}}\left[\frac{1}{2} \cdot 2sin\,\frac{(k+2)\theta}{2} \cdot cos\frac{(k+1)\theta}{2}\right] - 1$$

$$= sin\frac{(k+2)\theta}{2} \cdot cos\frac{(k+1)\theta}{2} \cdot cosec\frac{\theta}{2} - 1$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true. $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true for $\forall n \in$ N by P.M.I.

Exercise 5

Prove : **(1 to 15)**

1. $tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = sec\theta + tan\theta$

2. $\frac{cot\,\theta + cosec\,\theta - 1}{cot\,\theta - cosec\,\theta + 1} = cot\frac{\theta}{2}$

3. $tan\alpha = \sqrt{5}\,tan\beta \Rightarrow cos2\alpha = \frac{3cos2\beta - 2}{3 - 2cos2\beta}$

4. $tan\frac{\alpha}{2} = cos\theta \Rightarrow sin\alpha = \frac{1 - tan^4\frac{\theta}{2}}{1 + tan^4\frac{\theta}{2}}$

5. If $sin\theta = a$, then the roots of $a(1 + x^2) = 2x$ are $tan\frac{\theta}{2}$ and $cot\frac{\theta}{2}$.

6. If $cos\theta = a$, then the roots of $4x^2 - 4x + 1 = a^2$ are $cos^2\frac{\theta}{2}$ and $sin^2\frac{\theta}{2}$.

7. If α and β are the roots of the equation $acos\theta + bsin\theta = c$, then

(1) $cos\alpha + cos\beta = \frac{2ac}{a^2+b^2}$ and $cos\alpha \cdot cos\beta = \frac{c^2-b^2}{a^2+b^2}$

(2) $tan\alpha + tan\beta = \frac{-2ab}{b^2-c^2}$ and $tan\alpha \cdot tan\beta = \frac{a^2-c^2}{b^2-c^2}$

(3) $sin(\alpha + \beta) = \frac{2ab}{a^2+b^2}$.

8. $cos^5\theta = \frac{1}{16}[10cos\theta + 5cos3\theta + cos5\theta]$

9. $(2cos\theta + 1)(2cos\theta - 1)(2cos2\theta - 1)(2cos4\theta - 1) = 2cos8\theta + 1$

10. $cosec\theta + cosec2\theta + cosec4\theta + cot4\theta = cot\frac{\theta}{2}$

11. $(cos^2 48^\circ - sin^2 12^\circ) - (cos^2 66^\circ - sin^2 6^\circ) = \frac{1}{4}$

12. $\frac{sec\,8\theta - 1}{sec\,4\theta - 1} = \frac{tan\,8\theta}{tan\,2\theta}$

13. $cot\frac{\pi}{24} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$

14. $tan\frac{\pi}{16} = \sqrt{4+2\sqrt{2}} - (\sqrt{2} + 1)$

15. $4sin27^\circ = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$

16. If $x = sin\theta + cos\theta \cdot sin2\theta$ and $y = cos\theta + sin\theta \cdot sin2\theta$,
then prove that $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2$.

17. If A + B + C = π, prove that

(1) $sin(B + 2C) + sin(C + 2A) + sin(A + 2B) = 4sin\left(\frac{B - C}{2}\right) \cdot sin\left(\frac{C - A}{2}\right) \cdot sin\left(\frac{A - B}{2}\right)$

(2) $cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2} = 4cos\left(\frac{\pi - A}{4}\right) \cdot cos\left(\frac{\pi - B}{4}\right) \cdot cos\left(\frac{\pi - C}{4}\right)$

18. Prove : ΔABC is right angled triangle $\Leftrightarrow$

$cos^2A + cos^2B + cos^2C = 1 \Leftrightarrow sin^2A + sin^2B + sin^2C = 2$

Prove by principle of mathematical induction : **(19 to 22)**

19. $sinx + sin3x + sin5x + ... + sin(2n - 1)x = \frac{sin^2nx}{sinx}$

20. $\frac{1}{2}tan\frac{x}{2} + \frac{1}{4}tan\frac{x}{4} + ... + \frac{1}{2^n}\ tan\frac{x}{2^n} = \frac{1}{2^n}\ cot\ \frac{x}{2^n} - cotx$

21. $sin\theta + sin2\theta + ... + sinn\theta = sin\frac{(n+1)\theta}{2} \cdot sin\frac{n\theta}{2} \cdot cosec\frac{\theta}{2}$

22. $cos\alpha \cdot cos2\alpha \cdot cos4\alpha \cdot ... \cdot cos2^{n-1}\alpha = \frac{sin^n 2\alpha}{2^n \cdot sin\alpha}$

23. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) One root of $4x^3 - 3x = \frac{1}{2}$ is ... ☐

(a) $sin70^o$ (b) $sin10^o$ (c) $sin20^o$ (d) $cos70^o$

(2) The range of the function $cos^4\theta - sin^4\theta$ is ... ☐

(a) [0, 1] (b) [−1, 1] (c) (0, 1) (d) (−1, 1)

(3) The range of $sec^4\theta + cosec^4\theta$ is ... ☐

(a) $[1, \alpha]$ (b) R^+ (c) $[8, \infty)$ (d) R − (−1, 1)

(4) The value of $cos67\frac{1}{2}^o$ is ... ☐

(a) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (b) $\frac{\sqrt{2-\sqrt{2}}}{2}$ (c) $\sqrt{2} - 1$ (d) $\sqrt{2} + 1$

(5) The value of $3sin\frac{\pi}{9} - 4sin^3\frac{\pi}{9}$ is ... ☐

(a) $\frac{1}{2}$ (b) −1 (c) $\frac{\sqrt{3}}{2}$ (d) $-\frac{1}{2}$

(6) If $sin\theta = \frac{3}{5}, \frac{\pi}{2} < \theta < \pi$, then $P(2\theta)$ is in the ...... quadrant. ☐

(a) 1st (b) 2nd (c) 3rd (d) 4th

(7) One root of the equation $6x - 8x^3 = \sqrt{3}$ is ... ☐

(a) $sin20^o$ (b) $sin30^o$ (c) $sin10^o$ (d) $cos10^o$

(8) If α is the root of $25cos^2\theta + 5cos\theta - 12 = 0, \frac{\pi}{2} < \theta < \pi$ then $sin2\alpha$ is ... ☐

(a) $\frac{-24}{25}$ (b) $\frac{-13}{18}$ (c) $\frac{13}{18}$ (d) $\frac{24}{25}$

(9) $\dfrac{sin\,3\theta}{1+2cos\,2\theta}$ is equal to ...

(a) $-sin\theta$ (b) $-cos\theta$ (c) $cos\theta$ (d) $sin\theta$

(10) The value of $\left(\dfrac{1+sin\theta-cos\theta}{1+sin\theta+cos\theta}\right)^2$ is ...

(a) $tan^2\dfrac{\theta}{2}$ (b) $2cot\dfrac{\theta}{2}$ (c) $cot^2\dfrac{\theta}{2}$ (d) $2cosec\dfrac{\theta}{2}$

(11) The value of $12sin40^o - 16sin^3 40^o$ is ...

(a) $-3\sqrt{2}$ (b) $2\sqrt{3}$ (c) $-2\sqrt{3}$ (d) $3\sqrt{2}$

(12) If $sin\alpha = \dfrac{-3}{5}$, $\pi < \alpha < \dfrac{3\pi}{2}$, then the value of $cos\dfrac{\alpha}{2}$ is ...

(a) $\dfrac{-3}{\sqrt{10}}$ (b) $\dfrac{-1}{\sqrt{10}}$ (c) $\dfrac{1}{\sqrt{10}}$ (d) $\dfrac{3}{\sqrt{10}}$

(13) If $\dfrac{1+cos A}{1-cos A} = \dfrac{m^2}{n^2}$, then $tan A$ is equal to ...

(a) $\pm\dfrac{2mn}{m^2-n^2}$ (b) $\pm\dfrac{2mn}{m^2+n^2}$ (c) $\dfrac{m^2+n^2}{m^2-n^2}$ (d) $\dfrac{m^2-n^2}{m^2+n^2}$

(14) $cos^4\left(\dfrac{\pi}{24}\right) - sin^4\left(\dfrac{\pi}{24}\right)$ is equal to ...

(a) $\dfrac{\sqrt{5}-1}{2\sqrt{2}}$ (b) $\dfrac{\sqrt{5}-1}{4}$ (c) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\dfrac{\sqrt{2+\sqrt{2}}}{4}$

(15) If $cos\alpha = -0.6$ and $\pi < \alpha < \dfrac{3\pi}{2}$, then $tan\dfrac{\alpha}{4}$ is equal to ...

(a) $\dfrac{1-\sqrt{5}}{2}$ (b) $\dfrac{\sqrt{5}-1}{2}$ (c) $\dfrac{\sqrt{5}}{2}$ (d) $\dfrac{\sqrt{5}+1}{2}$

(16) If $0 < \theta < \dfrac{\pi}{2}$ is an acute angle and $2x \cdot sin^2\dfrac{\theta}{2} + 1 = x$, then $tan\theta$ is ...

(a) $\sqrt{x^2-1}$ (b) $\sqrt{x^2+1}$ (c) $\sqrt{x^2-2}$ (d) $\sqrt{x^2-\frac{1}{2}}$

(17) If $tanx = \dfrac{b}{a}$, then the value of $acos2x + bsin2x$ is ...

(a) $a - b$ (b) a (c) b (d) $a + b$

(18) The value of $cos6^o \cdot sin24^o \cdot cos72^o$ is ...

(a) $\dfrac{-1}{8}$ (b) $\dfrac{-1}{4}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{1}{4}$

(19) The maximum value of the expression $sin^6\theta + cos^6\theta$ is ...

(a) 1 (b) $\dfrac{1}{2}$ (c) $\dfrac{5}{8}$ (d) $\dfrac{13}{8}$

(20) If $cosA = \dfrac{3}{4}$, then the value of $32sin\dfrac{A}{2}\, sin\dfrac{5A}{2}$ is equal to ...

(a) -11 (b) $-\sqrt{11}$ (c) $\sqrt{11}$ (d) 11

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Summary

We studied following points in this chapter :

1. $sin2\alpha = 2sin\alpha\ cos\alpha$
2. $cos2\alpha = cos^2\alpha - sin^2\alpha = 2cos^2\alpha - 1 = 1 - 2sin^2\alpha$
3. $1 + cos2\alpha = 2cos^2\alpha$ and $1 - cos2\alpha = 2sin^2\alpha$
4. $sin2\alpha = \frac{2tan\alpha}{1+tan^2\alpha}$
5. $cos2\alpha = \frac{1-tan^2\alpha}{1+tan^2\alpha}$
6. $tan2\alpha = \frac{2tan\alpha}{1-tan^2\alpha}$ $\quad \alpha \in R - \left[\left\{(2k-1)\frac{\pi}{2}\right\} \cup \left\{(2k-1)\frac{\pi}{4}\right\}\right]\ k \in Z$
7. $cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha}$ $\quad \alpha \in R - \left\{\frac{k\pi}{2} \mid k \in Z\right\}$
8. $sin3\alpha = 3sin\alpha - 4sin^3\alpha$
9. $cos3\alpha = 4cos^3\alpha - 3cos\alpha$
10. $tan3\alpha = \frac{3tan\alpha - tan^3\alpha}{1-3tan^2\alpha}$ $\quad \alpha \in R - \left\{(2k-1)\frac{\pi}{6}, k \in Z\right\}$
11. $cot3\alpha = \frac{cot^3\alpha - 3cot\alpha}{3cot^2\alpha - 1}$ $\quad \alpha \in R - \left\{\frac{k\pi}{3} \mid k \in Z\right\}$
12. $sin^2\frac{\alpha}{2} = \frac{1-cos\alpha}{2}$
13. $cos^2\frac{\alpha}{2} = \frac{1+cos\alpha}{2}$
14. $tan^2\frac{\alpha}{2} = \frac{1-cos\alpha}{1+cos\alpha}$ $\quad \alpha \in R - \left\{(2k-1)\pi \mid k \in Z\right\}$
15. $sin18^{\circ} = \frac{\sqrt{5}-1}{4}$, $cos18^{\circ} = \sqrt{\frac{10+2\sqrt{5}}{16}}$
16. $sin36^{\circ} = \sqrt{\frac{10-2\sqrt{5}}{16}}$, $cos36^{\circ} = \frac{\sqrt{5}+1}{4}$
17. $sin22\frac{1}{2}^{\circ} = \frac{\sqrt{2-\sqrt{2}}}{2}$, $cos22\frac{1}{2}^{\circ} = \frac{\sqrt{2+\sqrt{2}}}{2}$, $tan22\frac{1}{2}^{\circ} = \sqrt{2} - 1$

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Chapter 6

TRIGONOMETRIC EQUATIONS AND PROPERTIES OF A TRIANGLE

If equations are trains threading the landscape of numbers, then no train stops at pi.
– Richard Preston

Pure mathematics is in its way, the poetry of logical ideas.
– Albert Einstein

6.1 Introduction

In the previous semester and in chapters 4, 5 we have studied about trigonometric functions, their graphs and their properties like zeros, range, periodic nature, identities. Trigonometry is useful in land surveying. We know that by using trigonometry we can find the height of a hill without actually measuring it. In 1852, **Radhanath Sikdar**, an *Indian mathematician* and a surveyor from Bengal, was the first to identify Mount Everest as the world's highest peak, using trigonometric calculations. Trigonometry is useful in modern navigation such as satellite systems, astronomy, aviation, oceanography.

In this chapter we will learn how to solve trigonometric equations and properties of a triangle using trigonometry.

6.2 Trigonometric Equations

A trigonometric equation is an equation containing trigonometric functions, e.g. $sin^2x - 4cosx = 1$ is a trigonometric equation.

A trigonometric equation that holds true for all values of the variable in its domain is called a trigonometric identity, e.g. $cos2\theta = 2cos^2\theta - 1$ is a trigonometric identity.

There are other equations, which are true only for some proper subsets of domain of functions involved. We will learn some techniques for solving such trigonometric equations, as well as how to obtain the complete set of solutions of an equation based on a single solution of that equation. The equations $sinx = \frac{1}{2}$ has not only the solution $x = \frac{\pi}{6}$ but also $x = \frac{5\pi}{6}$, $x = 2\pi + \frac{\pi}{6}$, $x = 3\pi - \frac{\pi}{6}$ etc. are also solutions of $sinx = \frac{1}{2}$. Thus, we can say that $x = \frac{\pi}{6}$ is a solution of $sinx = \frac{1}{2}$ but it is not the complete solution of the equation. **A general solution to an equation is the set of all possible solutions of that equation.** Note that some trigonometric equations may not have any solution, e.g. $sinx = \pi$. Due to periodic nature of trigonometric functions, if a trigonometric equation has a solution it may have infinitely many solutions. The set of all such solution is known as the **general solution**.

Look at the graph of $y = sinx$. Observe any of the horizontal line $y = k$ where k varies from -1 to 1. We can see that the graph of $y = k$ intersects the graph of $y = sinx$ in infinitely many points (figure 6.1). This means that if we take $a \in [-1, 1]$, then there are infinitely many real numbers x such that $sinx = a$. For a solution of a trigonometric equation, we need a unique real number α such that $sin\alpha = a$. For that we have to restrict the domain suitably. If we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, etc. then we get a unique number α such that $sin\alpha = a$. We assume that the restricted domain for $y = sinx$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In this domain any horizontal line $y = k$, $k \in [-1, 1]$ intersects the graph of $y = sinx$ only at one point (figure 6.2).

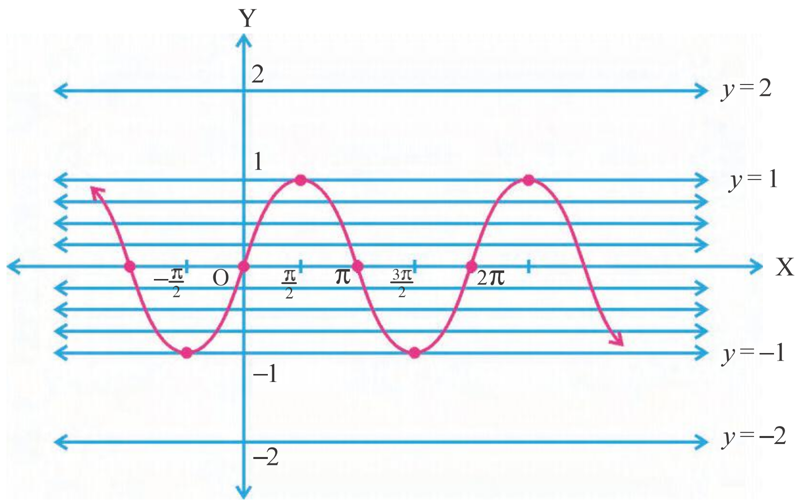

Figure 6.1

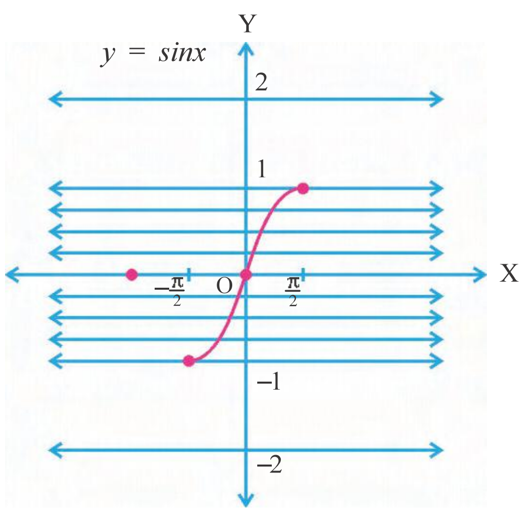

Figure 6.2

Similar situation arises for the function $y = cosx$. (figure 6.3)

We take the restricted domain $[0, \pi]$ for $y = cosx$. (figure 6.4)

Note that any horizontal line $y = a$ where $|a| > 1$ will not intersect the graph of $y = sinx$ or $y = cosx$. Thus $sinx = a$ or $cosx = a$ where $|a| > 1$ has no solution.

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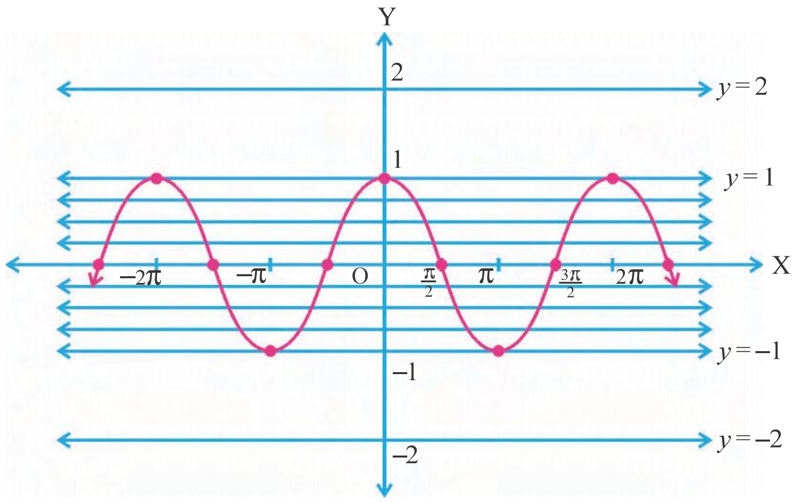

Figure 6.3

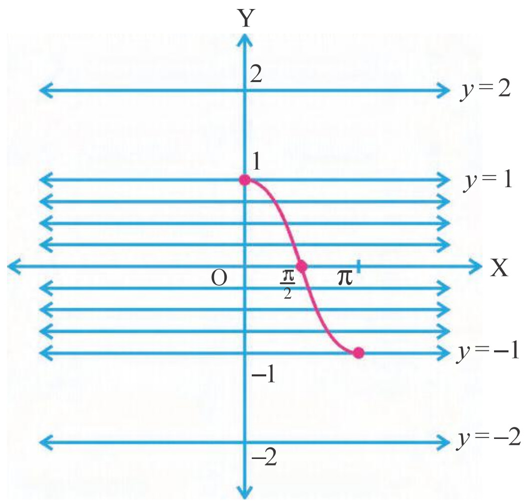

Figure 6.4

For the function $y = tanx$, if we draw any horizontal line in the plane it will intersect the graph of $y = tanx$ at infinitely many points (figure 6.5). This means that if we take any $a \in$ R, then there are infinitely many real number x such that $tanx = a$. we need a unique value α such that $tan\alpha = a$. So we have to restrict domain suitably. We take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as restricted domain of $y = tanx$. (figure 6.6). We shall discuss this in more detail when we study the concept of inverse trigonometric functions in the third semester in 12th standard.

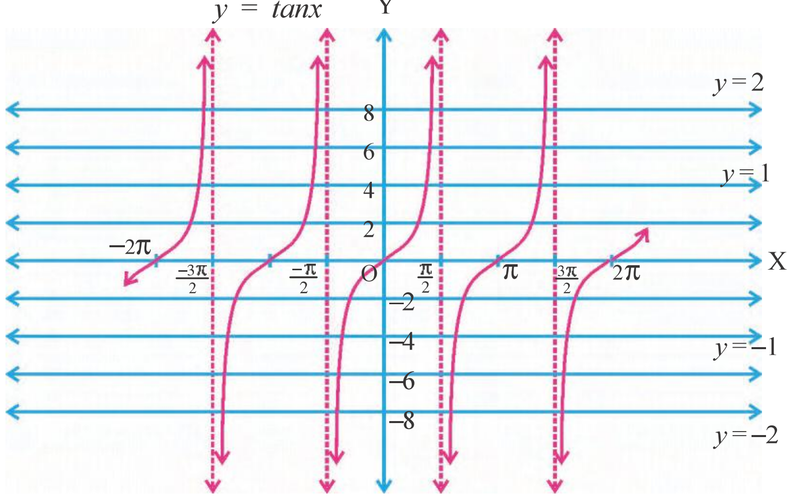

Figure 6.5

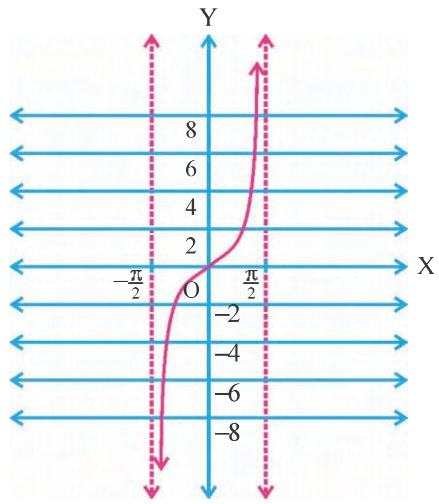

Figure 6.6

Thus, for any $a \in [-1, 1]$ there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that, $a = sin\alpha$.

Also, for any $a \in [-1, 1]$ there is a unique $\alpha \in [0, \pi]$, such that, $a = cos\alpha$.

Finally, for any $a \in$ R there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that, $a = tan\alpha$.

We know the set of zeros of sine, cosine and tangent functions. That actually means that we already know the general solutions of the equations $sin\theta = 0$, $cos\theta = 0$, $tan\theta = 0$.

$$sin\theta = 0 \Leftrightarrow \theta = k\pi,\ k \in Z$$

$$cos\theta = 0 \Leftrightarrow \theta = (2k + 1)\frac{\pi}{2},\ k \in Z$$

$$tan\theta = 0 \Leftrightarrow \theta = k\pi,\ k \in Z$$

We shall now solve the equations, $sin\theta = a$, $-1 \le a \le 1$, $cos\theta = a$, $-1 \le a \le 1$ and $tan\theta = a$, $a \in$ R.

6.3 General Solution of $sin\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that, $a = sin\alpha$.

Now, $sin\theta = a = sin\alpha$

$\therefore \quad sin\theta - sin\alpha = 0$

$\Leftrightarrow \quad 2cos\frac{\theta+\alpha}{2}\ sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad cos\frac{\theta+\alpha}{2} = 0$ or $sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = (2n + 1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = n\pi,\ n \in Z$ **(Why ?)**

$\Leftrightarrow \quad \theta = (2n + 1)\pi - \alpha$ or $\theta = 2n\pi + \alpha,\ n \in Z$

$\Leftrightarrow \quad \theta = (2n + 1)\pi + (-1)^{2n+1}\alpha$ or $\theta = 2n\pi + (-1)^{2n}\alpha,\ n \in Z$

Therefore, the general solution is given by $\theta = k\pi + (-1)^k\alpha,\ k \in Z$.

(We have replaced $2n + 1$, $2n$ by k because any integer is of the form either $2n + 1$ or $2n$)

Thus, $\boldsymbol{sin\theta = sin\alpha \Leftrightarrow \theta = k\pi + (-1)^k\alpha,\ k \in Z}$

Hence, the solution set of $\sin\theta = a, -1 \le a \le 1$ is given by $\{k\pi + (-1)^k\alpha \mid k \in Z\}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = a = \sin\alpha$.

(We may take any $\alpha \in R$ such that $a = \sin\alpha$. The solution remains same. This convention of taking $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is only for the uniformity of the form of the solution set.)

General Solution of $\cos\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in [0, \pi]$ such that, $a = \cos\alpha$.

Now, $\cos\theta = a = \cos\alpha$

$$\therefore \quad \cos\theta - \cos\alpha = 0 \Leftrightarrow -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow \sin\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow \frac{\theta+\alpha}{2} = k\pi \text{ or } \frac{\theta-\alpha}{2} = k\pi, k \in Z$$

$$\Leftrightarrow \theta = 2k\pi - \alpha \text{ or } \theta = 2k\pi + \alpha, k \in Z$$

Therefore the general solution is given by $\theta = 2k\pi \pm \alpha, k \in Z$.

Thus, $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2k\pi \pm \alpha, k \in Z$

Hence, **the solution set of $\cos\theta = a, -1 \le a \le 1$ is given by $\{2k\pi \pm \alpha \mid k \in Z\}$ where $\alpha \in [0, \pi]$ and $\cos\theta = a = \cos\alpha$.**

General Solution of $\tan\theta = a$, where $a \in R$

Here $a \in R$. Therefore, there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that, $a = \tan\alpha$.

Now, $\tan\theta = a = \tan\alpha$

$$\therefore \quad \tan\theta - \tan\alpha = 0 \Leftrightarrow \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0$$

$$\Leftrightarrow \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha} = 0$$

$$\Leftrightarrow \frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha} = 0$$

$$\Leftrightarrow \sin(\theta - \alpha) = 0$$

$$\Leftrightarrow \theta - \alpha = k\pi, k \in Z$$

$$\Leftrightarrow \theta = k\pi + \alpha, k \in Z$$

Thus, $\tan\theta = \tan\alpha \Leftrightarrow \theta = k\pi + \alpha, k \in Z$

Hence, **the solution set of $\tan\theta = a, a \in R$ is given by $\{k\pi + \alpha \mid k \in Z\}$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = a = \tan\alpha$.**

By the word 'solve' we shall mean to obtain the general solution set of the given equation.

Example 1 : Solve : (1) $2\sin2\theta - 1 = 0$ (2) $\sin^2\theta - \sin\theta - 2 = 0$

Solution : (1) $2\sin2\theta - 1 = 0$

$\therefore \quad sin2\theta = \frac{1}{2} = sin\left(\frac{\pi}{6}\right)$

We know that general solution of $sin\theta = sin\alpha$ is $k\pi + (-1)^k\alpha$, $k \in Z$.

$\therefore \quad 2\theta = k\pi + (-1)^k \frac{\pi}{6}, k \in Z$

$\therefore \quad \theta = \frac{k\pi}{2} + (-1)^k \frac{\pi}{12}, k \in Z$

Hence, the required solution set is $\left\{\frac{k\pi}{2} + (-1)^k \frac{\pi}{12} \,\middle|\, k \in Z\right\}$.

(2) $sin^2\theta - sin\theta - 2 = 0$

$\therefore \quad (sin\theta + 1)(sin\theta - 2) = 0$

$\therefore \quad sin\theta = -1$ or $sin\theta = 2$

But $sin\theta = 2$ is not possible. **(Why ?)**

So, $sin\theta = -1 = sin\left(-\frac{\pi}{2}\right)$

$\therefore \quad \theta = k\pi + (-1)^k \left(-\frac{\pi}{2}\right), k \in Z$

Hence, the required solution set is $\left\{k\pi + (-1)^{k+1} \frac{\pi}{2} \,\middle|\, k \in Z\right\}$.

Example 2 : Solve : (1) $2cos5\theta + \sqrt{3} = 0$ (2) $2cos^2\theta - \sqrt{3}cos\theta = 0$

Solution : (1) $2cos5\theta + \sqrt{3} = 0$

$\therefore \quad cos5\theta = -\frac{\sqrt{3}}{2} = cos\left(\pi - \frac{\pi}{6}\right) = cos\left(\frac{5\pi}{6}\right)$ $\left(\frac{5\pi}{6} \in [0, \pi]\right)$

We know that general solution of $cos\theta = cos\alpha$ is $\theta = 2k\pi \pm \alpha$, $k \in Z$.

$\therefore \quad 5\theta = 2k\pi \pm \frac{5\pi}{6}, k \in Z$

$\therefore \quad \theta = \frac{2k\pi}{5} \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{\frac{2k\pi}{5} \pm \frac{\pi}{6} \,\middle|\, k \in Z\right\}$.

(2) $2cos^2\theta - \sqrt{3}cos\theta = 0$

$\therefore \quad cos\theta(2cos\theta - \sqrt{3}) = 0$

$\therefore \quad cos\theta = 0$ or $cos\theta = \frac{\sqrt{3}}{2} = cos\left(\frac{\pi}{6}\right)$

$\therefore \quad \theta = (2k + 1)\frac{\pi}{2}, k \in Z$ or $\theta = 2k\pi \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{(2k + 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\} \cup \left\{2k\pi \pm \frac{\pi}{6} \,\middle|\, k \in Z\right\}$.

Example 3 : Solve : (1) $sin5x - sin3x - sinx = 0$ (2) $cosx + cos2x + cos3x = 0$

Solution : (1) $sin5x - sin3x - sinx = 0$

$\therefore \quad 2cos4x\ sinx - sinx = 0$

$\therefore \quad sinx(2cos4x - 1) = 0$

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$\therefore \quad sinx = 0$ or $cos4x = \frac{1}{2} = cos\left(\frac{\pi}{3}\right)$

$\therefore \quad x = k\pi,\ k \in Z$ or $4x = 2k\pi \pm \frac{\pi}{3},\ k \in Z$

$\therefore \quad x = k\pi,\ k \in Z$ or $x = \frac{k\pi}{2} \pm \frac{\pi}{12},\ k \in Z$

Hence, the required solution set is $\{k\pi \mid k \in Z\} \cup \left\{\frac{k\pi}{2} \pm \frac{\pi}{12} \;\middle|\; k \in Z\right\}$.

(2) $cosx + cos2x + cos3x = 0$

$\therefore \quad cos3x + cosx + cos2x = 0$

$\therefore \quad 2cos2x\ cosx + cos2x = 0$

$\therefore \quad cos2x\ (2cosx + 1) = 0$

$\therefore \quad cos2x = 0$ or $cosx = -\frac{1}{2} = cos\frac{2\pi}{3}$ $\quad \left(\frac{2\pi}{3} \in [0, \pi]\right)$

$\therefore \quad 2x = (2k + 1)\frac{\pi}{2},\ k \in Z$ or $x = 2k\pi \pm \frac{2\pi}{3},\ k \in Z$

$\therefore \quad x = (2k + 1)\frac{\pi}{4},\ k \in Z$ or $x = 2k\pi \pm \frac{2\pi}{3},\ k \in Z$

Hence, the required solution set is $\left\{(2k + 1)\frac{\pi}{4} \;\middle|\; k \in Z\right\} \cup \left\{2k\pi \pm \frac{2\pi}{3} \;\middle|\; k \in Z\right\}$.

Example 4 : Solve : (1) $tan^2\theta + (1 - \sqrt{3})tan\theta - \sqrt{3} = 0$

(2) $tan\theta + tan4\theta + tan7\theta = tan\theta\ tan4\theta\ tan7\theta$

Solution : (1) $tan^2\theta + (1 - \sqrt{3})tan\theta - \sqrt{3} = 0$

$\therefore \quad tan^2\theta + tan\theta - \sqrt{3}tan\theta - \sqrt{3} = 0$

$\therefore \quad tan\theta(tan\theta + 1) - \sqrt{3}(tan\theta + 1) = 0$

$\therefore \quad (tan\theta + 1)(tan\theta - \sqrt{3}) = 0$

$\therefore \quad tan\theta = -1$ or $tan\theta = \sqrt{3}$

$\therefore \quad tan\theta = tan\left(-\frac{\pi}{4}\right)$ or $tan\theta = tan\frac{\pi}{3}$

$\therefore \quad \theta = k\pi - \frac{\pi}{4},\ k \in Z$ or $\theta = k\pi + \frac{\pi}{3},\ k \in Z$

Hence, the required solution set is $\left\{k\pi - \frac{\pi}{4} \;\middle|\; k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{3} \;\middle|\; k \in Z\right\}$.

(2) $tan\theta + tan4\theta + tan7\theta = tan\theta\ tan4\theta\ tan7\theta$

$\therefore \quad tan\theta + tan4\theta = -tan7\theta + tan\theta\ tan4\theta\ tan7\theta$

$\therefore \quad tan\theta + tan4\theta = -tan7\theta\ (1 - tan\theta\ tan4\theta)$ $\quad$ (i)

First we prove that $1 - tan\theta\ tan4\theta \neq 0$.

If $1 - tan\theta\ tan4\theta = 0$ then by (i) we have $tan\theta + tan4\theta = 0$.

Thus, $tan\theta\ tan4\theta = 1$ and $tan4\theta = -tan\theta$ which gives $tan^2\theta = -1$ which is not possible in R.

Now, by (i) we have $\frac{\tan\theta + \tan 4\theta}{1 - \tan\theta \tan 4\theta} = -\tan 7\theta$

$\therefore \quad \tan(\theta + 4\theta) = -\tan 7\theta$

$\therefore \quad \tan 5\theta = \tan(-7\theta)$

$\therefore \quad 5\theta = k\pi - 7\theta, k \in Z$

$\therefore \quad \theta = \frac{k\pi}{12}, k \in Z$

Also $\tan\theta$, $\tan 4\theta$, $\tan 7\theta$, should be defined.

$\therefore \quad \theta \neq (2m + 1)\frac{\pi}{2}, 4\theta \neq (2m + 1)\frac{\pi}{2}, 7\theta \neq (2m + 1)\frac{\pi}{2}, \ k \in Z$

$\therefore$ If $\theta = \frac{k\pi}{12}, k \in Z$ then $k \neq 6, 18, 30, ...$

$4\theta = \frac{k\pi}{3} \neq (2m + 1)\frac{\pi}{2}$ for any $k \in Z - \{6, 18, ...\}$

$7\theta = \frac{7k\pi}{12} \neq (2m + 1)\frac{\pi}{2}$ for any $k \in Z - \{6, 18, ...\}$

$\therefore \quad k \neq 6, 18, ...$

$\therefore \quad k \neq 12n + 6, n \in Z$

$\therefore$ k is not odd multiple of 6.

$\therefore$ The solution set is $\left\{\frac{k\pi}{12} \,\middle|\, k \in Z \text{ where } k \neq 12n + 6\right\}, \quad n \in Z$

Example 5 : Solve : (1) $4\sin\theta = \operatorname{cosec}\theta$ (2) $\sec\theta + \tan\theta = 2 - \sqrt{3}$

Solution : (1) $4\sin\theta = \operatorname{cosec}\theta$

$\therefore \quad 4\sin\theta = \frac{1}{\sin\theta}$

$\therefore \quad 4\sin^2\theta = 1$

$\therefore \quad \sin\theta = \pm\frac{1}{2}$

$\therefore \quad \sin\theta = \sin\left(\frac{\pi}{6}\right)$ or $\sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \quad \theta = k\pi + (-1)^k \frac{\pi}{6}, k \in Z$ or $\theta = k\pi + (-1)^k\left(-\frac{\pi}{6}\right), k \in Z$

$\therefore \quad \theta = k\pi + (-1)^k \frac{\pi}{6}, k \in Z$ or $\theta = k\pi + (-1)^{k+1} \frac{\pi}{6}, k \in Z$

$\therefore \quad \theta = k\pi \pm \frac{\pi}{6}, k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{6} \,\middle|\, k \in Z\right\}$.

(2) $\sec\theta + \tan\theta = 2 - \sqrt{3}$ **(i)**

Since $\sec^2\theta - \tan^2\theta = 1$, we have $\sec\theta - \tan\theta = \frac{1}{2 - \sqrt{3}} = \frac{2 + \sqrt{3}}{(2 - \sqrt{3})(2 + \sqrt{3})} = 2 + \sqrt{3}$

$\therefore \quad \sec\theta - \tan\theta = 2 + \sqrt{3}$ **(ii)**

Solving (i) and (ii) we get, $\sec\theta = 2$ and $\tan\theta = -\sqrt{3}$

Note that the above is a simultaneous system of trigonometric equations.

Since $\cos\theta = \frac{1}{2} > 0$ and $\tan\theta = -\sqrt{3} < 0$, P($\theta$) is in the fourth quadrant.

$\therefore \quad \cos\theta = \cos\left(-\frac{\pi}{3}\right)$ and $\tan\theta = \tan\left(-\frac{\pi}{3}\right)$

$\therefore \quad \theta = 2k\pi - \frac{\pi}{3}, k \in Z$ **(P(θ) is in fourth quadrant.)**

Hence, required solution set is $\left\{2k\pi - \frac{\pi}{3} \;\middle|\; k \in Z\right\}$.

6.4 The General Solution of $a\cos x + b\sin x = c$, $a, b, c \in R$ and $a^2 + b^2 \neq 0$

For the given real numbers a and b, we can find $r > 0$ and $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$. **(chapter 4)**

$\therefore \quad a^2 + b^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = r^2$

$\therefore \quad r = \sqrt{a^2 + b^2}$ **($r > 0$)**

Now, $a\cos x + b\sin x = c$

$\therefore \quad r\cos\alpha\cos x + r\sin\alpha\sin x = c$

$\therefore \quad r\cos(x - \alpha) = c$

$\therefore \quad \cos(x - \alpha) = \frac{c}{r}$ **(i)**

The last equation will have a solution if and only if $\left|\frac{c}{r}\right| \leq 1$, that is if and only if $c^2 \leq r^2$, that is if and only if $c^2 \leq a^2 + b^2$.

If $\cos(x - \alpha) = \cos\beta$, where $\cos\beta = \frac{c}{r}$, $\beta \in [0, \pi]$, then the general solution of (i) is $x - \alpha = 2k\pi \pm \beta$, $k \in Z$ where $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$.

Thus, if $c^2 \leq a^2 + b^2$, the general solution of $a\cos x + b\sin x = c$ is $x = 2k\pi + \alpha \pm \beta$, $k \in Z$, where $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$ and $\cos\beta = \frac{c}{r}$, $\beta \in [0, \pi]$, $r = \sqrt{a^2 + b^2}$.

If $c^2 > a^2 + b^2$, the equation has no solution. In this case the solution set is $\emptyset$.

Example 6 : Solve : $\sqrt{3}\cos x + \sin x = \sqrt{2}$

Solution : Method 1 : Here $a = \sqrt{3}$, $b = 1$, $c = \sqrt{2}$.

$\therefore \quad r^2 = a^2 + b^2 = 3 + 1 = 4$

Hence, $r = 2$. Here $c^2 \leq a^2 + b^2$. So the given equation has a non-empty solution.

$a = r\cos\alpha$ and $b = r\sin\alpha$ gives $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$. Therefore $\alpha = \frac{\pi}{6}$

Now, $\cos\beta = \frac{c}{r} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore \quad \beta = \frac{\pi}{4}$

Hence, required solution set is $\{2k\pi + \alpha \pm \beta \mid k \in Z\} = \left\{2k\pi + \frac{\pi}{6} \pm \frac{\pi}{4} \;\middle|\; k \in Z\right\}$.

Method 2 : $\sqrt{3}cosx + sinx = \sqrt{2}$

$\therefore \quad \frac{\sqrt{3}}{2}cosx + \frac{1}{2}sinx = \frac{1}{\sqrt{2}}$

$\therefore \quad cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} = cos\left(\frac{\pi}{4}\right)$

$\therefore \quad x - \frac{\pi}{6} = 2k\pi \pm \frac{\pi}{4}, k \in Z$

Hence, the required solution set is $\left\{2k\pi + \frac{\pi}{6} \pm \frac{\pi}{4} \mid k \in Z\right\}$.

$\therefore$ The required solution set is $\left\{2k\pi + \frac{5\pi}{12} \mid k \in Z\right\} \cup \left\{2k\pi - \frac{\pi}{12} \mid k \in Z\right\}$.

Example 7 : Solve : $3cos\theta + 4sin\theta = 6$

Solution : Here $a = 3$, $b = 4$, $c = 6$.

$\therefore \quad r^2 = a^2 + b^2 = 25$. $c^2 = 36$. So, $c^2 > a^2 + b^2$

Hence, the solution set is $\emptyset$.

Exercise 6.1

Solve the following equations :

1. $2cos2\theta + \sqrt{2} = 0$
2. $2cos^2\theta + \sqrt{3}cos\theta = 0$
3. $2cos\theta + sec\theta = 3$
4. $4sin^2\theta - 8cos\theta + 1 = 0$
5. $\sqrt{2}\,cosec3\theta - 2 = 0$
6. $2sin^2\theta - sin\theta = 0$
7. $2sin\theta + cosec\theta = 3$
8. $sin2\theta + cos\theta = 0$
9. $sin7\theta = sin\theta + sin3\theta$
10. $cos^2\theta - cos\theta = 0$
11. $tan2\theta - \sqrt{3} = 0$
12. $\sqrt{3}cot\theta - cot^2\theta = 0$
13. $tan^2\theta - (\sqrt{3} + 1)tan\theta + \sqrt{3} = 0$
14. $cos\theta + sin\theta = 1$
15. $\sqrt{3}sin\theta - cos\theta = \sqrt{2}$
16. $2cos\theta + sin\theta = 3$
17. $3 - cot^2 5\theta = 0$
18. $cosec^2 2\theta - 2 = 0$
19. $\sqrt{2} + sec4\theta = 0$
20. $tan3\theta + cot\theta = 0$

*

6.5 Properties of a triangle

The literal meaning of the word trigonometry is "the science of measurement of (the parts of) a triangle." A triangle has three angles and three sides. Measures of angles and sides are not independent of each other. In this article we shall get the exact relationship between the parts of a triangle.

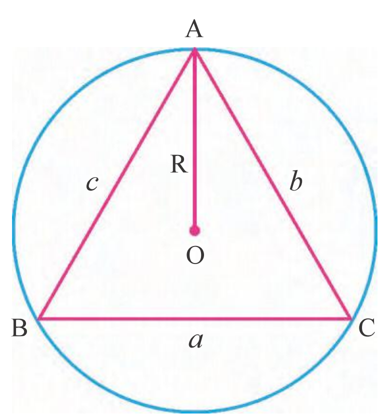

Figure 6.7

We will use following notation in relation to a triangle :

$m\angle BAC = A$, $m\angle ABC = B$, $m\angle BCA = C$

$A + B + C = \pi$

(A, B, C are taken in radian measures.)

$AB = c$, $BC = a$, $CA = b$

The radius of the circumcircle of the triangle, that is, circumradius = R

***sine* Rule :**

In ΔABC we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

We shall prove here that $\frac{a}{\sin A} = 2R$. The other two can be proved similarly.

There are three possibilities for A :

(1) $0 < A < \frac{\pi}{2}$ (2) $A = \frac{\pi}{2}$ (3) $\frac{\pi}{2} < A < \pi$

Case 1 : $0 < A < \frac{\pi}{2}$

Suppose O is the circumcentre of ΔABC. Let $\overrightarrow{BO}$ intersect the circumcircle at D. Here BD = 2OB = 2R and

$D = m\angle BDC = m\angle CAB = A$ **(i)**

(Angles in the same segment)

Now in ΔBCD, $m\angle BCD = \frac{\pi}{2}$ **(Angle in a semicircle)**

$\therefore \sin D = \frac{BC}{BD} = \frac{a}{2R}$

$\therefore \sin A = \frac{a}{2R}$ **(by (i))**

$\therefore \frac{a}{\sin A} = 2R$

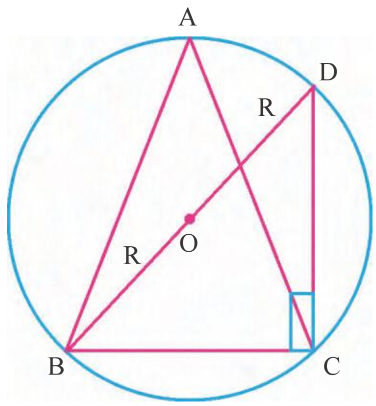

Figure 6.8

Case 2 : ΔABC is right angled and $A = \frac{\pi}{2}$

$\therefore \overline{BC}$ is a diameter of the circumcircle.

$\therefore BC = 2R$

Now, $a = BC = 2R = 2R\sin\frac{\pi}{2} = 2R\sin A$

$\therefore \frac{a}{\sin A} = 2R$

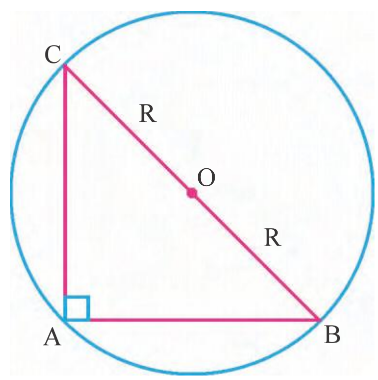

Figure 6.9

Case 3 : $\frac{\pi}{2} < A < \pi$

As $\angle BAC$ is obtuse, so vertex A is on the minor arc BC. Now take any Point A' on the major arc BC.

Here, $m\angle BA'C = (\pi - A) < \frac{\pi}{2}$ $\left(\frac{\pi}{2} < A < \pi\right)$

$\therefore$ By case (1) applied to $\Delta BA'C$, we get

$BC = a = 2R\sin A' = 2R\sin(\pi - A) = 2R\sin A$

$\therefore \frac{a}{\sin A} = 2R$

Thus, in each case, $\frac{a}{\sin A} = 2R$

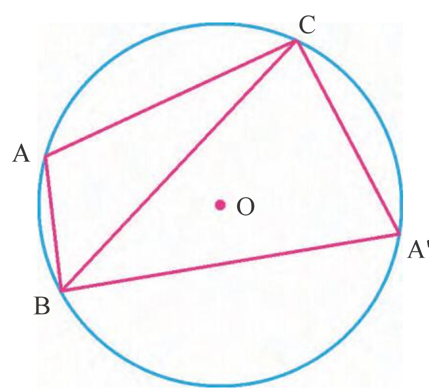

Figure 6.10

Similarly, we can prove that $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

***cosine* Rule :**

In ΔABC, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \; \cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We shall prove that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

As shown in the figure 6.11, without loss of generality we take vertex A as the origin and $\overrightarrow{AB}$ in the positive direction of the X-axis. Since AB = c, the coordinates of B are $(c, 0)$. Now AC = b and $m\angle CAB$ = A. So vertex C is $(b\cos A, b\sin A)$.

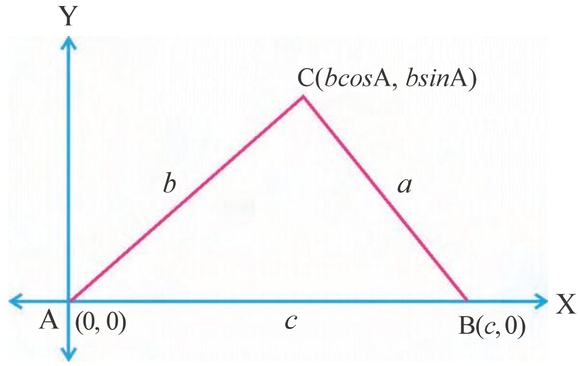

Figure 6.11

Now, a = BC

$$\therefore \quad a^2 = BC^2$$

$$= (b\cos A - c)^2 + (b\sin A - 0)^2$$

$$= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$

$$= b^2(\cos^2 A + \sin^2 A) - 2bc\cos A + c^2$$

$$\therefore \quad a^2 = b^2 - 2bc\cos A + c^2$$

$$\therefore \quad 2bc\cos A = b^2 + c^2 - a^2$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

In the same way, we can prove the results,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note :

(1) The above proof will not change even if $\angle BAC$ is a right angle or an obtuse angle.

(2) If the lengths of the three sides of a triangle are known, we can find the measure of all the angles using cosine rule. Similarly, if two sides and the included angle are given, then by *cosine* rule we can find the remaining sides and remaining angles.

Important Formula :

We shall obtain an important result by the use of sine and cosine rules.

Projection Formula :

$$a = b\cos C + c\cos B, \; b = c\cos A + a\cos C, \; c = a\cos B + b\cos A$$

We shall prove $a = b\cos C + c\cos B$

We prove the result using cosine rule. (Try to prove it using sine rule)

$$b\cos C + c\cos B = b\frac{a^2+b^2-c^2}{2ab} + c\frac{c^2+a^2-b^2}{2ca}$$

$$= \frac{a^2+b^2-c^2}{2a} + \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{a^2+b^2-c^2+c^2+a^2-b^2}{2a} = \frac{2a^2}{2a} = a$$

Thus, $a = b\cos C + c\cos B$

Similarly, the other two projection formulae can also be proved.

Example 8 : For ΔABC prove that,

(1) $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

(2) $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right) = 0$

Solution : (1)

$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$

$$= a\left(\frac{b}{2R} - \frac{c}{2R}\right) + b\left(\frac{c}{2R} - \frac{a}{2R}\right) + c\left(\frac{a}{2R} - \frac{b}{2R}\right)$$

$$= \frac{a(b-c)+b(c-a)+c(a-b)}{2R} = 0$$

(2) $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) = a\sin\left(\frac{\pi-(B+C)}{2}\right)\sin\left(\frac{B-C}{2}\right)$ **(A + B + C = π)**

$$= a\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= \frac{a}{2}(\sin B - \sin C)$$

$$= \frac{a}{2}\left(\frac{b}{2R} - \frac{c}{2R}\right)$$

$$= \frac{1}{4R}(ab - ac) \quad \textbf{(i)}$$

Similarly, $b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) = \frac{1}{4R}(bc - ab)$ **(ii)**

$c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right) = \frac{1}{4R}(ac - bc)$ **(iii)**

Adding (i), (ii) and (iii) we get

L.H.S. $= a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)$

$= \frac{1}{4R}(ab - ac + bc - ab + ac - bc) = 0 =$ R.H.S.

Example 9 : In any ΔABC, prove that

(1) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(2) $\frac{\tan C}{\tan A} = \frac{b^2+c^2-a^2}{a^2+b^2-c^2}$

Solution :

(1) L.H.S. $= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$$= \frac{b^2+c^2-a^2}{2bc} \times \frac{1}{a} + \frac{c^2+a^2-b^2}{2ca} \times \frac{1}{b} + \frac{a^2+b^2-c^2}{2ab} \times \frac{1}{c} \quad \textbf{(\textit{cosine} rule)}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc} = \text{R.H.S.}$$

(2) L.H.S. $= \frac{\tan C}{\tan A} = \frac{\sin C \cos A}{\cos C \sin A}$

$$= \frac{\frac{c}{2R}\left(\frac{b^2+c^2-a^2}{2bc}\right)}{\frac{a}{2R}\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$= \frac{b^2+c^2-a^2}{a^2+b^2-c^2} = \text{R.H.S.}$$

Example 10 : In ΔABC, prove that

$(a + b)\cos C + (b + c)\cos A + (c + a)\cos B = a + b + c$

Solution : L.H.S. $= (a + b)\cos C + (b + c)\cos A + (c + a)\cos B$

$= a\cos C + b\cos C + b\cos A + c\cos A + c\cos B + a\cos B$

$= b\cos C + c\cos B + c\cos A + a\cos C + a\cos B + b\cos A$

$= a + b + c =$ R.H.S.

Exercise 6.2

For ΔABC, prove **(1 to 9)** :

1. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$
2. $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$
3. $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$
4. $a^3\sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B) = 0$
5. $a\sin\left(\frac{A}{2}+C\right) = (b + c)\sin\frac{A}{2}$
6. $a\cos\left(\frac{B-C}{2}\right) = (b + c)\sin\frac{A}{2}$
7. $\sin\left(\frac{A-B}{2}\right) = \frac{a-b}{c}\cos\frac{C}{2}$
8. $\tan\left(\frac{A}{2}+B\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$
9. $\frac{1+\cos A \cos(B-C)}{1+\cos C \cos(A-B)} = \frac{b^2+c^2}{b^2+a^2}$
10. Prove : $\sin^2 A + \sin^2 B = \sin^2 C \Rightarrow \Delta$ABC is right angled at C.

11. Prove : $(a^2 + b^2) sin\,(A - B) = (a^2 - b^2) sin\,(A + B) \Rightarrow \Delta ABC$ is either isosceles or right angled.

12. Prove : $(b^2 - c^2) cot A + (c^2 - a^2) cot B + (a^2 - b^2) cot C = 0$

13. Prove : $\left(\frac{b^2 - c^2}{a^2}\right) sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) sin 2B + \left(\frac{a^2 - b^2}{c^2}\right) sin 2C = 0$

14. Prove : $2\left(a sin^2 \frac{C}{2} + c sin^2 \frac{A}{2}\right) = c + a - b$

15. Prove : $4\left(bc\, cos^2 \frac{A}{2} + ca\, cos^2 \frac{B}{2} + ab\, cos^2 \frac{C}{2}\right) = (a + b + c)^2$

16. Show that a triangle having sides equal to 3, 5, 7 is an obtuse angled triangle and determine the measure of the obtuse angle.

17. If the angles of a triangle are in the ratio 1 : 2 : 3, find the ratio of sides opposite to these angles.

18. The measures of angles A, B, C of a ΔABC are in A.P. and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, find A.

19. If in a ΔABC, $\frac{sin\,A}{sin\,C} = \frac{sin\,(A - B)}{sin\,(B - C)}$, prove that a^2, b^2, c^2 are in A.P.

20. In a ΔABC, $a = 2b$ and $|A - B| = \frac{\pi}{3}$. Find C.

*

Miscellaneous Problems :

Example 11 : Solve $sin 3\alpha = 4 sin\alpha\, sin(x + \alpha)\, sin(x - \alpha)$, where $\alpha \neq k\pi$, $k \in Z$

Solution : $sin 3\alpha = 4 sin\alpha\, sin(x + \alpha)\, sin(x - \alpha)$, where $\alpha \neq k\pi$, $k \in Z$

$\therefore \quad sin 3\alpha = 4 sin\alpha\, (sin^2 x - sin^2\alpha)$

$\therefore \quad 3 sin\alpha - 4 sin^3\alpha = 4 sin\alpha\, sin^2 x - 4 sin^3\alpha$

$\therefore \quad 3 sin\alpha = 4 sin\alpha\, sin^2 x$

$\therefore \quad sin^2 x = \frac{3}{4}$ **(Since $\alpha \neq k\pi$, $sin\alpha \neq 0$)**

$\therefore \quad sin x = \pm\frac{\sqrt{3}}{2} = sin\left(\pm\frac{\pi}{3}\right)$

$\therefore \quad x = k\pi + (-1)^k \frac{\pi}{3}, k \in Z$ or $x = k\pi + (-1)^k \left(-\frac{\pi}{3}\right), k \in Z$

$\therefore \quad x = k\pi \pm \frac{\pi}{3}, k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{3} \,\middle|\, k \in Z\right\}$.

Example 12 : Solve : $tan\left(\frac{\pi}{4} + \theta\right) + tan\left(\frac{\pi}{4} - \theta\right) = 4$

Solution : $tan\left(\frac{\pi}{4} + \theta\right) + tan\left(\frac{\pi}{4} - \theta\right) = 4$

$\therefore \quad \frac{1 + tan\,\theta}{1 - tan\,\theta} + \frac{1 - tan\,\theta}{1 + tan\,\theta} = 4$

$\therefore \quad \frac{(1 + tan\theta)^2 + (1 - tan\theta)^2}{(1 - tan\theta)(1 + tan\theta)} = 4$

$\therefore \quad \frac{2+2tan^2\theta}{1-tan^2\theta} = 4$

$\therefore \quad 2 + 2tan^2\theta = 4 - 4tan^2\theta$

$\therefore \quad 6tan^2\theta = 2$

$\therefore \quad tan^2\theta = \frac{1}{3}$

$\therefore \quad tan\theta = \pm\frac{1}{\sqrt{3}} = tan\left(\pm\frac{\pi}{6}\right)$

$\therefore \quad \theta = k\pi \pm \frac{\pi}{6},\ k \in Z$

Hence, the required solution set is $\left\{k\pi \pm \frac{\pi}{6} \mid k \in Z\right\}$.

Example 13 : If $\frac{sin\,A}{4} = \frac{sin\,B}{5} = \frac{sin\,C}{6}$, show that $\frac{cos\,A}{12} = \frac{cos\,B}{9} = \frac{cos\,C}{2}$ and hence find the value of $cosA + cosB + cosC$.

Solution : We have $\frac{sin\,A}{4} = \frac{sin\,B}{5} = \frac{sin\,C}{6}$

$\therefore \quad \frac{\frac{a}{2R}}{4} = \frac{\frac{b}{2R}}{5} = \frac{\frac{c}{2R}}{6}$

$\therefore \quad \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say), where $k > 0$

$\therefore \quad a = 4k,\ b = 5k,\ c = 6k$

Now, $cosA = \frac{b^2+c^2-a^2}{2bc} = \frac{25k^2+36k^2-16k^2}{2\cdot 5k \cdot 6k} = \frac{45k^2}{60k^2} = \frac{3}{4}$

$\therefore \quad \frac{cos\,A}{12} = \frac{1}{16}$

Similarly, $\frac{cos\,B}{9} = \frac{1}{16}$ and $\frac{cos\,C}{2} = \frac{1}{16}$

Hence, $\frac{cos\,A}{12} = \frac{cos\,B}{9} = \frac{cos\,C}{2}$

Also, $cosA + cosB + cosC = \frac{12}{16} + \frac{9}{16} + \frac{2}{16} = \frac{23}{16}$

Exercise 6

Solve **(1 to 10)** :

1. $2(sec^2\theta + sin^2\theta) = 5$
2. $2 - cosx = 2tan\frac{x}{2}$
3. $4sin\theta\ sin2\theta\ sin4\theta = sin3\theta$
4. $sin^2\theta - cos\theta = \frac{1}{4}$
5. $\sqrt{3}tan3\theta + \sqrt{3}tan2\theta + tan3\theta\ tan2\theta = 1$
6. $cosecx = 1 + cotx$
7. $sin^8x + cos^8x = \frac{17}{32}$

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8. $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

9. $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

10. $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

For ΔABC, prove **(11 to 14)** :

11. $a\cos A + b\cos B + c\cos C = 4R\sin A \sin B \sin C = \frac{abc}{2R^2}$

12. $a(\cos C - \cos B) = 2(b - c)\cos^2\frac{A}{2}$

13. $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc$

14. $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} \Rightarrow \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

15. Prove : *cosine* rule using sine rule.

16. Prove : $(a - b)^2\cos^2\frac{C}{2} + (a + b)^2\sin^2\frac{C}{2} = c^2$

17. Prove : $abc(\cot A + \cot B + \cot C) = R(a^2 + b^2 + c^2)$

18. If length of the sides of a triangle are 4, 5 and 6, prove that the largest measure of an angle is twice that of the angle with smallest measure.

19. If length of the sides of a triangle are m, n, $\sqrt{m^2 + mn + n^2}$, prove that the largest measure of an angle of the triangle is $\frac{2\pi}{3}$.

20. If length of the two sides of a triangle are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and if the included angle between them has measure $\frac{\pi}{3}$, then show that the perimeter of the triangle is $2\sqrt{3} + \sqrt{6}$.

21. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is ... ☐

(a) $\emptyset$ (b) $\left\{\frac{\pi}{4}\right\}$

(c) $\left\{k\pi + \frac{\pi}{4} \mid k \in Z\right\}$ (d) $\left\{2k\pi + \frac{\pi}{4} \mid k \in Z\right\}$

(2) Number of ordered pairs (a, x) satisfying the equation $\sec^2(a + 2)x + a^2 - 1 = 0$; $-\pi < x < \pi$ is ... ☐

(a) 2 (b) 1 (c) 3 (d) infinite

(3) The general solution of the equation $\sin^{50}x - \cos^{50}x = 1$ is ... ☐

(a) $2k\pi + \frac{\pi}{2}$, $k \in Z$ (b) $2k\pi + \frac{\pi}{3}$, $k \in Z$

(c) $k\pi + \frac{\pi}{3}$, $k \in Z$ (d) $k\pi + \frac{\pi}{2}$, $k \in Z$

(4) The number of solutions of the equation $3\sin^2x - 7\sin x + 2 = 0$, in the interval $[0, 5\pi]$ is ... ☐

(a) 0 (b) 5 (c) 6 (d) 10

(5) The real roots of the equation $cos^7x + sin^4x = 1$, in the interval $(-\pi, \pi)$, are ...

(a) $0, \frac{\pi}{3}, -\frac{\pi}{3}$ (b) $0, \frac{\pi}{4}, -\frac{\pi}{4}$ (c) $0, \frac{\pi}{2}, -\frac{\pi}{2}$ (d) $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$

(6) The number of points of intersection of $2y = 1$ and $y = sinx$, $-2\pi < x \leq 2\pi$ is ...

(a) 2 (b) 4 (c) 3 (d) 1

(7) The general solution of $sin\theta + cos\theta = 2$ is ...

(a) $k\pi, k \in Z$ (b) $2k\pi + \frac{\pi}{2}, k \in Z$

(c) $\emptyset$ (d) $(2k + 1)\frac{\pi}{2}, k \in Z$

(8) The general solution of $cos2\theta = cos^2\theta - sin^2\theta$ is ...

(a) R (b) $k\pi, k \in Z$

(c) $\emptyset$ (d) $(2k + 1)\frac{\pi}{2}, k \in Z$

(9) In a ΔABC, if $\frac{cos A}{a} = \frac{cos B}{b} = \frac{cos C}{c}$ and $a = 2$, then the area of the triangle is ...

(a) 1 (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

(10) In a ΔABC, $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, numbers of such triangles are ...

(a) 1 (b) 0 (c) 2 (d) infinite

(11) The perimeter of ΔABC is 6 times the arithmetic mean of the sines of its angles. If a is 1, then A is ...

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(12) In a ΔABC, $a = 2b$ and $A = 3B$, then A = ...

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

(13) If A, B, C in a ΔABC are in A.P. and the sides a, b, c are in G.P., then a^2, b^2, c^2 are in ...

(a) G.P. (b) A.P.

(c) $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in A.P. (d) no relation

(14) In ΔABC, $A = \frac{\pi}{4}$, $c = \frac{\pi}{3}$, then $a + c\sqrt{2}$ = ...

(a) b (b) $\sqrt{3}b$ (c) $\sqrt{2}b$ (d) $2b$

(15) In a ΔABC, $2ac\sin\frac{1}{2}(A - B + C)$ = ...

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 + a^2$ (d) $c^2 - a^2 - b^2$

*

Summary

We studied following points in this chapter :

1. $sin\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

2. $cos\theta = 0 \Leftrightarrow \theta = (2k+1)\frac{\pi}{2}$ $k \in Z$

3. $tan\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

4. Solution set of $sin\theta = a, -1 \le a \le 1$ is given by $\{k\pi + (-1)^k\alpha \mid k \in Z\}$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin\theta = a = sin\alpha$.

5. Solution set of $cos\theta = a, -1 \le a \le 1$ is given by $\{2k\pi \pm \alpha \mid k \in Z\}$, where $\alpha \in [0, \pi]$ and $cos\theta = a = cos\alpha$.

6. Solution set of $tan\theta = a, a \in R$ is given by $\{k\pi + \alpha \mid k \in Z\}$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\theta = a = tan\alpha$.

7. If $c^2 \le a^2 + b^2$, the general solution of $acosx + bsinx = c$ is

   $x = 2k\pi + \alpha \pm \beta, k \in Z$, where $\alpha \in [0, 2\pi)$ such that $a = rcos\alpha$ and $b = rsin\alpha$ and $cos\beta = \frac{c}{r}, \beta \in [0, \pi], r = \sqrt{a^2+b^2}$

   If $c^2 > a^2 + b^2$, the solution set is $\emptyset$.

8. The *sine* rule is : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

9. The *cosine* rule is :

   $cosA = \frac{b^2+c^2-a^2}{2bc}$, $cosB = \frac{c^2+a^2-b^2}{2ca}$ and $cosC = \frac{a^2+b^2-c^2}{2ab}$

10. Projection Formula :

    $a = bcosC + ccosB, b = ccosA + acosC, c = acosB + bcosA$

Aryabhata gave an accurate approximation for π. He wrote in the *Aryabhatiya* the following :

Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.

This gives $\pi = \frac{62832}{20000} = 3.1416$ which is a surprisingly accurate value. In fact $\pi = 3.14159265$ correct to 8 places.

He gave a table of *sines* calculating the approximate values at intervals of $90^\circ/24 = 3^\circ 45'$. In order to do this he used a formula for $sin(n+1)x - sin\, nx$ in terms of $sin\, nx$ and $sin(n-1)x$. He also introduced the versine (versin = 1 − *cosine*) into trigonometry.

Aryabhata gives the radius of the planetary orbits in terms of the radius of the Earth/Sun orbit as essentially their periods of rotation around the Sun. He believes that the Moon and planets shine by reflected sunlight. Incredibly he believes that the orbits of the planets are ellipses. He correctly explains the causes of eclipses of the Sun and the Moon. The Indian belief up to that time was that eclipses were caused by a demon called Rahu. His value for the length of the year at 365 days 6 hours 12 minutes 30 seconds is an overestimate since the true value is less than 365 days 6 hours.

Chapter 7

SEQUENCES AND SERIES

7.1 Introduction

The word 'sequence' used in the English language and in mathematics has the same sense. That is the sequence emphasises on the order of occurrence. When we talk about a sequence of events, it clearly indicates the order of occurrence of the events. For example, India won the ICC World Cup-2011. As we know that Indian team played a sequence of matches and won certain number of them and finally won the final match. Here, we can see the sequence of events taking place in a definite order. Similarly, in mathematics, when we talk about a sequence of numbers, it clearly indicates the first number, the second number, the third number and so on. Historically, ***Aryabhata*** was the first mathematician to give the formula for the sum of the squares of first n natural numbers, the sum of cubes of first n natural numbers etc. This is found in his work ***Aryabhatiyam***. Such kind of work is also observed in the work of famous Italian mathematician ***Fibonacci* (1175-1250). The numbers of Fibonacci sequence are also known as Fibonacci numbers** and they are applied in many fields of knowledge.

Now, let us discuss about sequences mathematically. Observe the sequence of even numbers 2, 4, 6, ..., we can easily see that the sequence is 2(1), 2(2), 2(3), ..., so we can generalise that nth even number must be $2(n)$. So we can think of a function $f : N \to R$, $f(n) = 2n$. Similarly the sequence 1, 4, 9, 16, ... can be written as $f : N \to R$, $f(n) = n^2$. So we define sequence as a function whose domain is N or $\{1, 2, 3, ..., n\}$.

Sequence : A function $f : N \to R$ or $f : \{1, 2, 3, ..., n\} \to R$ is called a sequence. $f : \{1, 2, 3, ..., n\} \to R$ is called a finite sequence. Here $n \in N$.

For instance, $f : N \to R$, $f(n) = 3n - 1$.

Taking $n = 1, 2, 3, ...$ we get $f(1) = 2$, $f(2) = 5$, $f(3) = 8$, ... 2, 5, 8, ... are called respectively first, the second, the third, ... term of the sequence. $f(n)$ is called the nth term or a general term.

$f(n)$ is also denoted by a_n or t_n or T_n or u_n etc.

$\{f(n)\}$ or $\{a_n\}$ or $\{t_n\}$ indicates the sequence having the nth term as $f(n)$ or a_n or t_n respectively.

According as codomain of the function is N, Z or R, the sequence is called a sequence of natural numbers, a sequence of integers or sequence of real numbers respectively.

The nth term of a sequence may be in the form of formula, but it is not necessary that every sequence is defined by means of some formula. For example, the sequence of prime numbers 2, 3, 5, 7, 11, 13, ... There is no formula to get the nth prime number, so the sequence is not expressed by defining a rule.

Let us see one interesting sequence, $f(n) = (n - 1) \cdot (n - 2) \cdot (n - 3) + (2n - 1)$. Obviously $f(1) = 1$, $f(2) = 3$, $f(3) = 5$. We may be tempted to say that $f(4) = 7$, but it is not so, it is 13. Thus **by using a few terms only we can not guess the general term of a sequence.**

Example 1 : Find first five terms of the sequence : $f : N \rightarrow R$, $f(n) = 2n^2 - 4$.

Solution : Here $f(n) = 2n^2 - 4$

$\therefore$ $f(1) = 2(1)^2 - 4 = -2$, $f(2) = 2(2)^2 - 4 = 4$,

$f(3) = 2(3)^2 - 4 = 14$, $f(4) = 2(4)^2 - 4 = 28$, $f(5) = 2(5)^2 - 4 = 46$.

Thus, the first five terms are -2, 4, 14, 28 and 46.

Example 2 : For $f : N \rightarrow R$, $f(n) = n(-1)^n$, find the difference between 17th and 16th terms.

Solution : Here $f(n) = n(-1)^n$

$\therefore$ $f(16) = 16(-1)^{16} = 16$ and $f(17) = 17(-1)^{17} = -17$

Now, $f(17) - f(16) = (-17) - (16) = -33$

$\therefore$ The difference $= | f(17) - f(16) | = 33$

Example 3 : $f : N \rightarrow R$, $f(n) = 8 - n^3$. Find the first four terms of the sequence.

Solution : $f(1) = 8 - (1)^3 = 7$, $f(2) = 8 - (2)^3 = 0$, $f(3) = 8 - (3)^3 = -19$ and

$f(4) = 8 - (4)^3 = -56$.

$\therefore$ The first four terms are 7, 0, -19 and -56.

Example 4 : Let the sequence $f : N \rightarrow R$ be defined by $f(1) = 1$ and $f(n) = f(n - 1) - 1$ for $n \geq 2$. Find the first five terms of the sequence.

Solution : Here $f(1) = 1$

Now $f(n) = f(n - 1) - 1$, for $n \geq 2$

$\therefore$ $f(2) = f(2 - 1) - 1 = f(1) - 1 = 1 - 1 = 0$

$f(3) = f(2) - 1 = -1$, $f(4) = f(3) - 1 = -2$, $f(5) = f(4) - 1 = -3$

$\therefore$ The first five terms are 1, 0, -1, -2 and -3.

Example 5 : If $f : N \rightarrow R$, $f(n) = \cos\frac{n\pi}{2}$, find the first six terms of the sequence f.

Solution : Here $f(n) = \cos\frac{n\pi}{2}$

$\therefore \quad f(1) = cos\frac{\pi}{2} = 0, \quad f(2) = cos\pi = -1, \quad f(3) = cos\frac{3\pi}{2} = 0$

$f(4) = cos2\pi = 1, \quad f(5) = cos\frac{5\pi}{2} = 0, \quad f(6) = cos3\pi = -1$

So the first six terms are 0, −1, 0, 1, 0 and −1.

Example 6 : What will be the 10th term of the sequence defined by

$f(n) = (n - 1)(n + 2)(n - 3)$?

Solution : Here, $f(n) = (n - 1)(n + 2)(n - 3)$

$\therefore \quad f(10) = (10 - 1)(10 + 2)(10 - 3) = 9 \cdot 12 \cdot 7 = 756$

$\therefore$ Hence the 10th term is 756.

7.2 Series :

Let $a_1, a_2, a_3, ..., a_n, ...$ be a given sequence. Let us think of the sequence formed by using the terms of the given sequence as follows :

$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, ... a_1 + a_2 + a_3 + ... + a_n, ...$ Such a new sequence is called a **series** derived from sequence $\{a_n\}$.

Usually, S_n denotes the sum of the first n terms of a sequence. So the sequence $S_1, S_2, S_3, ..., S_n$ becomes the series corresponding to the given original sequence.

Hence **every series is a sequence and *n*th term of the series is the sum of the first *n* terms of its corresponding sequence.**

For instance, take the sequence of odd natural numbers. i.e. 1, 3, 5, 7, 9, ...

$\therefore \quad S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 3 = 4$

$S_3 = a_1 + a_2 + a_3 = 1 + 3 + 5 = 9$

$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 3 + 5 + 7 = 16$

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We get the sequence 1, 4, 9, 16, ... which is the sequence of squares of natural numbers. i.e. $S_n = n^2$. It is called the **series** derived from the sequence $f(n) = 2n - 1$.

Let us obtain nth term a_n of a sequence from the sum of first n terms S_n of the same sequence.

We can derive the formula for a_n as follows, if we are given the formula of S_n :

$S_1 = a_1$

$S_2 = a_1 + a_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$

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$S_n = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n = S_{n-1} + a_n$

We observe that $S_n = S_{n-1} + a_n$ for $n = 2, 3, 4, ...$

$\therefore \quad S_n - S_{n-1} = a_n \quad \forall n \geq 2$ and $S_1 = a_1$

This gives the formula for a_n, when the sum of first n terms S_n is given.

Example 7 : For the sequence $\{a_n\}$, $S_n = n^3 - 2n$, find the first four terms and 8th term of $\{a_n\}$.

Solution : $S_n = n^3 - 2n$

$\therefore \quad S_1 = (1)^3 - 2(1) = 1 - 2 = -1, \qquad S_2 = (2)^3 - 2(2) = 8 - 4 = 4,$

$S_3 = (3)^3 - 2(3) = 27 - 6 = 21, \qquad S_4 = (4)^3 - 2(4) = 64 - 8 = 56$

So, $a_1 = S_1 = -1, \quad a_2 = S_2 - S_1 = 4 - (-1) = 5, \quad a_3 = S_3 - S_2 = 21 - 4 = 17,$

$a_4 = S_4 - S_3 = 56 - 21 = 35.$

$\therefore$ The first four terms of $\{a_n\}$ are -1, 5, 17 and 35.

The 8th term, $a_8 = S_8 - S_7$

$= [(8)^3 - 2(8)] - [(7)^3 - 2(7)]$

$= [512 - 16] - [343 - 14] = 167$

Example 8 : From the formula for the series, $S_n = 4^n - 1$, obtain the formula for the corresponding sequence.

Solution : $a_1 = S_1 = 4^1 - 1 = 3, \quad S_n = 4^n - 1$

$\therefore \quad S_{n-1} = 4^{n-1} - 1$

$a_n = S_n - S_{n-1}, \forall n \geq 2 = (4^n - 1) - (4^{n-1} - 1)$

$= 4^n - 4^{n-1}$

$= 4^{n-1}(4 - 1)$

$= 3 \cdot 4^{n-1}, \quad \forall n \geq 2$

Taking $n = 1$, $3 \cdot 4^{1-1} = 3 = a_1$

$\therefore \quad a_n = 3 \cdot 4^{n-1}, \forall n \geq 1$

Exercise 7.1

1. Write the first five terms of the following sequence :

   (1) $f(n) = 3n + 1$ (2) $f(n) = \frac{n-(-1)^n}{2}$ (3) $f(n) = n$th prime number

2. The Fibonacci sequence is defined by,

   $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$, find a_3, a_4, a_5, a_6.

3. Obtain a_2, a_3, a_4 for the following sequences :

   (1) $a_1 = -3$ and $a_n = 2a_{n-1} + 1, \forall n > 1.$

   (2) $a_1 = \frac{1}{2}$ and $a_n = 3a_{n-1} + (-1)^n, \forall n \geq 2.$

4. Find the first three terms and tenth term of the sequence $\{a_n\}$:

   (1) $S_n = n^2 - 1$ (2) $S_n = \frac{n(n+1)}{2}$

5. From the following formula for the series S_n, obtain the formula for corresponding sequence :

   (1) $S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1, a \neq 0$ (2) $S_n = 4\{1 - (-3)^{n-1}\}$

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7.3 Arithmetic Progression (A.P.)

Observe the sequence 1, 3, 5, 7, ... Here each term (after the first) is obtained by adding the same number 2 to its preceding term. The difference between two consecutive terms is a non-zero constant. Such a sequence is called an **arithmetic progression**. We define it as follows :

Arithmetic Progression : A sequence $f : N \to R$, $f(n) = an + b$, $a, b \in R$, $a \neq 0$ is called an arithmetic progression (A.P.). Thus an A.P. is a linear function of n, where $n \in N$.

For example, the sequence $f(n) = 3n - 4$, $n \in N$ is an A.P., its terms are $-1, 2, 5, 8, 11, ...$ Here difference between any two consecutive terms is 3, a constant.

In the above discussion, we can observe that the difference between any two consecutive terms is a non-zero constant and f is a linear function of n, $n \in N$. Now we shall combine these two properties in the following theorem.

Theorem 1 : Difference between any two successive terms in an A.P. is a non-zero constant.

Proof : Suppose $\{f(n)\} = \{an + b\}$ is an A.P., $a, b \in R$, $a \neq 0$.

For any $k \in N$, $f(k + 1) - f(k) = [a(k + 1) + b] - (ak + b)$

$$= ak + a + b - ak - b$$

$$= a, \text{ a non-zero constant}$$

Thus, the difference of between any two successive terms $f(k + 1)$ and $f(k)$ is a non-zero constant. We call it the **common difference** of the A.P. and usually denote it by 'd'. Now, onwards the common difference will be termed as difference. Here we take $d = f(k + 1) - f(k)$ which may be positive or negative.

The converse of above theorem is also true. Suppose the first term of a sequence $\{f(n)\}$ is 'a' and the difference $f(k + 1) - f(k) = d$, $d \neq 0$ for all $k \in N$. Then it is clear that the sequence is an A.P. In general we conclude that **nth term of the A.P. as $f(n) = a + (n - 1)d$, $d \neq 0$** and it is a linear function of n. We shall prove our conclusion by the method of mathematical induction.

Theorem 2 : If the first term of a sequence $\{f(n)\}$ is a and if the difference of two successive terms is $d \neq 0$, then $f(n) = a + (n - 1)d$, $\forall n \in N$ and so it is an A.P.

Proof : Let the statement $P(n) : f(n) = a + (n - 1)d$, $\forall n \in N$

(1) For $n = 1$, $f(1) = a$, the first term and

$a + (n - 1)d = a + (1 - 1)d = a$

$\therefore$ P(1) is true.

(2) Let $P(k) : f(k) = a + (k - 1)d$ be true for some $k \in N$. **(i)**

Then we shall prove that $P(k + 1)$ is true.

$f(k + 1) = f(k) + d$ **($f(k + 1) - f(k) = d$)**

$= [a + (k - 1)d] + d$ **(from (i))**

$\therefore$ $f(k + 1) = a + kd$

$= a + [(k + 1) - 1]d$

Thus $P(k)$ is true. $\Rightarrow$ $P(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Here, $f(n) = a + (n - 1)d = dn + (a - d)$ is a linear function of n (as $d \neq 0$), so f is an A.P.

We conclude from these two theorems that if 'a' is the first term and 'd' is the common difference of an A.P., then the A.P. can be written as $a, a + d, a + 2d, ..., a + (n - 1)d, ...$

Thus the formula for nth term of an A.P. is $f(n) = a + (n - 1)d$. a_n is also used for the last term of a finite A.P. having domain $\{1, 2, 3, ..., n\}$.

If we denote nth term by t_n, then $t_n = a + (n - 1)d$, where a is the first term and d is the common difference.

Note : a, b, c are consecutive terms in A.P. $\Leftrightarrow b - a = c - b$

$\Leftrightarrow 2b = a + c$

Example 9 : For an A.P. 3, 8, 13, 18, ... find the 17th and 40th terms.

Solution : $a = 3$, $d = 5$

nth term of the A.P. is $t_n = a + (n - 1)d$

$= 3 + (n - 1)5$

$= 5n - 2$

Taking $n = 17$, $t_{17} = 5(17) - 2 = 83$ and

taking $n = 40$, $t_{40} = 5(40) - 2 = 198$.

$\therefore$ 17th term is 83 and 40th term is 198.

Example 10 : Which term of the A.P. 3, 14, 25, 36, ... will be 121 less than its 37th term ?

Solution : Here $a = 3$, $d = 11$, given $m = 37$

mth term, $t_m = a + (m - 1)d$

$\therefore \quad t_{37} = 3 + (37 - 1)11$

$= 3 + 396 = 399$

Let t_n be the term 121 less then t_{37}.

$\therefore \quad t_n = t_{37} - 121 = 399 - 121 = 278$

$\therefore \quad a + (n - 1)d = 278$

$\therefore \quad 3 + (n - 1)11 = 278$

$\therefore \quad (n - 1)11 = 278 - 3 = 275$

$\therefore \quad n - 1 = 25$

$\therefore \quad n = 26$

Thus, the 26th term is 121 less than its 37th term.

Note : Order of the term 121 less is $\frac{121}{11} = 11$ less than 37th term (here $d = 11$).

So $37 - 11 = 26$th term is the required term.

Example 11 : If the 11th term of an A.P. is zero, then prove that its 31st term is double than the 21st term.

Solution : $t_n = a + (n - 1)d$

$\therefore \quad t_{11} = a + 10d$

$\therefore \quad 0 = a + 10d$ **(i)**

Now, $2 \cdot t_{21} = 2(a + 20d)$

$= 2a + 40d$

$= (a + 30d) + (a + 10d)$

$= t_{31} + 0 = t_{31}$ **(from (i))**

Thus, 31st term of the A.P. is double than the 21st term.

Example 12 : If the pth term of an A.P. is q and the qth term is p, $p \neq q$, then find the nth term of the A.P.

Solution : Here t_p is $a + (p - 1)d = q$ **(i)**

and t_q is $a + (q - 1)d = p$ **(ii)**

Solving (i) and (ii), we get

$(p - q)d = q - p$

$\therefore$ As $p \neq q$, $d = -1$ and $a = p + q - 1$

Now the nth term $t_n = a + (n - 1)d$

$= p + q - 1 + (n - 1)(-1)$

$= p + q - n$

Arithmetic Series :

The series corresponding to an A.P. is called an Arithmetic Series.

The nth term of the arithmetic series corresponding to the A.P.

$a, a + d, a + 2d, ..., a + (n - 1)d$ is

$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 1)d]$

Now we shall prove the expression for the sum of first n terms of the A.P.,

i.e. $S_n = \frac{n}{2}[2a + (n - 1)d]$ by the principle of mathematical induction.

Theorem 3 : If first term of an A.P. is a and d is the common difference, then the sum of first n terms is $S_n = \frac{n}{2}[2a + (n - 1)d], \forall n \in N$.

Proof : Let the statement $P(n) : S_n = \frac{n}{2}[2a + (n - 1)d], \forall n \in N$.

(1) For $n = 1$, $S_1 = \frac{1}{2}[2a + (1 - 1)d] = a$, i.e. the sum of the first term is the first term 'a' itself.

$\therefore$ P(1) is true.

(2) Let $P(k) : S_k = \frac{k}{2}[2a + (k - 1)d]$ be true for some $k \in N$. **(i)**

Let $n = k + 1$

$S_{k+1} = S_k + (k + 1)$th term

$= \frac{k}{2}[2a + (k - 1)d] + a + [(k + 1) - 1]d$ **(from (i))**

$= \frac{1}{2}[2ak + k(k - 1)d + 2a + 2kd]$

$= \frac{1}{2}[2a(k + 1) + k(k - 1 + 2)d]$

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$$= \frac{1}{2}[2a(k + 1) + kd(k + 1)]$$

$$= \frac{k+1}{2}[2a + \{(k + 1) - 1\}d]$$

Thus, P(k) is true. $\Rightarrow$ P($k + 1$) is true.

$\therefore$ By the principle of mathematical induction P(n) is true. $\forall n \in$ N.

Note : Formula for S_n of an A.P. of finite term is

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + \{a + (n - 1)d\}] = \frac{n}{2}(a + l)$$

where a is the first term and l is the last term, i.e. $l = t_n = a + (n - 1)d$.

Thus, the formula of S_n for A.P. $= \frac{\text{number of terms}}{2}$ [first term + last term]

Example 13 : Find the sum of the first fifteen terms of A.P. 15, 11, 7, 3, ...

Solution : Here $a = 15$, $d = 11 - 15 = -4$ and $n = 15$

Now, $S_n = \frac{n}{2}[2a + (n - 1)d]$

$$\therefore \quad S_{15} = \frac{15}{2}[2(15) + (15 - 1)(-4)]$$

$$= \frac{15}{2}[30 - 56] = \frac{15}{2}[-26] = -195$$

$\therefore$ The sum of the first 15 terms is -195.

Example 14 : The sum of n terms of two AP.s are in the ratio $(3n + 6) : (5n - 13)$, $\forall n \in$ N. Find the ratio of their 11th terms.

Solution : Suppose, the first term and common difference of one A.P. are a_1 and d_1 and the same for the other A.P. are a_2 and d_2 respectively.

According to the given condition,

$$\frac{\text{Sum of the first } n \text{ terms of first A.P.}}{\text{Sum of the first } n \text{ terms of second A.P.}} = \frac{3n+6}{5n-13}$$

$$\therefore \quad \frac{\frac{n}{2}[2a_1 + (n - 1)d_1]}{\frac{n}{2}[2a_2 + (n - 1)d_2]} = \frac{3n+6}{5n-13}$$

$$\therefore \quad \frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2} = \frac{3n+6}{5n-13} \quad \text{(i)}$$

$\therefore$ Let t_n and t_n' be the nth terms of given A.P.s.

$$\text{Now, } \frac{t_{11}}{t'_{11}} = \frac{a_1 + 10d_1}{a_2 + 10d_2}$$

$$= \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$$

$$= \frac{2a_1 + (21 - 1)d_1}{2a_2 + (21 - 1)d_2}$$

So, substituting $n = 21$ is (i), we have

$$\frac{t_{11}}{t'_{11}} = \frac{3(21)+6}{5(21)-13} = \frac{69}{92} = \frac{3}{4}$$

$\therefore$ The ratio of the 11th terms of the two A.P.s is 3 : 4.

Note : Sometimes we need to assume some consecutive terms of an A.P.

If **three** or **five** or **seven** consecutive terms are given, then we assume the middle term as 'a' and preceding terms decreasing by 'd' and succeeding terms increasing by 'd'.

So we assume,

The 3 consecutive terms in A.P. : $a - d, a, a + d$

The 5 consecutive terms in A.P. : $a - 2d, a - d, a, a + d, a + 2d$

The 7 consecutive terms in A.P. : $a - 3d, a - 2d, a - d, a, a + d, a + 2d, a + 3d$

If **four** or **six** consecutive terms are given, then there are two middle terms, so we assume them as $a - d$ and $a + d$. Here the difference between consecutive terms is taken as '$2d$', so preceding term is decreased by '$2d$' and succeeding term is increased by '$2d$'. So we assume,

The 4 consecutive terms in A.P. : $a - 3d, a - d, a + d, a + 3d$

The 6 consecutive terms in A.P. : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$

Example 15 : The sum and the product of three consecutive terms of an A.P. are 24 and 312 respectively. Find the three terms.

Solution : Suppose the three consecutive terms of the A.P. are $a - d, a, a + d$.

According to the given conditions,

$(a - d) + a + (a + d) = 24$ and $(a - d) \cdot a \cdot (a + d) = 312$

Thus, $3a = 24$.

So $a = 8$ and $(8 - d) \cdot 8 \cdot (8 + d) = 312$

$\therefore$ $64 - d^2 = 39$

$\therefore$ $d^2 = 25$

$\therefore$ $d = 5$ or $d = -5$

If $a = 8$ and $d = 5$, then the required terms are 3, 8, 13 and if $a = 8$ and $d = -5$, then they are 13, 8, 3.

Thus the required terms are 3, 8, 13.

Example 16 : The sum of four consecutive terms of an A.P. is 24 and the product of first and last terms is -45. Find the terms.

Solution : Suppose the four consecutive terms of the A.P. are

$a - 3d, a - d, a + d, a + 3d$.

Their sum $(a - 3d) + (a - d) + (a + d) + (a + 3d) = 24$

$\therefore$ $4a = 24$. So $a = 6$.

Also $(a - 3d)(a + 3d) = -45$

$\therefore$ $(6 - 3d)(6 + 3d) = -45$

$\therefore$ $36 - 9d^2 = -45$

$\therefore$ $9d^2 = 81$

$\therefore \quad d^2 = 9$

$\therefore \quad d = 3$ or $d = -3$

If $a = 6$ and $d = 3$, then the required terms are $-3, 3, 9, 15$ and if $a = 6$ and $d = -3$, then they are $15, 9, 3, -3$.

Example 17 : The income of a person is ₹ 3,50,000 in the first year. He receives an increment of ₹ 15,000 to his income per year. What will be his income at the end of 15th year ? How much amount he will receive in 15 years ?

Solution : Here we have an A.P. with $a = 3{,}50{,}000$ and $d = 15{,}000$

Now, $t_n = a + (n - 1)d$

$\therefore \quad t_{15} = 3{,}50{,}000 + 14(15{,}000) = 5{,}60{,}000$

and $\quad S_n = \frac{n}{2}(a + l)$

$= \frac{15}{2}(3{,}50{,}000 + 5{,}60{,}000) = 68{,}25{,}000$

$\therefore$ At the end of 15th year his income will be ₹ 5,60,000 and total amount he will receive in 15 years is ₹ 68,25,000.

Exercise 7.2

1. Find the desired terms in the following A.P.s :

   (1) 16th term in $-17, -13, -9, ...$

   (2) 31st term in $101, 96, 91, ...$

   (3) 10th term in $3, \frac{9}{2}, 6, \frac{15}{2}, ...$

2. If the nineth term of an A.P. is 30, find the sum of its first seventeen terms.
3. Find the sum of all natural numbers lying between 100 and 500, which are divisible by 5.
4. The first term of an A.P. is 4 and the sum of first five terms is one-sixth of the sum of the next five terms. Find the 8th term.
5. The sum of first n terms of an A.P. is $3n^2 + 5n$. Which term of it is 164 ?
6. If the sum of the first m terms of an A.P. is n and the sum of the first n terms is m, obtain the sum of the first $(m + n)$ terms.
7. If pth, qth and rth terms of an A.P. are l, m, n respectrively, then find the value of $l(q - r) + m(r - p) + n(p - q)$.
8. The ratio of the sum of first n terms of two A.P.s is $(3n - 13) : (5n - 1)$ for all $n \in$ N. Find the ratio of their 13th terms.
9. The ratio of the nth terms of two A.P.s is $(2n - 1) : (4n + 3)$ for all $n \in$ N. Find the ratio of the sum of the first 25 terms.
10. Find the sum of all integers from 100 to 200 which are divisible by 2 but not by 5.
11. If the 10th term of an A.P. is $\frac{1}{20}$ and the 20th term is $\frac{1}{10}$, then find the 200th term.
12. The sum of three consecutive terms of an A.P. is 9 and the sum of their squares is 59, find these terms.

13. If the sum of four consecutive terms of an A.P. is 32 and the product of whose 2nd and 3rd term is 60, find these terms.

14. A man starts repaying his loan with the initial instalment of ₹ 200. If he increases the instalment by ₹ 20 every month, how much total amount he will pay at the end of 20th instalment ?

15. Bhargav saves ₹ 50 in the first week of a year and then increases his weekly savings by ₹ 17.50. In the nth week, his weekly savings become ₹ 207.50. Find n and the amount he had saved.

16. A spiral is made up of successive semicircles, with centres alternately at P and Q. Semicircles start with centre P with radii 1 *cm*, 3 *cm*, 5 *cm*, ...; and centre Q with radii 2 *cm*, 4 *cm*, 6 *cm* ... What is the length of the spiral if it is made up of 20 such semicircles ? (See figure 7.1)

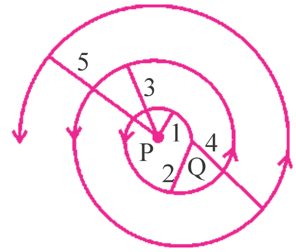

Figure 7.1

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7.3 Geometric Progression (G.P.)

Let us observe some sequences like :

(1) 3, 6, 12, 24,... (2) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$,... (3) 0.1, 0.01, 0.001, 0.0001, ...

We note that each term (except the first) progresses in a definite order.

We observe that in (1) the second term onwards each term is double than the preceding term; and in (3) each term is 0.1 times the preceding.

So, the ratio of any term to the preceding term is a constant, i.e. same for all terms (except the first). A Sequence with this property is called a **Geometric Progression (G.P.). The constant ratio is called the common ratio of the G.P. Thus, if the ratio of each term to the preceding is a non-zero constant, then the sequence is a G.P.** We define G.P. as :

Geometric Progression : A sequence $f : N \rightarrow R$, $f(n) = Ar^n$, $A \in R - \{0\}$, $r \in R - \{0\}$ is called a Geometric Progression. A G.P. is an exponential function.

Putting $n = 1, 2, 3,...$ we get the terms of a G.P. as $Ar, Ar^2, Ar^3, ...$

Theorem 4 : The ratio of any two consecutive terms of a G.P. is a non-zero constant.

Proof : Suppose $f : N \rightarrow R$ is a G.P., then for some $A \neq 0$ and some $r \neq 0$, we have $f(n) = Ar^n$, $\forall n \in N$.

The ratio of two consecutive terms $f(k + 1)$ and $f(k)$ is $\frac{f(k+1)}{f(k)} = \frac{Ar^{k+1}}{Ar^k} = r$, a non-zero constant.

Converse of this theorem is also true.

Suppose the first term of a sequence is $a \neq 0$ and the ratio of two consecutive terms is r, where $r \neq 0$, then the terms of the sequence are $a, ar, ar^2, ..., ar^{n-1}$. Thus, nth term $t_n = ar^{n-1}$.

We will prove this by the principle of mathematical induction.

Theorem 5 : If the ratio of two consecutive terms of a sequence is a non-zero constant, then the sequence is a G.P.

Proof : Consider P(n) : $f(n) = ar^{n-1}$; $a, r \in R - \{0\}$

(1) For $n = 1$, $f(1) = ar^0 = a$, the first term

$\therefore$ P(1) is true.

(2) Let P(k) : $f(k) = ar^{k-1}$ be true for some $k \in$ N.

Now we shall prove that P(k + 1) is also true.

$\therefore \frac{f(k+1)}{f(k)} = r$ **(given)**

$\therefore f(k+1) = r \cdot f(k) = r \cdot (ar^{k-1}) = ar^k = ar^{(k+1)-1}$

Thus P(k) is true. $\Rightarrow$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction P(n) is true for all $n \in$ N.

Here, $\{f(n)\}$ is a G.P.

We note that the G.P., whose first term is a and common ratio is r, is $a, ar, ar^2, ..., ar^{n-1}, ...$

Here also we write nth term of a G.P. as t_n.

So, $t_n = ar^{n-1}$, $a \neq 0$, $r \neq 0$.

Note : (1) Now onwards common ratio will be termed as the 'ratio'.

(2) If a, b, c are consecutive terms in G.P. then $\frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$.

Example 18 : Find nth and the 8th term of the G.P. 54, 36, 24, 16, ...

Solution : Here $a = 54$, $r = \frac{t_2}{t_1} = \frac{36}{54} = \frac{2}{3}$

Now $t_n = ar^{n-1} = 54\left(\frac{2}{3}\right)^{n-1} = \frac{2 \times 3^3 \times 2^{n-1}}{3^{n-1}}$

$\therefore t_n = 2^n \cdot 3^{4-n}$

Taking $n = 8$ in t_n, we have

$t_8 = 2^8 \cdot 3^{-4}$

$= \frac{256}{81}$

$\therefore$ The nth term of the G.P. is $2^n \cdot 3^{4-n}$ and the 8th term is $\frac{256}{81}$.

Example 19 : If the third term of a G.P. is 18 and its sixth term is 486, find the 9th term.

Solution : Here $t_3 = 18$ and $t_6 = 486$

Now $t_n = ar^{n-1}$

$\therefore t_3 = ar^2 = 18$ and $t_6 = ar^5 = 486$

$\therefore \frac{t_6}{t_3} = \frac{ar^5}{ar^2} = \frac{486}{18}$

$\therefore r^3 = 27$

$\therefore r = 3$

Also $ar^2 = 18$. So $9a = 18$

$\therefore$ $a = 2$

$\therefore$ $t_9 = ar^8 = 2(3)^8 = 13122$

$\therefore$ The 9th term of the G.P. is 13122.

Geometric Series :

The series corresponding to a G.P. is called a geometric series.

If first term of a G.P. is 'a' and ratio is 'r', then the nth term of the geometric series is $S_n = a + ar + ar^2 + ... + ar^{n-1}$.

Now we shall prove the formula for S_n by the principle of mathematical induction.

Theorem 6 : If first term of a G.P. is a and the ratio is r, then the sum of first n term is $S_n = \frac{a(r^n - 1)}{r - 1}$, $a \neq 0$, $r \neq 0$, $r \neq 1$, $n \in N$ and $S_n = na$ if $r = 1$.

Proof : Let the statement $P(n) : S_n = \frac{a(r^n - 1)}{r - 1}$, $a \neq 0$, $r \neq 0$, $r \neq 1$, $n \in N$.

(1) For $n = 1$, $S_1 = \frac{a(r-1)}{r-1} = a$, i.e. the sum of the first term is the first term 'a' itself. Thus, P(1) is true.

(2) Let $P(k) : S_k = \frac{a(r^k - 1)}{r - 1}$ be true for some $k \in N$.

Let $n = k + 1$

$$S_{k+1} = \frac{a(r^k - 1)}{r - 1} + ar^{(k+1)-1}$$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k$$

$$= \frac{a}{r-1}[r^k - 1 + r^{k+1} - r^k]$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

Thus, $P(k)$ is true. $\Rightarrow P(k + 1)$ is true.

Also we have seen that P(1) is true.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all $n \in N$.

If $r = 1$, it is clear that $S_n = a + a + ... + a(n \text{ times}) = na$

Note : The formula for S_n can also be written as $S_n = \frac{a(1 - r^n)}{1 - r}$, usually we use this form when $r < 1$.

Example 20 : For a G.P. $t_2 = 6$ and $t_5 = 48$, find S_6.

Solution : $t_2 = 6$ and $t_5 = 48$

$\therefore$ $ar = 6$ and $ar^4 = 48$

$\therefore$ $\frac{ar^4}{ar} = \frac{48}{6}$

$\therefore$ $r^3 = 8 = 2^3$

$\therefore$ $r = 2$ and $ar = 6$

$\therefore \quad a = 3$

Now $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$\therefore \quad S_6 = \dfrac{3(2^6 - 1)}{2 - 1} = 189$

Example 21 : The third term of a G.P. is $\frac{3}{4}$ and the sum of the first five terms is $\frac{32}{33}$ times the sum of the first ten terms. Find the sum of the first four terms.

Solution : Here $t_3 = \frac{3}{4}$ and $S_5 = \frac{32}{33} \cdot S_{10}$

$\therefore \quad ar^2 = \frac{3}{4}$ and $\dfrac{a(r^5 - 1)}{r - 1} = \dfrac{32}{33} \cdot \dfrac{a(r^{10} - 1)}{r - 1}$

$\therefore \quad \frac{33}{32} = r^5 + 1$

$\therefore \quad r^5 = \frac{33}{32} - 1$

$\therefore \quad r^5 = \frac{1}{32} = \left(\frac{1}{2}\right)^5$

$\therefore \quad r = \frac{1}{2}$

As $ar^2 = \frac{3}{4}$, $a\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$\therefore \quad a = 3$

Now, $S_4 = \dfrac{a(1 - r^4)}{1 - r} = \dfrac{3\left(1 - \frac{1}{16}\right)}{1 - \frac{1}{2}} = 6\left(\frac{15}{16}\right)$

$\therefore \quad S_4 = \frac{45}{8}$

$\therefore$ The sum of the first four terms is $\frac{45}{8}$.

Note : Sometimes we need to assume some consecutive terms of a G.P.

If **three** or **five** or **seven** consecutive terms of a G.P. are given, then we assume the middle term as 'a' and preceding terms are obtained dividing a by r, r^2, r^3, ... and succeeding terms are obtained, multiplying a by r, r^2, r^3, ...

So we assume,

The 3 consecutive terms for G.P. : $\frac{a}{r}, a, ar$

The 5 consecutive terms for G.P. : $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

The 7 consecutive terms for G.P. : $\frac{a}{r^3}, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, ar^3$

If **four** or **six** consecutive terms of a G.P. are given, then there are two middle terms, so we assume them as $\frac{a}{r}$ and ar. The terms preceding to $\frac{a}{r}$ are obtained by dividing $\frac{a}{r}$ by r^2, r^4, r^6, ... and the terms succeeding to ar are obtained by multiplying r^2, r^4, r^6, ... So we assume, The 4 consecutive terms for G.P. : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

The 6 consecutive terms for G.P. : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

Example 22 : Three numbers are consecutive term of a G.P. Their sum and product are $\frac{31}{5}$ and 1 respectively, find the numbers.

Solution : Let the three numbers in a G.P. be $\frac{a}{r}, a, ar$

Their product $\left(\frac{a}{r}\right)(a)(ar) = 1$ and sum $\left(\frac{a}{r}\right) + (a) + (ar) = \frac{31}{5}$

$\therefore \quad a^3 = 1$ and $\frac{1}{r} + 1 + r = \frac{31}{5}$

$\therefore \quad a = 1$

$\therefore \quad 5r^2 - 26r + 5 = 0$

$\therefore \quad (5r - 1)(r - 5) = 0$

$\therefore \quad r = \frac{1}{5}$ or $r = 5$

Taking $a = 1$ and $r = \frac{1}{5}$, the numbers are 5, 1, $\frac{1}{5}$

(If we take $r = 5$, then the same numbers can be obtained.)

Example 23 : Find the sum of the sequence 5, 55, 555, ... upto first n terms.

Solution : $S_n = 5 + 55 + 555 + \ldots\ n$ terms

$= \frac{5}{9}[9 + 99 + 999 + \ldots\ n \text{ terms}]$

$= \frac{5}{9}[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots\ n \text{ terms}]$

$= \frac{5}{9}[(10 + 10^2 + 10^3 + \ldots\ n \text{ terms}) - (1 + 1 + 1 + \ldots\ n \text{ terms})]$

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$ **(here $a = 10$, $r = 10$)**

$= \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

$= \frac{50}{81}(10^n) - \frac{50}{81} - \frac{5n}{9}$

Example 24 : The number of bacteria in a certain culture increase at the rate 4 % every hour. If initially there are 40 bacteria present, then how many bacteria will be present at the end of the 4th hour ? How many bacteria have increased during the 4th hour ?

Solution : Initially number of bacteria is 40. Bacteria increase at the rate of 4 % at the end of each hour.

So at the end of the first hour, the number of bacteria will be

$40 + 40\left(\frac{4}{100}\right) = 40(1 + 0.04) = 40(1.04)$

At the end of the second hour, they are $40(1.04)^2$. At the end of the third hour they are $40(1.04)^3$.

Thus, number of bacteria present in the successive hours form a G.P. with $T_1 = a = 40$ and $r = 1.04$

At the end of the fourth hour they are $T_5 = 40(1.04)^4 = 46.7943$

i.e. at the end of the fourth hour number of bacteria present is approximately 47.

During the fourth hour the increase in number of bacteria

= Number of bacteria present at the end of the fourth hour − Number of bacteria present at the end of the third hour

$= 40[(1.04)^4 - (1.04)^3]$

$= 40(1.04)^3\ (1.04 - 1)$

$= 40(1.04)^3\ (0.04)$

$= 1.7987$

∴ During the fourth hour approximately 2 bacteria have increased.

Exercise 7.3

1. Find the indicated terms of the following G.P.s :

(1) The 12th term of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, ...$

(2) The 11th term of $7, \frac{-7}{2}, \frac{7}{4}, \frac{-7}{8}, ...$

(3) The 8th term of $-2, -2\sqrt{2}, -4, -4\sqrt{2}, ...$

2. Do as directed for the following G.P.s :

(1) $t_7 = 96$, $r = 2$, find t_{10}.

(2) $a = 2$, $r = \sqrt{2}$, $t_n = 128$, find n.

(3) $a = 3$, $r = 3$, $S_n = 363$, find n.

(4) $r = \frac{1}{3}$, $S_3 = \frac{585}{4}$, find a.

3. The sum of the first three terms of a G.P. is 21 and the sum of the next three terms is 168, find the sum of the first five terms.

4. If the sum of the first two terms of a G.P. is $\frac{9}{2}$ and the sixth term is 8 times the third term, find G.P.

5. Sum to first n terms of a sequence :

(1) 7, 77, 777, 7777, ... (2) 3, 33, 303, 3003, ...

6. Find the sum of : $a(a + b) + a^2(a^2 + b^2) + a^3(a^3 + b^3) + ...$ n terms $(a, b \neq 0, \pm 1)$

7. The product of five positive numbers in G.P. is 32 and ratio of the greatest number to the smallest number is 81 : 1, find the numbers.

8. In a G.P., the $(p + q)$th term is m and the $(p - q)$th term is n. Find its pth term in terms of m and n.

9. If $1, a, b, c, 2$ are consecutive terms in a G.P., then find the value of abc.

10. If pth, qth and rth terms of a G.P. are themselves consecutive terms of a G.P., prove that p, q, r are in A.P.

11. If x, y, z are three consecutive terms in a G.P., prove that $\frac{1}{x+y} + \frac{1}{y+z} = \frac{1}{y}$.

12. Find four positive consecutive terms in a G.P. such that their product is 16 and having sum of second and third terms equal to 5.

13. A motorcycle was purchased for ₹ 60,000. If its price goes down by 10 % each year, what would be its price at the end of the fourth year.

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7.4 Means

Arithmetic Mean (A.M.) : If three distinct numbers a, A, b are consecutive terms of an A.P., then A is called an arithmetic mean of the two numbers a and b.

a, A and b are in A.P.

$\therefore \quad A - a = b - A$

$\therefore \quad 2A = a + b$

$\therefore \quad A = \frac{a+b}{2}$

Thus, A.M. of a and b is $A = \frac{a+b}{2}$, i.e. it is the average of a and b.

For instance, the A.M. of 4 and 12 is $A = \frac{4+12}{2} = 8$.

Arithmetic Means : For given two distinct real numbers a and b, if numbers $a, A_1, A_2, A_3, \ldots A_n, b$ are consecutive terms of an A.P., we say that $A_1, A_2, A_3, \ldots, A_n$ are n arithmetic means between a and b.

Suppose $A_1, A_2, A_3, \ldots, A_n$ are n arithmetic means between a and b.

Here we have $n + 2$ terms in an A.P.. The first term is a and $(n + 2)$nd term is b.

$\therefore \quad t_{n+2} = b = a + [(n + 2) - 1]\,d$

$\therefore \quad b - a = (n + 1)d$

$\therefore \quad \frac{b-a}{n+1} = d$

Here, arithmetic mean $A_1 = a + d$

$$= a + \left(\frac{b-a}{n+1}\right)$$

Thus $A_1 = a + \left(\frac{b-a}{n+1}\right)$, $A_2 = a + 2\left(\frac{b-a}{n+1}\right)$, $A_3 = a + 3\left(\frac{b-a}{n+1}\right)$, ...

n arithmetic means between a and b are $A_k = a + k\left(\frac{b-a}{n+1}\right)$, where $k = 1, 2, 3, \ldots n$.

Here, A_k denotes kth A.M. out of n means between a and b.

For $n = 1$, $A_1 = a + \frac{b-a}{n+1} = \frac{a+b}{2}$, the A.M. of a and b.

Thus, A.M. of two distinct real numbers a and b is $A = \frac{a+b}{2}$.

Example 25 : Find four arithmetic means between 8 and 23.

Solution : Here, $a = 8$, $b = 23$ and $n = 4$

So $d = \frac{b-a}{n+1} = \frac{23-8}{4+1} = \frac{15}{5} = 3$

$\therefore$ The four arithmetic means between 8 and 23 are :

$8 + 3$, $8 + 2(3)$, $8 + 3(3)$, $8 + 4(3)$. They are 11, 14, 17 and 20.

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Example 26 : If n arithmetic means are inserted between 1 and 31 such that the ratio of $(n-1)$th mean to the 7th mean is 9 : 5, find n.

Solution : Here, $a = 1$ and $b = 31$

Common difference $d = \frac{b-a}{n+1} = \frac{31-1}{n+1} = \frac{30}{n+1}$

$$\frac{(n-1)\text{th A.M.}}{\text{7th A.M.}} = \frac{9}{5} \qquad \textbf{(given)}$$

$$\therefore \quad \frac{1+(n-1)\left(\frac{30}{n+1}\right)}{1+7\left(\frac{30}{n+1}\right)} = \frac{9}{5}$$

$$\therefore \quad \frac{n+1+30n-30}{n+1+210} = \frac{9}{5}$$

$\therefore$ $5(31n - 29) = 9(n + 211)$

$\therefore$ $155n - 145 = 9n + 1899$

$\therefore$ $146n = 2044$

$\therefore$ $n = 14$

Geometric Mean (G.M.) : Given distinct positive real numbers a and b, if G is a positive number such that a, G, b are consecutive terms of G.P., then G is called a geometric mean of a and b.

a, G, b are in G.P.

$\therefore$ $\frac{G}{a} = \frac{b}{G}$

$\therefore$ $G^2 = ab$

$\therefore$ $G = \sqrt{ab}$

For instance, the G.M. of 2 and 18 is $G = \sqrt{2 \times 18} = 6$.

Geometric Means : For given distinct positive real numbers a and b, if the positive numbers $a, G_1, G_2, G_3, \ldots G_n, b$ are consecutive terms of a G.P., then $G_1, G_2, G_3, \ldots G_n$ are called the geometric means between a and b.

Now, we shall find the formula for n geometric means between a and b.

Suppose $G_1, G_2, G_3, \ldots G_n$ are n geometric means between a and b, then $a, G_1, G_2, G_3, \ldots G_n, b$ are consecutive terms in a G.P. in which the first term is a and the $(n+2)$nd term is b.

$\therefore$ $t_{n+2} = b = ar^{n+1}$ where r is the common ratio of G.P.

$\therefore$ $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Thus, $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, G_3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \ldots$

$\therefore$ n geometric means between a and b are

$$G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}, \text{ where } k = 1, 2, 3, \ldots n$$

Here, G_k denotes kth G.M. out of n G.M.s between a and b.

For $n = 1$, $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{1+1}} = \sqrt{ab}$, the G.M. of a and b.

Thus, G.M. of two distinct positive real numbers a and b is $G = \sqrt{ab}$.

Example 27 : Find three G.M.s between 2 and $\frac{2}{81}$.

Solution : Here, $a = 2$, $b = \frac{2}{81}$ and $n = 3$

$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{2}{81} \times \frac{1}{2}\right)^{\frac{1}{3+1}} = \left(\frac{1}{81}\right)^{\frac{1}{4}} = \frac{1}{3}$

Now, $G_1 = ar = 2 \cdot \frac{1}{3} = \frac{2}{3}$, $G_2 = ar^2 = 2 \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{9}$, $G_3 = ar^3 = 2 \cdot \left(\frac{1}{3}\right)^3 = \frac{2}{27}$

$\therefore$ The three geometric means between 2 and $\frac{2}{81}$ are $\frac{2}{3}$, $\frac{2}{9}$ and $\frac{2}{27}$.

Example 28 : If the A.M. and G.M. of two positive real numbers are 7 and $2\sqrt{6}$ respectively, find the numbers.

Solution : Let a and b be two numbers whose A.M., A = 7 and G.M., G = $2\sqrt{6}$

$\therefore \quad A = \frac{a+b}{2} = 7$ and $G = \sqrt{ab} = 2\sqrt{6}$

$\therefore \quad a + b = 14$ and $ab = 24$

$\therefore \quad b = \frac{24}{a}$

$\therefore \quad a + \frac{24}{a} = 14$

$\therefore \quad a^2 - 14a + 24 = 0$

$\therefore \quad (a - 12)(a - 2) = 0$

$\therefore \quad a = 12$ or $a = 2$

Now if $a = 12$, then $b = 2$ and if $a = 2$ then $b = 12$

$\therefore$ The required numbers are 2 and 12.

Example 29 : If G is the G.M. of a and b and A_1, A_2 are the two A.M.s between a and b, then prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

Solution : A_1, A_2 are two A.M.s between a and b.

$\therefore \quad a, A_1, A_2, b$ are consecutive terms of A.P.

$\therefore \quad A_1 = \frac{a + A_2}{2}$ and $A_2 = \frac{A_1 + b}{2}$

$\therefore \quad 2A_1 - A_2 = a$ and $2A_2 - A_1 = b$

Now $G^2 = ab = (2A_1 - A_2)(2A_2 - A_1)$

Theorem 7 : If A and G are respectively the A.M. and G.M. of two distinct positive numbers a and b, then prove that $A > G$.

Proof : a and b are distinct and positive.

So, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$

$= \frac{1}{2}(a + b - 2\sqrt{ab})$

$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$ **($\because$ $a \neq b$; $a, b > 0$)**

$\therefore \quad A > G$

Example 30 : The difference between A.M. and G.M. of two positive real numbers is 12 and the ratio of these numbers is 1 : 9, find the numbers.

Solution : Suppose required numbers are a and b, $a, b \in R^+$.

$\therefore \quad A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

As $A > G$, so $A - G = 12$ and $\frac{a}{b} = \frac{1}{9}$ (given)

$\therefore \quad \frac{a+b}{2} - \sqrt{ab} = 12$ and $b = 9a$

$\therefore \quad \frac{a+9a}{2} - \sqrt{a \cdot 9a} = 12$

$\therefore \quad 5a - 3a = 12$

$\therefore \quad a = 6$ and $b = 54$

$\therefore$ The numbers are 6 and 54.

Example 31 : For positive real numbers a, b, c prove that $(a + b)(b + c)(c + a) \geq 8abc$.

Solution : We know that $\frac{a+b}{2} \geq \sqrt{ab}$.

Similarly $\frac{b+c}{2} \geq \sqrt{bc}$ and $\frac{c+a}{2} \geq \sqrt{ca}$.

Multiplying the respective sides of the above results, we get

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) \geq abc$$

$\therefore \quad (a + b)(b + c)(c + a) \geq 8abc$

Exercise 7.4

1. Place 5 A.M.s between 3 and 4.
2. Place 3 A.M.s between -3 and 29.
3. Insert 5 G.M.s between $\frac{1}{8}$ and 8.
4. Insert 3 G.M.s between 2 and $\frac{1}{2}$.
5. Find two positive numbers whose A.M. and G.M. are 25 and 15 respectively.
6. If A.M. and G.M. of the roots of a quadratic equation are 10 and 8 respectively, then obtain the quadratic equation.

7. If $sec(x + y)$, $secx$, $sec(x - y)$ are in A.P. then prove that $cosx = \pm\sqrt{2}\ cos\frac{y}{2}$, where $cosx \neq 1$; $cosy \neq 1$.

8. If $\frac{1}{q}$ is A.M. of $\frac{1}{p}$ and $\frac{1}{r}$, then prove that $\frac{r+p}{q}$ is the A.M. of $\frac{p+q}{r}$ and $\frac{q+r}{p}$, where $p, q, r \neq 0$.

*

7.5 Sums of Some Special Series

Sequence of powers of Natural Numbers : We have obtained the formula for the sum of first n terms of an A.P. and a G.P., but it is not possible to find such formula for every sequence. There are some important sequences which are neither A.P. nor G.P. and we are able to calculate the sum of their n terms. We shall consider some such special sequences. We shall find formula for the sum of first n natural numbers, their squares and their cubes.

We shall introduce the notation 'Σ' pronounced as 'sigma', which is to be specially used for such series. It is the enlarged form of the upright capital Greek letter sigma, Σ means summation.

$\sum_{n=2}^{n=6} t_n$ reads as 'sigma' t_n where n runs from 2 to 6 and it denotes sum of t_n for $n = 2, 3, 4, 5$ and 6. i.e. $\sum_{n=2}^{n=6} t_n = t_2 + t_3 + t_4 + t_5 + t_6$.

For example,

$$\sum_{n=2}^{n=6} (2n + 3) = \{2(2) + 3\} + \{2(3) + 3\} + \{2(4) + 3\} + \{2(5) + 3\} + \{2(6) + 3\}$$

$$= 7 + 9 + 11 + 13 + 15$$

$$= 55$$

i.e. we have to substitute $n = 2, 3, 4, 5$ and 6 in $(2n + 3)$ and we shall add the resulting numbers.

It is customary to write $\sum_{n=2}^{6} t_n$ instead of writing $\sum_{n=2}^{n=6} t_n$.

The symbol Σ has the following properties which can easily be proved. (Try yourself !)

(1) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(2) $\sum_{i=1}^{n} ma_i = m \sum_{i=1}^{n} a_i$, where m is a constant not depending on i.

(3) $\sum_{i=1}^{n} 1 = \sum_{i=1}^{n} (i)^0 = 1^0 + 2^0 + 3^0 + ... + n^0$

$$= 1 + 1 + 1 + ... + 1 \ (n \text{ times})$$

$$= n$$

(4) $\sum_{i=1}^{n} m = m \sum_{i=1}^{n} 1 = mn$, where m is constant.

Note : (1) $\sum_{i=1}^{n} (a_i \cdot b_i) \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$

(2) $\sum_{i=1}^{n} \left(\frac{a_i}{b_i}\right) \neq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$, $b_i \neq 0 \ \forall i \in N$

Now we shall find the sum of first n terms of some special series, namely

(1) $\sum_{r=1}^{n} r$ (2) $\sum_{r=1}^{n} r^2$ (3) $\sum_{r=1}^{n} r^3$.

Now, $\sum_{i=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, we can prove it by the formula for sum of first n terms in A.P. (Try it !).

Note : It is believed that the great mathematician Gauss obtained the sum $1 + 2 + 3 + ... + 100$ at the age of 5 to 6 years. When his teacher had asked to sum the numbers 1 to 100, he had answered the question in no time. He had paired the numbers and added numbers in pairs as $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, ... $50 + 51 = 101$, each pair giving the same sum 101 and there are such fifty pairs, so the sum of 1 to 100 is $50 \times 101 = 5050$.

From this also, we can have $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$.

Theorem 8 : $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in N$

Proof : Here $S_n = 1^2 + 2^2 + 3^2 + ... + n^2$.

We consider the identity : $x^3 - (x-1)^3 = 3x^2 - 3x + 1$

Putting $x = 1, 2, 3, ..., n$, we have

$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$

⋮

$n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$

Adding all the above results, we have

$n^3 - 0^3 = 3[1^2 + 2^2 + 3^2 + ... + n^2] - 3[1 + 2 + 3 + ... + n] + [1 + 1 + 1 + ... + 1, n \text{ times}]$

$\therefore \quad n^3 = 3 \cdot S_n - 3 \sum_{r=1}^{n} r + n$

$\therefore \quad 3S_n = n^3 + \frac{3n(n+1)}{2} - n$

$\therefore \quad S_n = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right)$

$= \frac{1}{6}\left(2n^3 + 3n^2 + n\right)$

$= \frac{1}{6} \cdot n\left(2n^2 + 3n + 1\right)$

$= \frac{1}{6} \cdot n\,(n + 1)(2n + 1)$

$\therefore \quad \sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6} \cdot n\,(n + 1)(2n + 1)$

Theorem 9 : $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}, \; n \in N$

Proof : Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

We consider the identity : $x^4 - (x - 1)^4 = 4x^3 - 6x^2 + 4x - 1$

Putting $x = 1, 2, 3, ..., n$, we have

$1^4 - 0^4 = 4(1)^3 - 6(1)^2 + 4(1) - 1$

$2^4 - 1^4 = 4(2)^3 - 6(2)^2 + 4(2) - 1$

$3^4 - 2^4 = 4(3)^3 - 6(3)^2 + 4(3) - 1$

$\vdots$

$n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1$

Adding all the above results, we have

$n^4 - 0^4 = 4[1^3 + 2^3 + 3^3 + ... + n^3] - 6[1^2 + 2^2 + 3^2 + ... + n^2]$

$\qquad + 4[1 + 2 + 3 + ... + n] - [1 + 1 + 1 + ... + 1, n \text{ times}]$

$\therefore \quad n^4 = 4 \cdot S_n - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - n$

$\therefore \quad 4S_n = n^4 + 6\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + n$

$\therefore \quad S_n = \frac{1}{4}\left[n^4 + n(n + 1)(2n + 1) - 2n(n + 1) + n\right]$

$= \frac{1}{4} \cdot n\left[n^3 + (n + 1)(2n + 1) - 2(n + 1) + 1\right]$

$= \frac{1}{4} \cdot n\left(n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1\right)$

$= \frac{1}{4} \cdot n\left(n^3 + 2n^2 + n\right)$

$= \frac{1}{4}n \cdot n\left(n^2 + 2n + 1\right)$

$= \frac{1}{4}n^2(n + 1)^2$

$\therefore \quad S_n = \frac{1}{4} \cdot n^2(n + 1)^2 \quad \text{or} \quad S_n = \left[\frac{1}{2}\,n(n + 1)\right]^2$

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$\therefore \quad \sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r^3$ are also denoted by Σn, Σn^2, Σn^3 respectively.

Example 32 : Obtain the following sums :

(1) $\sum_{r=7}^{16} 2r^3$, (2) $\sum_{r=10}^{20} (3r - r^2)$

Solution : (1) $\sum_{r=7}^{16} 2r^3 = 2\sum_{r=7}^{16} r^3$

$= 2\left[\sum_{r=1}^{16} r^3 - \sum_{r=1}^{6} r^3\right]$

$= 2\left[\frac{(16)^2 \cdot (17)^2}{4} - \frac{(6)^2 \cdot (7)^2}{4}\right]$

$= 2\ [18496 - 441]$

$= 2\ [18055] = 36110$

(2) $\sum_{r=10}^{20} (3r - r^2) = 3\left[\sum_{r=1}^{20} r - \sum_{r=1}^{9} r\right] - \left[\sum_{r=1}^{20} r^2 - \sum_{r=1}^{9} r^2\right]$

$= 3\left[\frac{20(20+1)}{2} - \frac{9(9+1)}{2}\right] - \left[\frac{20(20+1)\ (20(2)+1)}{6} - \frac{9(9+1)\ (2(9)+1)}{6}\right]$

$= 3\left[\frac{(20)\ (21)}{2} - \frac{9(10)}{2}\right] - \left[\frac{20(21)\ (41)}{6} - \frac{9(10)\ (19)}{6}\right]$

$= 3\ (210 - 45) - (2870 - 285)$

$= 495 - 2585 = -2090$

Example 33 : Sum to n terms $1^3 + 3^3 + 5^3 + ...$

Solution : Let us think of the nth term (general term) of the series 1, 3, 5, ... It is an A.P. with the first term $a = 1$ and $d = 2$.

$\therefore \quad t_n = a + (n-1)d = 1 + (n-1)2 = 2n - 1$

$\therefore$ the nth term of the given sequence is $(2n-1)^3$.

Note : To get general term, it is not necessary to show the method how we get it. For example in this question it is clear that 1, 3, 5,... is a sequence of odd natural numbers, so the nth odd natural number is obvious $2n - 1$.

Now, $S_n = 1^3 + 3^3 + 5^3 + ... + (2n-1)^3$

$= \sum_{r=1}^{n} (2r-1)^3$

$= \sum_{r=1}^{n} (8r^3 - 12r^2 + 6r - 1)$

$$= 8 \sum_{r=1}^{n} r^3 - 12 \sum_{r=1}^{n} r^2 + 6 \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8 \cdot \frac{n^2(n+1)^2}{4} - 12 \cdot \frac{n(n+1)(2n+1)}{6} + 6 \cdot \frac{n(n+1)}{2} - n$$

$$= n(n + 1) [(2n(n + 1) - 2(2n + 1) + 3)] - n$$

$$= n(n + 1) (2n^2 + 2n - 4n - 2 + 3) - n$$

$$= n(n + 1) (2n^2 - 2n + 1) - n$$

$$= n[(n + 1)(2n^2 - 2n + 1) - 1]$$

$$= n(2n^3 + 2n^2 - 2n^2 - 2n + n + 1 - 1)$$

$$= n(2n^3 - n)$$

$$= n^2(2n^2 - 1)$$

Example 34 : Sum to n terms the series $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + ...$ and hence obtain the sum of first 50 terms.

Solution : Here, to begin with we see that the nth term of 1, 2, 3, ... is n and that of 4, 5, 6, ... is $(n + 3)$.

$$\therefore \quad t_n = n(n + 3)$$

$$\therefore \quad S_n = \sum_{r=1}^{n} r(r + 3)$$

$$= \sum_{r=1}^{n} (r^2 + 3r)$$

$$= \sum_{r=1}^{n} r^2 + 3 \sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1) + 9n(n+1)}{6}$$

$$= \frac{n(n+1)(2n+1+9)}{6}$$

$$= \frac{n(n+1) \cdot (2n+10)}{6}$$

$$= \frac{n(n+1)(n+5)}{3}$$

Substituting $n = 50$, we have

$$S_{50} = \frac{50(51)(55)}{3} = 46750$$

Thus, sum of the first n terms, $S_n = \frac{n(n+1)(n+5)}{3}$ and the sum of the first fifty terms is 46750.

Example 35 : Sum the series :

$(1 + x) + (1 + x + x^2) + (1 + x + x^2 + x^3) + \ldots n$ terms. $(x \neq 1)$

Solution : Here, $S_n = \frac{1-x^2}{1-x} + \frac{1-x^3}{1-x} + \frac{1-x^4}{1-x} + \ldots n$ terms

$$= \frac{1}{1-x}[(1 + 1 + 1 + \ldots n \text{ terms}) - (x^2 + x^3 + x^4 + \ldots n \text{ terms})]$$

$$= \frac{1}{1-x}\left[n - \frac{x^2(1-x^n)}{1-x}\right]$$

($x^2 + x^3 + x^4 + \ldots$ n terms is a geometric series; $a = x^2$, $r = x$)

Example 36 : If the sum of the first n terms of the series,

$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + \ldots$ is $\frac{n(n+1)^2}{2}$ when n is even, then what is the sum of the series when n is odd ?

Solution : When n is odd, then last term will be n^2.

$\therefore$ Series is $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + \ldots + 2(n-1)^2 + n^2$

$$= \frac{(n-1)((n-1)+1)^2}{2} + n^2 \qquad \text{((n - 1) is even)}$$

$$= \frac{(n-1) \cdot n^2 + 2n^2}{2}$$

$$= \frac{n^2(n-1+2)}{2}$$

$$= \frac{n^2(n+1)}{2}$$

(Verify your answer !)

Example 37 : Sum the series, $1 + \frac{1}{2}(1 + 2) + \frac{1}{3}(1 + 2 + 3) + \ldots 16$ terms.

Solution : Here, $t_n = \frac{1}{n}(1 + 2 + 3 + \ldots + n) = \frac{1}{n}\Sigma n$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

$$\therefore \quad S_n = \sum_{r=1}^{16}\left(\frac{r+1}{2}\right) = \frac{1}{2}\left(\sum_{r=1}^{16} r + \sum_{r=1}^{16} 1\right)$$

$$= \frac{1}{2}\left[\frac{16(17)}{2} + 16\right]$$

$$= \frac{1}{2}(136 + 16)$$

$$= \frac{1}{2}(152) = 76$$

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Exercise 7.5

1. Obtain the following sums :

(1) $\sum_{r=1}^{10} (2r^2 + 3)$ (2) $\sum_{r=2}^{10} (4r^2 - 28r + 49)$ (3) $\sum_{r=6}^{15} (r^2 - r - 1)$ (4) $\sum_{r=8}^{20} (2 - r^2)$

2. Find the sum of the first n terms of each series below :

(1) $3^2 + 7^2 + 11^2 + ...$ (2) $1^3 + 4^3 + 7^3 + ...$

(3) $2 \cdot 1 + 5 \cdot 3 + 8 \cdot 5 + ...$ (4) $3 \cdot 4 \cdot 5 + 4 \cdot 6 \cdot 5 + 5 \cdot 8 \cdot 5 + ...$

(5) $(5^4 - 1^4) + (8^4 - 4^4) + (11^4 - 7^4) + ...$ (6) $1^2 + \left(\frac{1^2 + 2^2}{2}\right) + \left(\frac{1^2 + 2^2 + 3^2}{3}\right) + ...$

(7) $(2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + ...$ (8) $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + ...$

(9) $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + ...$

3. Sum the following series :

(1) $2^3 - 3^3 + 4^3 - 5^3 + ... + 22^3 - 23^3$

(2) $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + 29^2 - 30^2$

*

Miscellaneous Examples :

Example 38 : If α_1 is the sum of the first n natural numbers, α_2 denotes the sum of their squares and α_3 is the sum of their cubes, prove that $9\alpha_2^2 = \alpha_3 (1 + 8\alpha_1)$.

Solution : $\alpha_1 = \frac{n(n+1)}{2}$, $\alpha_2 = \frac{n(n+1)(2n+1)}{6}$, $\alpha_3 = \frac{n^2(n+1)^2}{4}$

$$\text{Now, } \alpha_3 [1 + 8\alpha_1] = \frac{n^2(n+1)^2}{4} \left[1 + 8 \cdot \frac{n(n+1)}{2}\right]$$

$$= \frac{n^2(n+1)^2}{4} [4n^2 + 4n + 1]$$

$$= \frac{n^2(n+1)^2(2n+1)^2}{4} \times \frac{9}{9}$$

$$= \left(\frac{n(n+1)(2n+1)}{6}\right)^2 \cdot 9 = 9\alpha_2^2$$

Example 39 : Obtain the sum of the series,

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + ... \ n \text{ terms } (x \neq 0, x \neq \pm 1)$$

Solution : $\left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x^4 + 2 + \frac{1}{x^4}\right) + \left(x^6 + 2 + \frac{1}{x^6}\right) + ... \ n$ terms

$$= (x^2 + x^4 + x^6 + ... + n \text{ terms}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + ... \ n \text{ terms}\right) + (2 + 2 + 2 + ... \ n \text{ terms})$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{\frac{1}{x^2}\left(1 - \frac{1}{x^{2n}}\right)}{1 - \frac{1}{x^2}} + 2n$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{x^{2n} - 1}{(x^2 - 1) \cdot x^{2n}} + 2n$$

Exercise 7

1. Find the 30th term of the sequence 5, 0, -5, -10, ... Also find which term would be -200, if any.

2. For an A.P., the 12th term is 64 and the 20th term is 112, find the A.P.

3. Ramu travels at the speed of 40 *km*/*hr*. He reduces his speed every hour by 4 *km*. How much time would he take to travel 216 *km* ?

4. 200 wooden blocks are stacked in such a way that 20 blocks are in the bottom row, 19 are in the next upper row, 18 are in the upper row next to it and the process is continued. How many rows will be formed ? How many blocks are there in the upper most row ?

5. 1, 5, 25 are the pth, the qth and the rth terms respectively of a G.P. Prove that p, q, r are in A.P.

6. If a, b, c are consecutive terms of A.P. and $a, c - b, b - a$ are consecutive terms of G.P., then find $a : b : c$.

7. Sum to n terms of the series : $6 + 6.6 + 6.66 + 6.666 + ...$

8. If the sums of first n, $2n$, $3n$ terms of A.P. are α, β, γ respectively, prove that $\gamma = 3(\beta - \alpha)$.

9. Find S_{20} for an A.P. having t_3 as 7 and t_7 is 2 more than three times its t_3.

10. If a, b, c are consecutive terms of A.P., a^2, b^2, c^2 are consecutive terms of G.P., $a + b + c = \frac{3}{2}$ and $a < b < c$, find a.

11. If $a_n = 3 - 5n$, find S_n.

12. If S_{30} of an A.P. is 1635 and t_{30} is 98, find A.P.

13. If A.M. is three times G.M. of two positive numbers a and b, find $a : b$.

14. Sum the series : $1 + \frac{1^3 + 2^3}{2} + \frac{1^3 + 2^3 + 3^3}{3} + ... + \frac{1^3 + 2^3 + 3^3 + ... + 20^3}{20}$.

15. Sum of six consecutive terms of an A.P. is 48 and the product of the first and the last numbers is 39. Find these numbers.

16. Product of five consecutive terms of G.P. is 243. If the sum of the second and the fourth number is $\frac{51}{4}$, find the numbers.

17. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) In a sequence $\left\{\frac{n + (-1)^n}{2}\right\}$, the difference of the 12th and the 21st term is ... ☐

(a) 0 (b) $\frac{-1}{2}$ (c) $\frac{7}{2}$ (d) $\frac{33}{2}$

(2) If the 5th term of an A.P. is 7, then the sum of the first 9 terms is ... ☐

(a) 36 (b) 49 (c) 45 (d) 63

(3) If the third term of an A.P. is 9 and its tenth term is 21, then the sum of its first 12 terms is ... ☐

(a) 180 (b) 360 (c) 150 (d) 210

(4) A.M. of two positive numbers is 2. If a larger number is increased by 1, then their G.M. is also 2, then the numbers are ... ☐

(a) 1, 3 (b) $\frac{1}{2}, \frac{7}{2}$ (c) $\frac{2}{3}, \frac{10}{3}$ (d) 0.7, 3.3

(5) If for a G.P., $r = \frac{1}{3}$ and $S_4 = \frac{80}{27}$, then a = ...

(a) $\frac{2}{3}$ (b) 3 (c) 2 (d) $\frac{3}{2}$

(6) If 25, $x - 6$ and $x - 12$ are consecutive terms of G.P., then x = ...

(a) 8 (b) 12 (c) 16 (d) 20

(7) $\sum_{r=1}^{n}\left(\sum_{m=1}^{r} m\right)$ = ...

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)(n+2)}{6}$ (c) $\frac{n^2(n+1)^2}{4}$ (d) $\frac{n(n+1)(2n+1)}{12}$

(8) If S_1, S_2 and S_3 are the sums of the first n_1, n_2, n_3 terms of an A.P. respectively then

$\frac{2S_1}{n_1}(n_2 - n_3) + \frac{2S_2}{n_2}(n_3 - n_1) + \frac{2S_3}{n_3}(n_1 - n_2)$ = ...

(a) 0 (b) 1 (c) $S_1\ S_2\ S_3$ (d) $n_1\ n_2\ n_3$

(9) If the first term of a G.P. is 3 and the common ratio is 2, then the sum of first five to ten terms is ......

(a) 2976 (b) 3024 (c) 1488 (d) 3114

(10) 3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + ... 20 terms is ...

(a) 511 (b) 536 (c) 549 (d) 520

(11) If third term of a G.P. is 3, then the product of first five terms is ...

(a) 3^5 (b) 5^3 (c) 3^3 (d) 5^5

(12) A_1 and A_2 are two A.M.s inserted between a and b, while G_1 and G_2 are two G.M.s inserted between a and b, then $\frac{G_1G_2}{A_1 + A_2}$ = ...

(a) $\frac{a+b}{2ab}$ (b) $\frac{a+b}{ab}$ (c) $\frac{2ab}{a+b}$ (d) $\frac{ab}{a+b}$

(13) If the length of the sides of a right triangle are in A.P. then the *cosines* of their acute angles are ...

(a) $\frac{\sqrt{3}}{2}, \frac{1}{2}$ (b) $\frac{5}{13}, \frac{12}{13}$ (c) $\frac{3}{5}, \frac{4}{5}$ (d) $\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}$

(14) Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n - 1}$, $n \in N$, then...

(a) $S_{100} < 100$ (b) $S_{100} > 100$ (c) $S_{200} = 100$ (d) $S_{200} > 200$

(15) If a_1, a_2, a_3, ... a_n are in A.P. with common difference d, then

$\sin d\,[\operatorname{cosec} a_1 \cdot \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \cdot \operatorname{cosec} a_3 + ... + \operatorname{cosec} a_{n-1} \cdot \operatorname{cosec} a_n]$ = ...

(a) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$ (b) $\sec a_1 - \sec a_n$

(c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$

(16) For an A.P., if $4t_4 = 7t_7$, then t_{11} = ...

(a) -1 (b) 0 (c) 11 (d) 44

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(17) $0 < \theta \leq \frac{\pi}{2}$, then the minimum value of $sin^3\theta + cosec^3\theta$ is ...

(a) 2 (b) 1 (c) 0 (d) not possible

(18) a, b, c, d, e, f are in A.P., then $d - b = ...$

(a) $2(c - a)$ (b) $2(f - c)$ (c) $2(d - c)$ (d) $2(f - b)$

*

Summary

We studied following points in this chapter :

1. If S_n is given then to find a_n by $a_1 = S_1$; $a_n = S_n - S_{n-1}$, $n > 1$.

2. The nth term of A.P. is $t_n = a + (n - 1)d$ where, a is the first term and d is the common difference.

3. The sum of the first n terms of an A.P. is $S_n = \frac{n}{2}\{2a + (n - 1)d\}$.

4. The nth term of a G.P. is $t_n = ar^{n-1}$, $a \neq 0$, $r \neq 0$, where a is the first term and r is the common ratio.

The sum of the first n terms of G.P. is $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1} & r \neq 1 \\ na & r = 1 \end{cases}$

5. The arithmetic mean of two numbers a and b is $A = \frac{a+b}{2}$. If there are n arithmetic means inserted between a and b, then $d = \frac{b-a}{n+1}$ and the kth mean is given by $A_k = a + k\left(\frac{b-a}{n+1}\right)$, where $k = 1, 2, 3, ... n$.

6. The geometric mean of two positive numbers a and b is $G = \sqrt{ab}$. If there are n geometric means inserted between a and b, then $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ and the kth mean is given by $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$, where $k = 1, 2, 3, ... n$ ($G > 0$, $G_i > 0 \ \forall i \in N$)

7. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

Bhaskara (1114–1185), also known as Bhaskara II and Bhaskaracharya ("Bhaskara the teacher"), was an Indian mathematician and astronomer. He was born near Vijayvada. Bhaskara is said to have been the head of an astronomical observatory at Ujjain, the leading mathematical center of ancient India.

Bhaskara and his works represent a significant contribution to mathematical and astronomical knowledge in the 12th century. He has been called the greatest mathematician of medieval India. His main work the *Siddhanta Shiromani*, Sanskrit for "Crown of treatises," is divided into four parts called *Lilavati*, *Bijaganita*, *Grahaganita* and *Goladhyaya*. These four sections deal with arithmetic, algebra, mathematics of the planets and spheres respectively.

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Chapter 8

CONICS

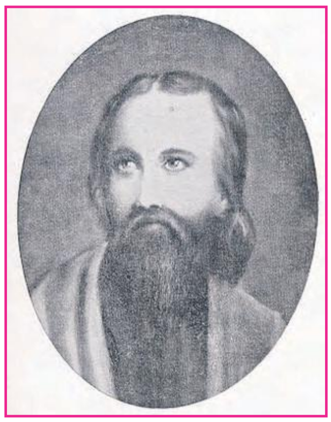
Proof is an idol before whom the pure mathematician tortures himself.

– Arthur Stanely Eddington

In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.

– Hermann Hankel

8.1 Introduction



Apollonius (262 BC - 190 BC)

We shall study about some special curves, viz., a circle, an ellipse, a parabola and a hyperbola in this chapter. The curves mentioned above can be obtained by taking intersection of a plane with a double napped right circular cone. These curves are called **conic sections** or more commonly **conics**. The names parabola and hyperbola are given by **Apollonius**. He is in fact considered as pioneer of studying such curves. These curves have a very wide range of applications in many fields of physics, optics etc. In the sixteenth century, **Galileo** observed that the path of a projectile is a parabola. This fact is now used in the design of an artillery. In the seventeenth century after prolonged observations **Keplar** gave laws of planetary motion in which it is said that the orbits of the earth and other planets around the sun are ellipses. Afterwards, **Newton** gave theoretical proof of **Keplar's** laws in a more general situation. Now a days a dish antenna for television and for other communication is also designed using the concept of conics. Thus, the study of conics is very important; and it has got applications in mechanics, space science, communication, optics etc. In this chapter we will discuss these curves, their equations and their properties.

8.2 Circle

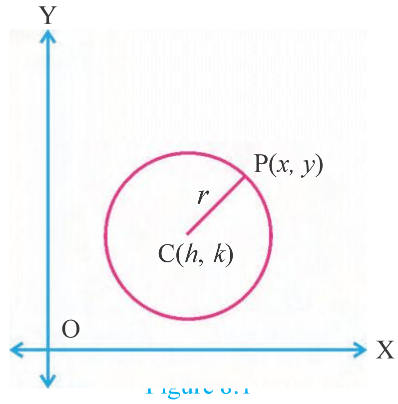
We know that the set of all points in a plane at the same distance from a fixed point is called a **circle**. The fixed point is called the **centre** and the fixed distance is called the **radius** of the circle.

Cartesian Equation of a Circle Centered at (h, k) and Radius r :

Let the point $C(h, k)$ be the centre of a circle and $P(x, y)$ be any point on the circle. Now since the radius of the circle is given to be r, we get

$$CP = r \Leftrightarrow CP^2 = r^2$$

$$\Leftrightarrow (x - h)^2 + (y - k)^2 = r^2$$



Thus the cartesian equation of a circle with centre $C(h, k)$ and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

This form of the equation is also called the **centre-radius form of the equation of a circle.**

8.3 Standard Form of the Equation of a Circle

The **standard equation** of a circle is obtained by taking origin as the centre. Thus in this form of the equation of the circle, centre is the origin and radius is, say r. Hence we put $h = 0$, $k = 0$ in above equation of a circle and get $x^2 + y^2 = r^2$. This form of the equation is called the **standard form of the equation of a circle.**

Further, if radius $r = 1$, the standard form reduces to $x^2 + y^2 = 1$. This is called the **equation of the unit circle.**

Example 1 : Obtain the equation of the circle with centre $(1, -1)$ and radius 2.

Solution : Here the centre is $(1, -1)$ and the radius is 2. So, the equation of the circle is

$$(x - 1)^2 + (y + 1)^2 = 2^2 = 4$$

That is, $x^2 + y^2 - 2x + 2y - 2 = 0$.

Example 2 : Show that the point, $(2\,sin\alpha, 2\,cos\alpha)$; $\alpha \in R$, lies on the circle, $x^2 + y^2 = 4$.

Solution : We know that a point lies on the circle, if the coordinates of the point satisfy the equation of the circle. Substituting $x = 2\,sin\alpha$, $y = 2\,cos\alpha$ in the given equation, we get

$$L.H.S. = (2\,sin\alpha)^2 + (2\,cos\alpha)^2$$

$$= 4\,sin^2\alpha + 4\,cos^2\alpha = 4 = R.H.S.$$

$\therefore$ $(2\,sin\alpha, 2\,cos\alpha)$, $\alpha \in R$ is on the circle $x^2 + y^2 = 4$.

Example 3 : Find the equation of the circle whose radius is 5 and centre is point of intersection of the lines $x + y = 1$ and $4x + 3y = 0$.

Solution : The point of intersection of the lines is the point satisfying both the equations $x + y = 1$ and $4x + 3y = 0$. Solving them $(-3, 4)$ is the centre of the circle.

Also radius is 5. So the equation of the required circle is,

$$(x + 3)^2 + (y - 4)^2 = 5^2, \text{ that is } x^2 + y^2 + 6x - 8y = 0.$$

Note : If the centre is the point of the intersection of two lines, then the lines contain diameters of the circle.

Example 4 : Find k, if the circle $x^2 + y^2 - 2x + 448y + k = 0$ passes through the origin.

Solution : The circle passes through $(0, 0)$. Substituting $x = 0 = y$ in the equation of the circle $0 + 0 - 0 + 0 + k = 0$. Thus we get $k = 0$.

Note : A circle passes through the origin if and only if the constant term in the equation is equal to zero.

Example 5 : Find the equation of the set of complex numbers $z = x + iy$, so that $| z - z_1 | = 5$, where $z_1 = 1 - 2i$.

Solution : We have $| z - z_1 | = 5$

$\therefore \quad | z - z_1 |^2 = 5^2$

$\therefore \quad | (x + iy) - (1 - 2i) |^2 = 25$

$\therefore \quad | (x - 1) + i(y + 2) |^2 = 25$

$\therefore \quad (x - 1)^2 + (y + 2)^2 = 25 \qquad$ (i)

$\therefore \quad x^2 + y^2 - 2x + 4y - 20 = 0$

From (i) it is clear that the set is a circle with centre (1, −2) and radius 5.

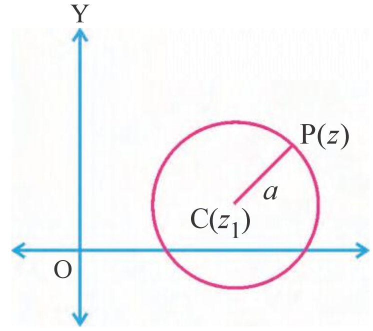


Figure 8.2

Note : In general the set of all complex numbers z satisfying $| z - z_1 | = a$, $a \in R^+$ represents a circle with radius a, centered at z_1. The Argand diagram of the given circle is shown in figure 8.2. In fact if C and P represent z_1 and z respectively in the Argand plane and if $CP = | z - z_1 | = a$ then P is on the circle with centre C and radius a.

Example 6 : Find the equation of the circles which touch X-axis.

Solution : If the radius of the circle is a, then the coordinates of the centre C are $(h, \pm a)$ or $(-h, \pm a)$ (figure 8.3). The equations of the circles are,

$(x - h)^2 + (y \pm a)^2 = a^2 \qquad$ or

$(x + h)^2 + (y \pm a)^2 = a^2$

$x^2 + y^2 - 2hx \pm 2ay + h^2 = 0 \qquad$ or

$x^2 + y^2 + 2hx \pm 2ay + h^2 = 0$

Thus, these four equations represent required circles.

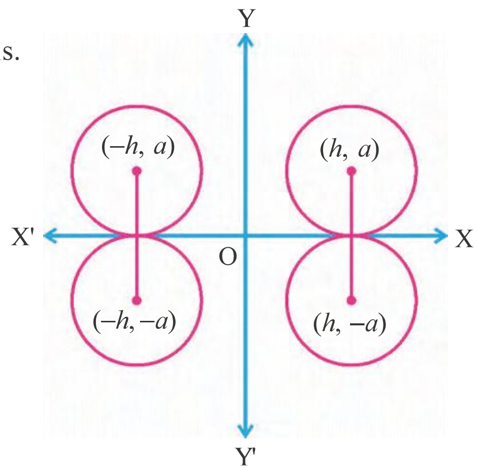


Figure 8.3

Note : If a circle of radius a touches Y-axis (figure 8.4) then its centre will be $(\pm a, k)$ or $(\pm a, -k)$, and hence the equations of such circles are of the form,

$$x^2 + y^2 \pm 2ax + 2ky + k^2 = 0$$

or

$$x^2 + y^2 \pm 2ax - 2ky + k^2 = 0$$

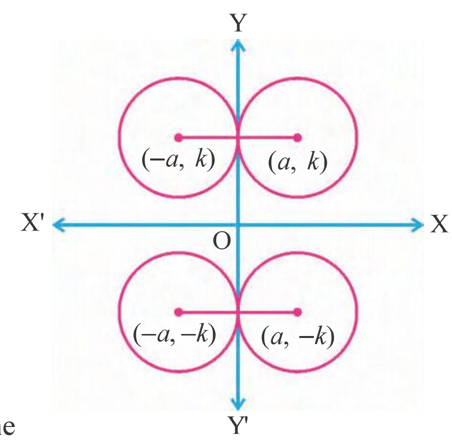


Figure 8.4

Example 7 : Find the equation of the circle with radius a in the first quadrant, if it touches both the axes.

Solution : If the circle touches both the axes in first quadrant, then its centre will be C(a, a) (figure 8.5) and radius a. Hence its equation is $(x - a)^2 + (y - a)^2 = a^2$.

$\therefore \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$ is the equation.

Note : For the circle of radius a, touching both the axes in the other quadrants, the centre has coordinates as given in the following table. (figure 8.5)

Quadrant	Centre
I	(a, a)
II	$(-a, a)$
III	$(-a, -a)$
IV	$(a, -a)$

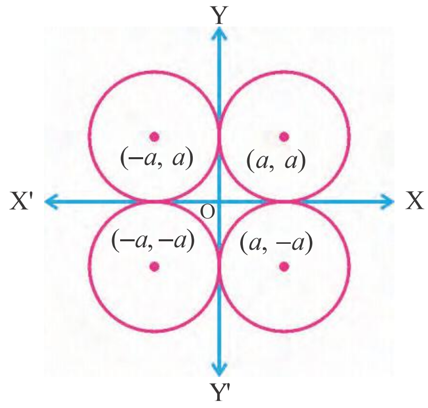


Figure 8.5

Exercise 8.1

1. Find the equation of the circle which radius and centre given below :

No.	Centre	Radius
1.	$(-2, 3)$	5
2.	$(-1, 1)$	$\sqrt{2}$
3.	$(-4\cos\alpha, 4\sin\alpha)$	5
4.	$(-\sqrt{2}, -\sqrt{5})$	$\sqrt{5}$
5.	$(1, 0)$	1

2. Find the equation of the circle for which lines containing the diameters are $x - y = 5$, $2x + y = 4$ and radius is 5.

3. Find the equation of the circle which touches Y-axis and has centre $(-2, -5)$.

4. Find the equation of the circle in the third quadrant having radius 3 and touching both the axes.

5. Find the equation of the circle passing through the origin, having radius $\sqrt{5}$ and having centre on $\overrightarrow{OX}$.

*

8.4 General Form of the Equation of a Circle

As discussed above each circle has unique centre and radius. Let for a circle the centre be (h, k) and radius be r. So, any circle has an equation of the form $(x - h)^2 + (y - k)^2 = r^2$ or $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$. Here h, k are real numbers and r is a positive number. From this equation, we observe the following :

(1) The equation of any circle is a quadratic equation in two variables.

(2) In the equation of a circle, coefficients of x^2 and y^2 are non-zero and equal (We will take these coefficients equal to 1)

(3) There is no xy-term, i.e. the coefficient of the xy-term is 0.

Thus, we take the general form of the equation of a circle in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Now if the equation of a circle is given in the above form, then we wish to determine the centre and radius of the circle. For this we will rearrange the terms, so that the equation reduces to the centre-radius form. Thus,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

$$\Leftrightarrow \quad x^2 + 2gx + g^2 + y^2 + 2fy + f^2 - g^2 - f^2 + c = 0$$

$$\Leftrightarrow \quad (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

If $g^2 + f^2 - c > 0$, then above equation can be written as,

$$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

In fact, above relation tells us that the distance of the point P(x, y) from the point C$(-g, -f)$ is $\sqrt{g^2 + f^2 - c}$.

Thus the equation (i) represents a circle if the constants g, f and c satisfy $g^2 + f^2 - c > 0$; and in this case the centre of the circle is C$(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$. Equation (i) is called the general form of the equation of a circle.

Note : If $g^2 + f^2 - c = 0$, only $(-g, -f)$ satisfies equation (i).

Example 8 : Does the equation $x^2 + y^2 + 6x - 8y + 20 = 0$ represent a circle ? If yes, find its centre and radius.

Solution : Comparing the equation with the general form, we find that $g = 3, f = -4$ and $c = 20$. Thus, $g^2 + f^2 - c = 3^2 + (-4)^2 - 20 = 5 > 0$.

Hence the given equation represents a circle.

The centre of the circle is $(-g, -f) = (-3, 4)$ and the radius is $\sqrt{g^2 + f^2 - c} = \sqrt{5}$.

Another Method :

Adjusting the terms in the equation to get sum of squares, we write

$$x^2 + y^2 + 6x - 8y + 20 = 0$$

$$\therefore \quad x^2 + 6x + 9 + y^2 - 8y + 16 - 5 = 0$$

$$\therefore \quad (x + 3)^2 + (y - 4)^2 = 5$$

This is the equation of the circle centered at C$(-3, 4)$ and radius as $r = \sqrt{5}$.

Example 9 : Determine which of the following equations represents a circle. Find the centre and radius for those which represent a circle.

(1) $x^2 + 2y^2 - 2x + 6y - 8 = 0$

(2) $2x^2 + 2y^2 - 2x + 6y - 8 = 0$

(3) $x^2 + y^2 - 2\sqrt{2}x + y - \frac{91}{4} = 0$

(4) $x^2 + y^2 - 2x\cos\beta + 2y\sin\beta = 0;\ \beta \in R$

(5) $2x^2 + 2y^2 - 2xy + 6y + 22x - 1008 = 0$

(6) $x^2 + y^2 - 4x - 6y + 13 = 0$

Solution : (1) In this equation coefficients of x^2 and y^2 are not equal and hence it is not an equation of a circle.

(2) Dividing the equation by 2 gives $x^2 + y^2 - x + 3y - 4 = 0$. This gives $g = -\frac{1}{2}$, $f = \frac{3}{2}$ and $c = -4$. Now, $g^2 + f^2 - c = \left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - (-4) = \frac{13}{2} > 0$, hence the equation represents a circle with centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and radius $\sqrt{\frac{13}{2}}$.

(3) Here, $g = -\sqrt{2}$, $f = \frac{1}{2}$ and $c = -\frac{91}{4}$. Now, $g^2 + f^2 - c = 2 + \frac{1}{4} + \frac{91}{4} = 25 > 0$. Hence the equation represents a circle with centre $\left(\sqrt{2}, -\frac{1}{2}\right)$ and radius 5.

(4) Here, $g = -cos\beta$, $f = sin\beta$ and $c = 0$.

Now, $g^2 + f^2 - c = cos^2\beta + sin^2\beta = 1 > 0$. Hence the equation represents a circle with centre $(cos\beta, -sin\beta)$ and radius 1.

(5) This equation contains a term with xy and hence it does not represent a circle.

(6) Here, $g = -2$, $f = -3$ and $c = 13$. Now, $g^2 + f^2 - c = (-2)^2 + (-3)^2 - 13 = 0$, hence this equation does not represent a circle.

Note : In above examples for equations given in (2), (3), (4) and (6), we can use the method of sum of squares. Also see that if $c < 0$, then $g^2 + f^2 - c$ is always positive. Hence $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle, if $c < 0$.

Example 10 : Find the equation of the circle passing through the points (1, 1) and (−5, 1) and having centre on the line $x + 3y - 1 = 0$.

Solution : Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ **(i)**

We need to determine the values of constants g, f and c using given conditions. The centre of the circle given in (i) is $(-g, -f)$. Now given that the centre of the required circle lies on the line $x + 3y - 1 = 0$, $(-g, -f)$ satisfies the equation of the line and hence, we get

$-g - 3f - 1 = 0$ or $g + 3f + 1 = 0$ **(ii)**

Also using coordinates of given points in (i), we get

$2g + 2f + c + 2 = 0$ **(iii)**

$-10g + 2f + c + 26 = 0$

i.e., $10g - 2f - c - 26 = 0$ **(iv)**

In (ii), (iii) and (iv), we have a system of three linear equations in three unknowns g, f and c.

Using (iii) + (iv), we get $12g - 24 = 0$

$\therefore$ $g = 2$

$\therefore$ $g + 3f + 1 = 0$ gives $f = -1$

Further, $2g + 2f + c + 2 = 0$

$\therefore$ $4 - 2 + c + 2 = 0$ **(by taking $g = 2$ and $f = -1$)**

$\therefore$ $c = -4$

$\therefore$ $x^2 + y^2 + 4x - 2y - 4 = 0$ is the equation of the required circle.

Example 11 : $A(x_1, y_1)$ and $B(x_2, y_2)$ are given end-points of a diameter of a circle. Find the equation of the circle.

Solution : As shown in the figure 8.6, let $A(x_1, y_1)$ and $B(x_2, y_2)$ be given end-points of a diameter of a circle and $P(x, y)$ be any point on the circle other than A or B. Now as we have studied in standard 10, the angle inscribed in a semi-circle is a right angle. Thus, ΔPAB is a right angled triangle with right angle at P (figure 8.6). According to Pythagoras theorem we have,

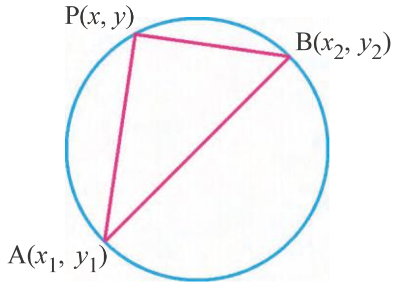


Figure 8.6

$PA^2 + PB^2 = AB^2$

Also we have, $PA^2 = (x - x_1)^2 + (y - y_1)^2$

$PB^2 = (x - x_2)^2 + (y - y_2)^2$

$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$\therefore \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2$

$\Leftrightarrow \quad x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2$

$= x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 + x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$

$\Leftrightarrow \quad -2x_1x_2 - 2y_1y_2 = x^2 - 2xx_1 + y^2 - 2yy_1 + x^2 - 2xx_2 + y^2 - 2yy_2$

$\Leftrightarrow \quad 2x^2 + 2y^2 - 2xx_1 - 2yy_1 - 2xx_2 - 2yy_2 + 2x_1x_2 + 2y_1y_2 = 0$

$\Leftrightarrow \quad x^2 + y^2 - xx_1 - yy_1 - xx_2 - yy_2 + x_1x_2 + y_1y_2 = 0$

This equation can also be written as $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ **(i)**

$A(x_1, y_1)$ and $B(x_2, y_2)$ also satisfy equation (i).

Hence equation (i) represents the circle having diameter $\overline{AB}$.

Other Method :

$\overline{AB}$ is diameter. Hence the centre is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Radius is $\sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} = \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{2}$

$\therefore$ The equation of the circle is

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4}$$

$\Leftrightarrow \quad x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + \frac{(x_1 + x_2)^2 + (y_1 + y_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2}{4} = 0$

$\Leftrightarrow \quad x_1^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$

$\Leftrightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

This is called the diameter form of the equation of a circle.

Note : $\overline{AP}$ and $\overline{PB}$ are perpendicular, hence product of their slopes is equal to -1. Using this also the equation of circle having diameter $\overline{AB}$ can be derived.

Exercise 8.2

1. Which of the following equations represents a circle ? Find the centre and the radius if the equation represents a circle :

(1) $x - y + 4 = 0$ (2) $x^2 + y^2 = 1$

(3) $x^2 + y^2 - 2x - 2y + 1 = 0$ (4) $x^2 - y^2 - 2x + 2y = 1008$

(5) $x^2 + 3y^2 - 6x + 8y = 0$ (6) $3x^2 + 3y^2 - 5x + 6y + 8 = 0$

(7) $x^2 + y^2 - x + y = 0$ (8) $9x^2 - 6x + 9y - 35 = 0$

(9) $x^2 + y^2 - 2x\tan\alpha + 2y\sec\alpha + 2\tan^2\alpha = 0; \left(\alpha \in R, \alpha \neq \frac{(2n+1)\pi}{2}; n \in Z\right)$

(10) $x^2 + y^2 - 2xy\tan\alpha + 2y\sec\alpha + 2\tan^2\alpha = 0; \alpha \in \left[0, \frac{\pi}{2}\right)$

2. Find the equation of the circle with centre (3, 4) and passing through the origin.

3. Find the equation of the circle which passes through the point (2, −1) and whose centre lies on both the lines $x + y = 5$ and $4x + y = 5$.

4. Obtain the equation of the circle that touches both the axes and passes through the point (−6, 3).

5. Show that the centres of the circles $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 1 = 0$ are collinear. Also show that their radii are in G.P.

6. Obtain the equation of the circle for given extremities of diameter using slopes of line segments.

*

8.5 Eccentricity

Geometric Definition of Conics : The set of points whose distance from a fixed point and whose perpendicular distance from a fixed line not passing through the given point are in a constant ratio is called a conic. The fixed point is called a focus of the conic and the fixed line is called a directrix of the conic. This constant ratio is called eccentricity of the conic and it is represented by symbol *e*.

8.6 Parabola

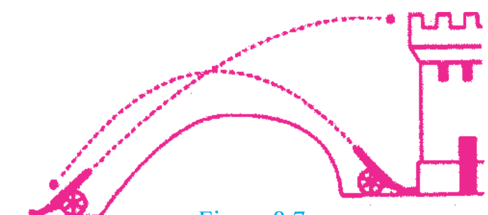


Figure 8.7

In 17th century, Galieo discovered that when an object, say a stone, is thrown in the air, then it follows parabolic path. Here 'para' means 'for' and bola means 'throwing'. Hence the name parabola. This discovery by Galileo made it possible for cannoneers to work out the kind of path a cannonball would travel if it were hurtled through the air at a specific angle. A formal definition of a parabola is as follows :

Definition : A parabola is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane (figure 8.8). Here by the distance of a point from the line, we mean its perpendicular distance.

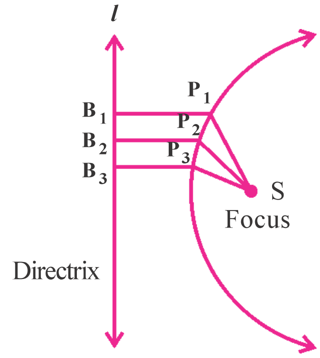


Figure 8.8

Points P_1, P_2 and P_3 are shown on the parabola.

By definition, $B_1P_1 = SP_1$, $B_2P_2 = SP_2$, $B_3P_3 = SP_3$. Similarly for all points on the parabola.

Let S be a fixed point and let l be a fixed line not passing through S. S is called the focus and l the directrix of the parabola.

If P is a point on a parabola and perpendicular distance of P from the directrix is PM, then according to definition of parabola SP = PM.

$\therefore \quad \frac{SP}{PM} = 1$

$\therefore$ According to definition of eccentricity of a conic, parabola is a conic with eccentricity 1.

Let Z be the foot of the perpendicular from S to the line l. Let A be mid-point of $\overline{ZS}$. Thus SA = AZ and $\overline{ZS}$ is perpendicular to the directrix l. Hence A lies on parabola.

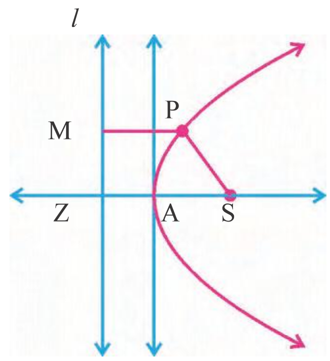


Figure 8.9

Choose A as origin, $\overleftrightarrow{AS}$ as X-axis and direction of $\overrightarrow{AS}$ as positive direction of the X-axis. Further, we take distance ZS = $2a$. Then S would be $(a, 0)$ and $Z(-a, 0)$. The equation of the directrix l would be $x = -a$.

Let $P(x, y)$ be any point on the parabola and let M be the foot of the perpendicular drawn from P to the directrix l. Then the coordinates of M are $(-a, y)$. Now as the point P is on the parabola,

$SP = PM$

$\therefore \quad SP^2 = PM^2$

$\therefore \quad (x - a)^2 + y^2 = (x + a)^2$

$\therefore \quad y^2 = (x + a)^2 - (x - a)^2$

$\therefore \quad y^2 = 4ax$ **(i)**

If a point $P(x, y)$ satisfies the equation $y^2 = 4ax$, then by taking above steps in the reverse order, we get SP = PM, i.e. the point P is on the parabola.

$\therefore$ The equation of the parabola is $y^2 = 4ax$.

This equation is called the standard form of the equation of a parabola.

Example 12 : Find the standard equation of the parabola whose focus is (4, 0) and equation of the directrix is $x + 4 = 0$.

Solution : As the focus is at (4, 0) and the directrix is $x + 4 = 0$, $a = 4$. The equation of the parabola is, $y^2 = 4(4)x$.

$\therefore \quad y^2 = 16x$ is the equation of the parabola.

8.7 Some definitions and results related to parabola

(1) **The axis of a parabola is defined as the line passing through the focus and perpendicular to the directrix. Accordingly for the parabola $y^2 = 4ax$, X-axis is axis of the parabola.**

(2) **The point of intersection of the parabola with its axis is called the vertex of the parabola. For $y^2 = 4ax$, the origin is the vertex.**

(3) **If the axis of the parabola is chosen as Y-axis and vertex at the origin, then the equation of the parabola comes out to be $x^2 = 4by$ (figure 8.10(ii)).** In this case focus is $(0, b)$ and the equation of the directrix is $y = -b$. Here $| b |$ is the distance between the vertex of the parabola and its focus.

(4) If the point P(x, y) is on the parabola $y^2 = 4ax$, then P(x, $-y$) is also a point on the parabola. Thus the parabola $y^2 = 4ax$ is symmetric about X-axis. (i.e. replacing y by $-y$ there is no change in the equation.) Similarly, the parabola $x^2 = 4by$ is symmetric about Y-axis. (i.e. replacing x by $-x$, there is no change in the equation.)

(5) Parabolas $y^2 = 4ax$ and $x^2 = 4by$ are shown in the figures ($a \neq 0$, $b \neq 0$).

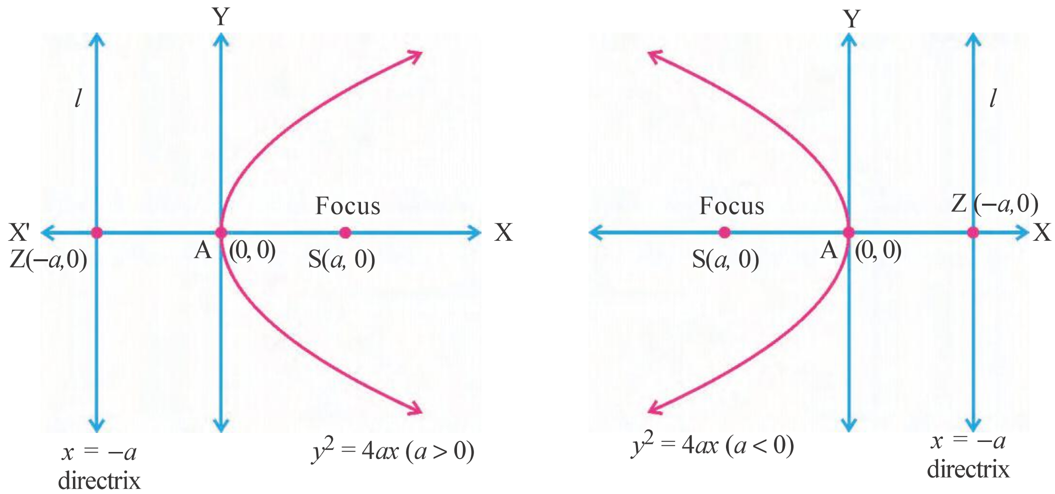


Figure 8.10 (i)

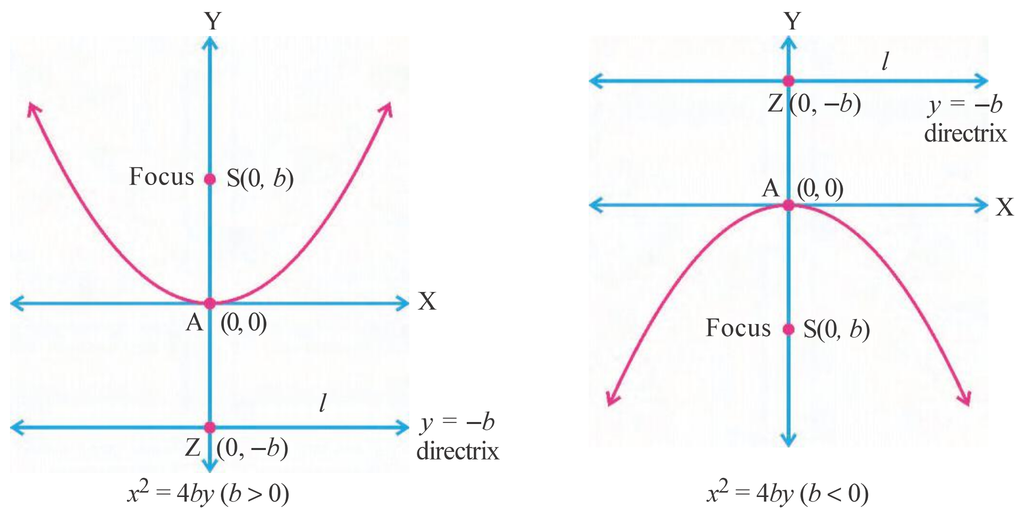


Figure 8.10(ii)

(6) **The line segment joining any two points of a parabola is called a chord of parabola. A chord passing through the focus is called a focal-chord. A focal-chord perpendicular to the axis, (or equivalently, parallel to the directrix), is called the latus-rectum of the parabola.**

8.8 Latus-rectum of a parabola

Let the end-points of the latus-rectum of a parabola $y^2 = 4ax$ be L and L'. Hence the line $\overleftrightarrow{LL'}$ is a vertical line. Since it passes through the focus (a, 0), its equation is $x = a$. Now as the points L and L' are on parabola also, using equation of the parabola we get, $y^2 = 4ax = 4a \cdot a = 4a^2$, which means $y = \pm 2a$. Thus, the coordinates of the end-points of the latus-rectum are L(a, $2|a|$) and L'(a, $-2|a|$). The length of the latus-rectum is the distance LL'.

Length of the latus-rectum $= LL'$

$$= \sqrt{(a-a)^2 + (2|a| + 2|a|)^2}$$

$$= 4|a|$$

Note : For the parabola $x^2 = 4by$, the end-points of the latus-rectum would be $L(2|b|, b)$ and $L'(-2|b|, b)$; and hence the length of the latus-rectum would be $4|b|$.

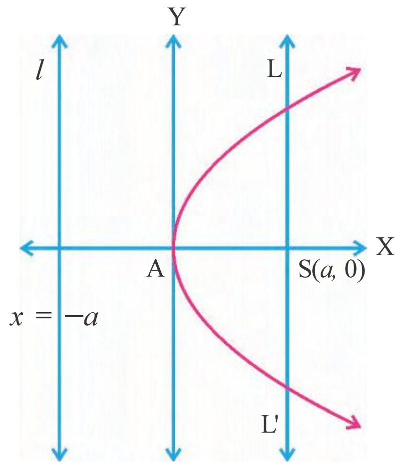


Figure 8.11

8.9 Parametric equations of a parabola

For any real parameter t, $x = at^2$ and $y = 2at$ satisfy the equation $y^2 = 4ax$. Conversely, suppose a point (x_1, y_1) lies on the parabola $y^2 = 4ax$. Let $t = \frac{y_1}{2a}$, then $x_1 = at^2$. In other words corresponding to any point on the parabola $y^2 = 4ax$, there exists a real number t such that $x = at^2$ and $y = 2at$.

Thus, $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ and any point on $y^2 = 4ax$ is of the form $(at^2, 2at)$ for some $t \in R$.

$x = at^2$, $y = 2at$ are called parametric equations of the parabola $y^2 = 4ax$. The point $P(at^2, 2at)$ is called a t-point of the parabola and it is denoted by $P(t)$.

Example 13 : Find the equation of the parabola whose focus is (2, 3) and directrix is $3x + 4y - 10 = 0$.

Solution : Let $P(x, y)$ be a point on the parabola. Now by the definition of a parabola, if focus is S and PM is perpendicular distance of P from the directrix, then

$SP = PM$, i.e. $SP^2 = PM^2$

$$\therefore \quad (x-2)^2 + (y-3)^2 = \left(\frac{3x+4y-10}{\sqrt{9+16}}\right)^2 = \frac{(3x+4y-10)^2}{25}$$

$$\therefore \quad 25(x^2 - 4x + 4 + y^2 - 6y + 9) = 9x^2 + 16y^2 + 24xy - 60x - 80y + 100$$

$\therefore \quad 16x^2 - 24xy + 9y^2 - 40x - 70y + 225 = 0$ is the equation of the required parabola.

Example 14 : By shifting the origin to (4, 3) find the coordinates of the focus and the equation of the directrix for the parabola $(y-3)^2 = 16(x-4)$.

Solution : Let (x', y') be coordinates of $P(x, y)$ with respect to the new origin, then

$x = x' + h = x' + 4, \quad y = y' + k = y' + 3$

$\therefore$ The equation of parabola becomes $(y')^2 = 16x'$.

$\therefore \quad 4a = 16$, i.e. $a = 4$

New coordinates of the focus (x', y') are $(a, 0)$ i.e. $(4, 0)$.

Now, $x = x' + 4, \quad y = y' + 3$

$\therefore$ Original coordinates of the focus are $(8, 3)$.

$\therefore$ The equation of the directrix in new coordinate system : $x' + a = 0$

$x' + 4 = 0$

$\therefore$ Its equation is $x - 4 + 4 = 0$.

$\therefore$ The equation of the directrix is $x = 0$.

Verification : From SP = PM, $(x - 8)^2 + (y - 3)^2 = x^2$,

$\therefore$ $(y - 3)^2 = x^2 - (x^2 - 16x + 64) = 16(x - 4)$.

Example 15 : For each equation given below, find the coordinates of the focus, the equation of the directrix, length of the latus-rectum and coordinates of the end-points of the latus-rectum for the parabola :

(1) $x^2 = -8y$ (2) $y^2 = 8x$ (3) $x^2 = 3y$ (4) $y^2 = -10x$

Solution : (1) Comparing $x^2 = -8y$ with the standard equation $x^2 = 4by$ we get, $4b = -8$. So $b = -2$. Here the axis of the parabola is Y-axis. Hence the focus is $(0, b) = (0, -2)$.

The equation of the directrix is $y = -b$, so $y = 2$ is the equation of the directrix.

The length of the latus-rectum is $4 \mid b \mid = 8$.

The end-points of the latus-rectum are $L(2|b|, b) = L(4, -2)$ and $L'(-2|b|, b) = L'(-4, -2)$.

(2) Comparing $y^2 = 8x$ with the equation $y^2 = 4ax$, we get $a = 2$. Here the axis of the parabola is X-axis.

Focus is $(a, 0) = (2, 0)$.

The equation of the directrix is $x = -a$, i.e. $x = -2$ or $x + 2 = 0$ is the equation of the directrix.

The length of the latus-rectum is $4 \mid a \mid = 8$.

The end-points of the latus-rectum are $L(a, 2|a|) = L(2, 4)$ and $L'(a, -2|a|) = L'(2, -4)$.

(3) Comparing $x^2 = 3y$ with the standard equation $x^2 = 4by$ we get, $4b = 3$, i.e. $b = \frac{3}{4}$. Here the axis of the parabola is Y-axis.

Focus is $(0, b) = \left(0, \frac{3}{4}\right)$.

The equation of the directrix is $y = -b$, so $y = -\frac{3}{4}$ or $4y + 3 = 0$ is the equation of the directrix.

The length of the latus-rectum is $4 \mid b \mid = 3$.

The end-points of the latus-rectum are $L(2|b|, b) = L\left(\frac{3}{2}, \frac{3}{4}\right)$ and $L'(-2|b|, b) = L'\left(-\frac{3}{2}, \frac{3}{4}\right)$.

(4) Comparing $y^2 = -10x$ with the equation $y^2 = 4ax$, we get $a = -\frac{5}{2}$. Here the axis of the parabola is X-axis.

Focus is $(a, 0) = \left(-\frac{5}{2}, 0\right)$.

The equation of the directrix is $x = -a$. So $x = \frac{5}{2}$ or $2x - 5 = 0$ is the equation of the dirextrix.

The length of the latus-rectum is $4 \mid a \mid = 10$.

The end-points of the latus-rectum are $L(a, 2|a|) = L\left(-\frac{5}{2}, 5\right)$ and

$L'(a, -2|a|) = L'\left(-\frac{5}{2}, -5\right)$.

Example 16 : Find the standard equation of the parabola having vertex at origin, focus at $(0, -3)$ and directrix $y = 3$.

Solution : Here focus $(0, -3)$ is on Y-axis and the directrix $y = 3$ is parallel to X-axis. Thus the equation of the parabola is of the form $x^2 = 4by$, with $b = -3$. Thus the equation of the required parabola is $x^2 = -12y$.

Example 17 : Find the standard equation of the parabola symmetric about X-axis, vertex at origin and passing through $(5, -5)$.

Solution : Given that the parabola is symmetric about X-axis and vertex is at origin. Hence the standard form of the equation is $y^2 = 4ax$. Further since the parabola passes through the point $(5, -5)$, we get $(-5)^2 = 4a(5)$

$\therefore \quad 25 = 20a$

$\therefore \quad a = \frac{5}{4}$

Hence the equation of the parabola is $y^2 = 5x$.

8.10 Properties of a Parabola

Property 1 : Let $P(t_1)$ and $Q(t_2)$ be two points on a parabola $y^2 = 4ax$. If $\overline{PQ}$ is a focal chord, then $t_1t_2 = -1$.

Proof : If $\overline{PQ}$ is the latus-rectum, then $P(a, 2a)$.

$\therefore \quad at_1^2 = a, \quad 2at_1 = 2a$

$\therefore \quad t_1 = 1$

Similarly for $Q(a, -2a)$, $t_2 = -1$

$\therefore \quad t_1t_2 = -1$

Now, suppose $\overline{PQ}$ is not the latus-rectum.

$\therefore \quad at_1^2 \neq a, \quad at_2^2 \neq a.$

Now, slope of $\overleftrightarrow{SP}$ = slope of $\overleftrightarrow{SQ}$

$$\therefore \quad \frac{2at_1}{at_1^2 - a} = \frac{2at_2}{at_2^2 - a}$$

$$\therefore \quad \frac{t_1}{t_1^2 - 1} = \frac{t_2}{t_2^2 - 1}$$

$\therefore \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$

$\therefore \quad t_1t_2^2 - t_1 = t_2t_1^2 - t_2$

$\therefore \quad t_1t_2^2 - t_2t_1^2 = t_1 - t_2$

$\therefore \quad t_1t_2 (t_2 - t_1) = -(t_2 - t_1)$

$\therefore \quad t_1t_2 = -1 \qquad (\because t_1 \neq t_2)$

Property 2 : Let S be the focus of the parabola $y^2 = 4ax$, $(a > 0)$ and $\overline{PQ}$ be a focal chord. Then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.

Proof : Let $P(t_1)$ and $Q(t_2)$ be the end-points of a focal chord. The coordinates of the focus are $(a, 0)$. The coordinates of the points P and Q are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively.

$$SP^2 = (at_1^2 - a)^2 + (2at_1)^2$$
$$= (at_1^2 - a)^2 + 4a^2t_1^2$$
$$= (at_1^2 + a)^2$$

$\therefore$ $SP = a(t_1^2 + 1)$ Similarly, $SQ = a(t_2^2 + 1)$ $(a > 0)$

$$\text{Now, } \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t_1^2+1)} + \frac{1}{a(t_2^2+1)}$$
$$= \frac{1+t_1^2+t_2^2+1}{a(t_1^2+1)(t_2^2+1)}$$
$$= \frac{1+t_1^2+t_2^2+t_1^2t_2^2}{a(t_1^2+1)(t_2^2+1)} \qquad (t_1t_2 = -1)$$
$$= \frac{(1+t_1^2)(1+t_2^2)}{a(t_1^2+1)(t_2^2+1)} = \frac{1}{a}$$

Property 3 : Let P be any point on a parabola and S be the focus of the parabola $y^2 = 4ax$. Let $\overleftrightarrow{PQ}$ be a line parallel to the axis of the parabola. Let the bisector of the angle $\angle SPQ$ intersect axis of the parabola in point G. Then $\overline{SP} \cong \overline{SG}$.

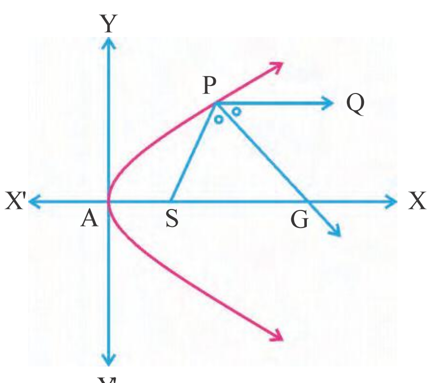


Figure 8.12

Proof : Here $\overleftrightarrow{PQ}$ is parallel to the axis of the parabola, that is to X-axis. Also $\overrightarrow{PG}$ is the bisector of $\angle SPQ$. Thus as shown in the figure 8.12, $\overleftrightarrow{PG}$ is a transversal to parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{SG}$. Hence $m\angle SGP = m\angle QPG$. Also, $m\angle SPG = m\angle QPG$. Thus, $m\angle SGP = m\angle SPG$. Hence ΔSPG is an isosceles triangle with $\overline{SP} \cong \overline{SG}$.

Note : This property has some applications in optics, for designing the mirrors. If a light source is placed at the focus of a parabolic mirror, then the light will travel parallel to the axis of the mirror. This fact is used in head-light of a vehicle, whereas any light ray which is parallel to axis of a parabolic mirror is reflected in to the focus. This is used in a dish antenna of television.

Exercise 8.3

1. Obtain the coordinates of foci, the equations of directrices and draw a rough sketch for following parabola :

(1) $2y^2 = x$ (2) $x^2 = -4y$ (3) $4x^2 = -y$ (4) $y^2 = 12x$

2. Find the standard equation of the parabola satisfying conditions given below :

(1) Vertex (0, 0), focus (0, −2).

(2) Vertex (0, 0), X-axis as axis of the parabola and pasing through (1, −4).

3. (1) Find the equation of the parabola whose focus is (−1, 2) and directrix is $x - y + 1 = 0$.

(2) Find the equation of the parabola whose focus is (−3, −4) and directrix is $3x - 4y - 5 = 0$.

4. Find the length of the latus-rectum and the equation of the directrix of the parabola $(x + 1)^2 = 4(y + 2)$ by shifting the origin to (−1, −2).

5. Find the area of the triangle formed by the end-points of the latus-rectum and the vertex of the parabola $x^2 = 12y$.

6. One end-point of a focal-chord of the parabola $y^2 = 4ax$ is $(at_1^2, 2at_1)$, find its other end-point. From this show that the length of the focal-chord is $\left(t_1 + \frac{1}{t_1}\right)^2$.

7. Distance SP of a point P on the parabola $y^2 = 12x$ from its focus S is 6 units. Find the coordinates of the point P.

*

8.11 Ellipse

Any cylinder sliced at an angle will reveal an ellipse in cross-section. To demonstrate this tilt a glass of water and the surface of the liquid acquires an elliptical outline (figure 8.14). Also in salads, cucumber is often cut obliquely to obtain elliptical slices.

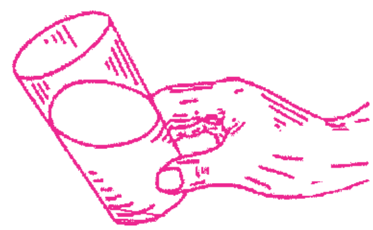


Figure 8.13

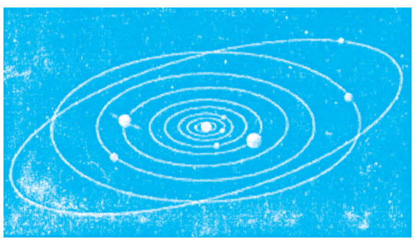


Figure 8.14

The early Greek astronomers thought that the planets moved in circular orbits about the unmoving earth, since the circle is the simplest mathematical curve. In the 17th century, Johannes Kepler eventually discovered that each planet travels around the sun in an elliptical orbit with the sun at one of its foci.

We have discussed eccentricity of a conic section. A conic section with $e < 1$ is called an ellipse.

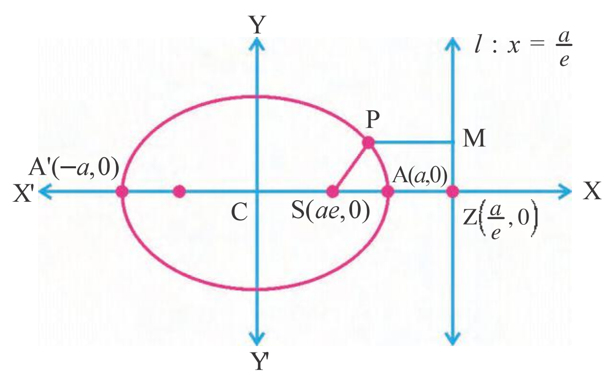


Figure 8.15

Standard equation of an ellipse :

Suppose S is the focus, l is the directrix and e is the eccentricity of an ellipse. Let P be a point on the ellipse. Let M be the foot of perpendicular from P to l.

By definition $e = \frac{SP}{PM}$ **(i)**

Let Z be the foot of the perpendicular from S to l. Let A and A' be two points that divide $\overline{SZ}$ from S in the ratio $e : 1$ and $-e : 1$ respectively.

Thus, $\frac{SA}{AZ} = e$. Also, $\frac{SA'}{A'Z} = e$

SA = distance of focus S from A

AZ = perpendicular distance of A from l. This holds for A' also.

Thus, $\frac{SA}{AZ} = \frac{SA'}{A'Z} = e$ and hence A and A' are both on the ellipse. Suppose C is the mid-point of $\overline{AA'}$. Let C be the origin and direction of $\overrightarrow{CA}$ as the positive direction of the X-axis. Let CA = a. Hence coordinates of A and A' are $(a, 0)$ and $(-a, 0)$ respectively. Let the coordinates of S be $(p, 0)$ and coordinates of Z be $(q, 0)$. As A$(a, 0)$ divides $\overline{SZ}$ from S in ratio $e : 1$, we get

$$a = \frac{eq + p}{e + 1} \quad \textbf{(ii)}$$

Similarly for A' the ratio of division is $-e : 1$.

$$-a = \frac{-eq + p}{-e + 1} \quad \textbf{(iii)}$$

From (ii) and (iii) we have,

$eq + p = ae + a$ and $-eq + p = ae - a$. Solving these equations for p and q,

$p = ae$ and $q = \frac{a}{e}$

Thus focus is S$(ae, 0)$ and coordinates of Z are $\left(\frac{a}{e}, 0\right)$. The directrix passes through Z and it is a vertical line. Its equation is $x = \frac{a}{e}$.

Let P(x, y) be any point on the ellipse. Then from (i)

$\frac{SP}{PM} = e \Leftrightarrow SP = e(PM)$

$\Leftrightarrow SP^2 = e^2(PM^2)$ **(iv)**

Here PM = distance of P(x, y) from the line l,

= distance of P(x, y) from the line $x - \frac{a}{e} = 0$

$$= \frac{\left|x - \frac{a}{e}\right|}{\sqrt{1 + 0}} \quad \left(\textbf{by the formula } \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\right)$$

$$= \left|x - \frac{a}{e}\right|$$

$$\therefore \quad PM^2 = \left(x - \frac{a}{e}\right)^2 \quad \textbf{(v)}$$

Also, $SP^2 = (x - ae)^2 + y^2$ **(vi)**

Using (v) and (vi) in (iv), we get

$$\frac{SP}{PM} = e \Leftrightarrow (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Leftrightarrow (x - ae)^2 + y^2 = e^2\left(x^2 - \frac{2ax}{e} + \frac{a^2}{e^2}\right)$$

$$\Leftrightarrow x^2 - 2aex + y^2 + a^2e^2 = e^2x^2 - 2aex + a^2$$

$$\Leftrightarrow x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad \text{(vii)}$$

Now, as $a > 0$ and $e < 1$, $a^2(1 - e^2) > 0$

Thus, we can choose $b > 0$ such that $a^2(1 - e^2) = b^2$. So (vii) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called the standard equation of the ellipse.

Conclusion :

(1) If the equation of an ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the relation $b^2 = a^2(1 - e^2)$ $(a > b)$ can be used to determine eccentricity of the ellipse.

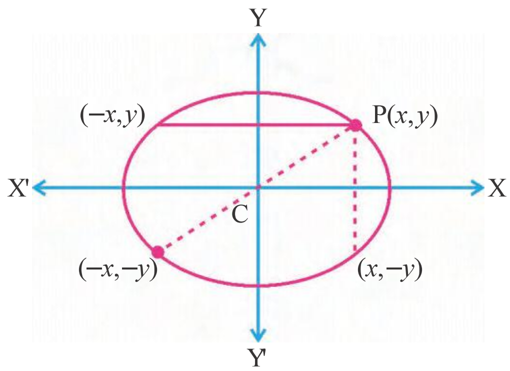


Figure 8.16

(2) Symmetry :

From the standard equation of an ellipse we observe that for any point P(x, y) on the ellipse

(i) the point (x, $-y$) is also on the ellipse, that is, the ellipse is symmteric about X-axis.

(ii) the point ($-x$, y) is on the ellipse, that is, the ellipse is symmetric about Y-axis.

(iii) the point ($-x$, $-y$) is on the ellipse, that is the ellipse is symmetric about the origin C(0, 0). This point C is called centre of the ellipse. And hence ellipse is also called a **central conic.**

(3) Intersection with coordinate axes :

In the derivation of the equation of an ellipse we have taken A(a, 0) and A'($-a$, 0) on the ellipse. Thus the ellipse intersects X-axis at $x = \pm a$. To find the intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with Y-axis, we put $x = 0$ and hence we get $y = \pm b$. Thus the ellipse intersects Y-axis in point B(0, b) and B'(0, $-b$) as shown in the figure 8.16. Similarly it can be observed that the ellipse intersects X-axis in A and A' by taking $y = 0$ in the equation of the ellipse. These points A, A', B and B' are called vertices of the ellipse.

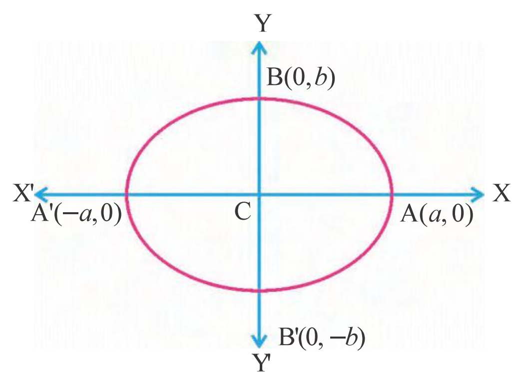


Figure 8.17

(4) Two pairs of focus and directrix :

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ **(i)**

We know that, $b^2 = a^2(1 - e^2)$

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

$\therefore \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\therefore \quad x^2 - x^2e^2 + y^2 = a^2 - a^2e^2$

$\therefore \quad x^2 + 2aex + a^2e^2 + y^2 = x^2e^2 + a^2 + 2aex$

$\therefore \quad (x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$ **(ii)**

To interprete (ii) we take S' = $(-ae, 0)$ and l' the line $x + \frac{a}{e} = 0$.

Now, the perpendicular distance of P(x, y) from l' (say PM') is given by

$$PM' = \frac{\left|x + \frac{a}{e}\right|}{\sqrt{1+0}} = \left|x + \frac{a}{e}\right|$$

$\therefore \quad PM'^2 = \left(x + \frac{a}{e}\right)^2$ **(iii)**

Also, $S'P^2 = (x + ae)^2 + y^2$ **(iv)**

From (iii) and (iv), (ii) gives,

$(S'P)^2 = e^2\,(PM')^2$

$\therefore \quad \frac{S'P}{PM'} = e$

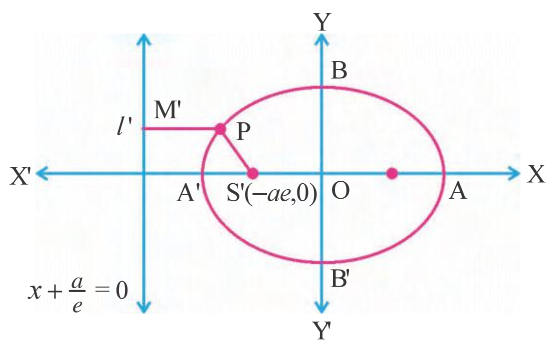


Figure 8.18

By the definition of eccentricity, S' can be taken as focus and l' as directrix. Thus an ellipse has two foci $(\pm ae, 0)$ and two corresponding directrices $x \mp \frac{a}{e} = 0$.

(5) It was seen that an ellipse is symmetric about $\overline{AA'}$ and $\overline{BB'}$. These line segments are called **axes of the ellipse**. Also AA' = $2a$ and BB' = $2b$ and $b < a$. Thus **$\overline{AA'}$ is called major axis** and **$\overline{BB'}$ is called minor axis** and b is called the length of **semi-minor axis, a is called the length of semi-major axis.**

Here major axis is along X-axis. If major axis is along Y-axis. Then foci of the ellipse are on Y-axis and directrices are parallel to X-axis. The equation of such an ellipse is,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $b > a$ and also $a^2 = b^2(1 - e^2)$.

Also, the coordinates of foci are $(0, \pm be)$, the equations of corresponding directrices are $y \mp \frac{b}{e} = 0$.

(6) In analogy with the case of a parabola, chord and focal chord of an ellipse are defined. But, as an ellipse has two foci, it has two latera-recta (figure 8.19). As shown in the figure end-points of latera-recta in different quadrants are denoted by L_1, L_2, L_3 and L_4. $\overline{L_1L_4}$ and $\overline{L_2L_3}$ are two latera-recta.

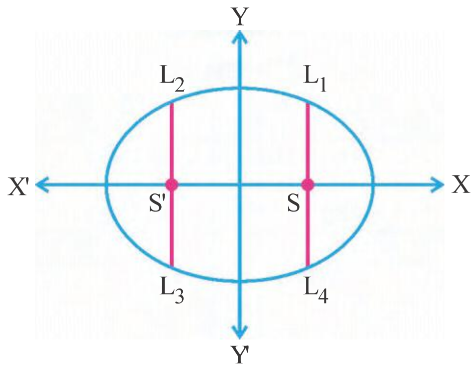


Figure 8.19

(7) Length of latera-recta :

Consider a latus-rectum $\overline{L_1L_4}$ passing through the focus S(ae, 0). Since $\overline{L_1L_4}$ is parallel to Y-axis, its length is the difference of y-coordinates of L_1 and L_4. To determine y-coordinates of L_1 and L_4, we put $x = ae$ in the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Thus, we get

$$e^2 + \frac{y^2}{b^2} = 1$$

$\therefore \quad y^2 = b^2(1 - e^2)$

But $1 - e^2 = \frac{b^2}{a^2}$

$\therefore \quad y^2 = \frac{b^4}{a^2}$

$\therefore \quad y = \pm \frac{b^2}{a}$

$\therefore$ y-coordinates of L_1 and L_4 are $\frac{b^2}{a}$ and $-\frac{b^2}{a}$ respectively. Hence

$$L_1L_4 = \frac{b^2}{a} - \left(-\frac{b^2}{a}\right) = \frac{2b^2}{a}$$

$$L_1\left(ae, \frac{b^2}{a}\right) \text{ and } L_4\left(ae, -\frac{b^2}{a}\right).$$

Similarly $L_2 = \left(-ae, \frac{b^2}{a}\right)$ and $L_3 = \left(-ae, -\frac{b^2}{a}\right)$.

$\therefore$ **The length of a latus-rectum** $= \frac{2b^2}{a}$

Example 18 : Obtain the equation of the ellipse whose focus has coordinates (2, 0), the equation of corresponding directrix is $x - 5 = 0$ and eccentricity is $\frac{1}{\sqrt{2}}$.

Solution : Let P(x, y) be any point on the ellipse, S be the focus and PM the perpendicular distance of P from directrix.

$\therefore$ $SP^2 = e^2PM^2$

$\therefore$ $(x - 2)^2 + y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 (x - 5)^2$

$\therefore$ $2(x^2 - 4x + 4 + y^2) = x^2 - 10x + 25$

$\therefore$ $x^2 + 2y^2 + 2x - 17 = 0$ is the equation of required ellipse.

Example 19 : By shifting the origin to (1, 2), prove that $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$ is the equation of an ellipse. Also find the coordinates of foci and the equation of directrices.

Solution : In standard notations taking $x = x' + 1$, $y = y' + 2$,

the transformed equation takes the form $\frac{(x')^2}{16} + \frac{(y')^2}{9} = 1$, which represents an ellipse.

$a^2 = 16$, $b^2 = 9$

As $b^2 = a^2(1 - e^2)$, we get $9 = 16(1 - e^2)$

$\therefore$ $e^2 = 1 - \frac{9}{16} = \frac{7}{16}$

$\therefore$ $e = \frac{\sqrt{7}}{4}$ **$(e > 0)$**

The coordinates of foci $(\pm ae, 0) = (\pm\sqrt{7}, 0)$ and the equations of corresponding directrices are

$x' \mp \frac{16}{\sqrt{7}} = 0$ **(in $x' - y'$ coordinate system)**

$\therefore$ In the original coordinate system the coordinates of foci are $\left(1 \pm \frac{\sqrt{7}}{4}, 2\right)$ and

the equations of corresponding directrices are $x - 1 \mp \frac{16}{\sqrt{7}} = 0$.

Example 20 : Find the coordinates of foci, the equations of directrices, eccentricity and length of the latus-rectum for each of the following ellipses :

(1) $\frac{x^2}{9} + y^2 = 1$ (2) $4x^2 + y^2 = 25$

Solution : (1) $\frac{x^2}{9} + y^2 = 1$ gives $a^2 = 9$, $b^2 = 1$. So $a = 3$, $b = 1$.

As $a > b$, the major axis is along X-axis.

(i) Eccentricity : We have $b^2 = a^2(1 - e^2)$

$\therefore$ $1 = 9(1 - e^2)$

$\therefore$ $\frac{1}{9} = 1 - e^2$

$\therefore$ $e^2 = \frac{8}{9}$

$\therefore$ $e = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

(ii) Foci : $(\pm ae, 0) = \left(\pm 3\left(\frac{\sqrt{8}}{3}\right), 0\right) = (\pm 2\sqrt{2}, 0)$

(iii) Directrices : $x = \pm\frac{a}{e}$

$\therefore$ $x = \pm 3\left(\frac{3}{\sqrt{8}}\right) = \pm\frac{9}{\sqrt{8}} = \pm\frac{9}{2\sqrt{2}}$

The equations of directrices are $x \pm \frac{9}{2\sqrt{2}} = 0$.

(iv) Length of latus-rectum : $\frac{2b^2}{a} = \frac{2}{3}$

(2) From the given equation, we get $\frac{4x^2}{25} + \frac{y^2}{25} = 1$ i.e. $\frac{x^2}{\left(\frac{25}{4}\right)} + \frac{y^2}{25} = 1$

Thus, $a^2 = \frac{25}{4}$, $b^2 = 25$

$\therefore$ $a = \frac{5}{2}$, $b = 5$. Hence $b > a$

$\therefore$ The major axis is along Y-axis.

(i) Eccentricity : $a^2 = b^2(1 - e^2)$

$\therefore$ $\frac{25}{4} = 25(1 - e)^2$

$\therefore$ $1 - e^2 = \frac{1}{4}$

$\therefore$ $e^2 = \frac{3}{4}$

$\therefore$ $e = \frac{\sqrt{3}}{2}$

(ii) Foci : $(0, \pm be) = \left(0, \pm 5\left(\frac{\sqrt{3}}{2}\right)\right) = \left(0, \pm \frac{5\sqrt{3}}{2}\right)$

(iii) Directrices : $y = \pm\frac{b}{e}$

So, $y = \pm 5\left(\frac{2}{\sqrt{3}}\right)$

$\therefore$ $y \pm \frac{10}{\sqrt{3}} = 0$ are the equations of directrices.

(iv) Length of latus-rectum : $\frac{2a^2}{b} = 2\left(\frac{25}{4}\right)\left(\frac{1}{5}\right) = \frac{5}{2}$

Example 21 : In each of the following cases, find the standard equation of the ellipse :

(1) Length of the major axis 6, eccentricity $\frac{1}{3}$ and major axis along X-axis.

(2) Length of the latus-rectum 8, eccentricity $\frac{1}{\sqrt{2}}$, major axis along Y-axis.

Solution : (1) Here major axis is along X-axis and length of the major axis is 6.

$\therefore$ $2a = 6$. So, $a = 3$

Hence $a^2 = 9$. Further $e = \frac{1}{3}$

Now, $b^2 = a^2(1 - e^2)$

$\therefore$ $b^2 = 9(1 - e^2) = 9\left(1 - \frac{1}{9}\right) = 9\left(\frac{8}{9}\right) = 8$

$\therefore$ The equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{8} = 1$.

(2) Here the major axis is along Y-axis.

$\therefore$ Length of the latus-rectum $\frac{2a^2}{b} = 8$. Hence $a^2 = 4b$ **(i)**

Also, eccentricity $e = \frac{1}{\sqrt{2}}$ and $a^2 = b^2(1 - e^2) = b^2\left(1 - \frac{1}{2}\right)$

$\therefore$ $a^2 = \frac{1}{2}b^2$ **(ii)**

From (i) and (ii), we get

$$\frac{1}{2}b^2 = 4b$$

$\therefore \quad b^2 - 8b = 0$

$\therefore \quad b = 8$ as $b \neq 0$.

$\therefore \quad b^2 = 64$

$\therefore \quad a^2 = \frac{b^2}{2} = \frac{64}{2} = 32$

Thus, the equation of the ellipse is $\frac{x^2}{32} + \frac{y^2}{64} = 1$.

Example 22 : Find the equation of ellipse whose major axis is along X-axis, length of semi-minor axis is 4 and distance between two foci is 5.

Solution : Here, length of the semi-minor axis $b = 4$. Major axis is along X-axis

Let S$(ae, 0)$, S'$(-ae, 0)$ be foci. Then the distance between them is SS' $= 2ae = 5$.

$\therefore \quad ae = \frac{5}{2}$ **(i)**

Also, $b^2 = a^2(1 - e^2) = a^2 - a^2e^2$

$16 = a^2 - \left(\frac{5}{2}\right)^2 = a^2 - \frac{25}{4}$ **(from (i))**

$\therefore \quad a^2 = 16 + \frac{25}{4} = \frac{89}{4}$

Thus, the equation of the required ellipse is $\frac{x^2}{\frac{89}{4}} + \frac{y^2}{16} = 1$.

$\therefore \quad \frac{4x^2}{89} + \frac{y^2}{16} = 1$

Exercise 8.4

1. Find the standard equation of the ellipse in each of the following :

(1) Foci $(\pm 2, 0)$, eccentricity $= \frac{1}{2}$

(2) Foci $(\pm 4, 0)$, vertices $(\pm 5, 0)$

(3) Length of the semi-minor axis 6, eccentricity $\frac{4}{5}$, major axis along X-axis.

(4) A focus $(0, 4)$, eccentricity $\frac{4}{5}$

(5) Eccentricity $\frac{2}{3}$, length of a latus-rectum 5, major axis along X-axis.

(6) Length of semi-major axis 4, eccentricity $\frac{1}{2}$, major axis along X-axis.

(7) Length of semi-minor axis 8, a focus $(0, 6)$.

2. If possible, find the equation of the ellipse whose foci are $(\pm 3, 0)$ and which passes through the point $(4, 1)$.

3. Find the coordinates foci, eccentricity, the equations of directrices and length of the latus-rectum for the following ellipses :

(1) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (2) $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (3) $x^2 + 2y^2 = 100$

(4) $\frac{x^2}{43} + \frac{7y^2}{688} = 1$ (5) $5x^2 + 9y^2 = 81$

4. Find the eccentricity of the ellipse in which the distance between the two directrices is three times the distance between the foci.

5. Find the equations of directrices of the ellipse $16x^2 + 25y^2 = 1600$. Show that the point $(5\sqrt{3}, 4)$ lies on the ellipse. Find the ratio of distance of this point from a directrix to its distance from the corresponding focus.

6. Show that the line $x + y = 3$ contains to a focal chord of the ellipse $20x^2 + 36y^2 = 405$.

7. Find the equation of the ellipse passing through the points (4, 3) and (−1, 4).

8. Find the equation of the ellipse having eccentricity $\frac{1}{2}$, a focus (3, 2) and corresponding directrix $y = 5$.

9. Shift the origin (2, 1) and prove that $\frac{(x-2)^2}{4} + \frac{(y-1)^2}{9} = 1$ represents an ellipse. Find the coordinates of foci and the equations of directrices.

*

8.12 Parametric Equations of an Ellipse

The equation of an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence $\left(\frac{x}{a}, \frac{y}{b}\right)$ is on the unit circle.

Sum of two squares is 1.

$\therefore \quad \exists\, \theta \in (-\pi, \pi]$ such that $\frac{x}{a} = cos\theta, \frac{y}{b} = sin\theta$

$\therefore \quad x = acos\theta, y = bsin\theta$

Further elimination of θ from $x = acos\theta$, $y = bsin\theta$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Thus we see that $x = acos\theta$, $y = bsin\theta$, $\theta \in (-\pi, \pi]$ are parametric equations of the ellipse. The point $(a\, cos\theta, b\, sin\theta)$ on the ellipse is called the θ-point.

Properties of an Ellipse :

Property 1 : The distance of a focus of an ellipse from an end-point of the minor axis is equal to the length of the semi-major axis.

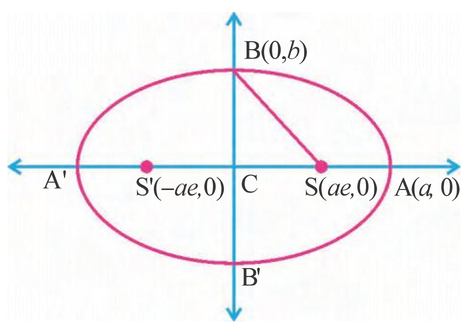


Figure 8.20

Proof : An end-point of the minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is B(0, b). The coordinates of one of the focus S are (ae, 0).

$\therefore \quad SB^2 = a^2e^2 + b^2 = a^2e^2 + a^2(1 - e^2) = a^2$

$\therefore \quad SB = a$

Similarly, for S'($-ae$, 0); $S'B^2 = a^2e^2 + b^2 = a^2$

$\therefore \quad S'B = a$

Also, the other end-point of the minor axis is B'(0, $-b$). For this point we also can show that, SB' = a = S'B'.

Property 2 : If S is a focus and A and A' are extremities of the major axis, then AS · A'S = b^2.

Proof : Here focus is S(ae, 0), A(a, 0) and A'($-a$, 0).

$\therefore$ AS · A'S $= \sqrt{(a-ae)^2}\sqrt{(a+ae)^2}$

$= a(1 - e)\, a(1 + e)$ **($0 < e < 1$)**

$= a^2(1 - e^2) = b^2$

Property 3 : For every point P(x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, SP + S'P = $2a$, where S and S' are foci and $b < a$.

Proof : The directricies of the ellipse are $x \pm \frac{a}{e} = 0$. Thus the distance of the point P(x, y) from respective directrices is $\left|\frac{a}{e} \mp x\right|$. By definition of the ellipse we have,

$SP = e\left|\frac{a}{e} - x\right| = |a - ex|$

$S'P = e\left|\frac{a}{e} + x\right| = |a + ex|$

Also as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so $\frac{x^2}{a^2} \le 1$

$\therefore$ $|x| \le a$. Also $e < 1$

$\therefore$ $|ex| < a$ or $-a < ex < a$

$\therefore$ $a - ex > 0$ and also $a + ex > 0$

$\therefore$ SP $= a - ex$, S'P $= a + ex$

$\therefore$ SP + S'P $= 2a$

The converse of above property is also true. That is, the set of all points in the plane, the sum of whose distances from two fixed points in the plane is a constant is an ellipse whose major axis has the same length as the constant.

To prove this result we proceed as follows :

Suppose S(c, 0) and S'($-c$, 0) are two fixed points in the plane. These points are selected so that the origin C the is mid-point of $\overline{SS'}$ and the direction of $\overrightarrow{CS}$ is the positive direction of the X-axis. Suppose P is a point in the plane such that SP + S'P = $2a$, where a is a constant. ($a \ne c$)

P $\notin \overline{SS'}$ **(If P $\in \overline{SS'}$, SP + S'P = SS' i.e. $2a = 2c$)**

$\therefore$ SP + S'P > SS'

$\therefore$ $2a > 2c$ **(i)**

Now, SP + S'P $= 2a$

$\therefore$ $\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

$\therefore$ $\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

$\therefore$ $(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x - c)^2 + y^2$

$\therefore \quad a\sqrt{(x-c)^2+y^2} = a^2 - cx$

$\therefore \quad \sqrt{(x-c)^2+y^2} = a - \frac{c}{a}x$

$\therefore \quad$ Taking $\frac{c}{a} = e$, $\sqrt{(x-c)^2+y^2} = a - ex$

$\therefore \quad \sqrt{(x-ae)^2+y^2} = a - ex \qquad (c = ae)$

$\therefore \quad (x - ae)^2 + y^2 = (a - ex)^2$

$\therefore \quad x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$

$\therefore \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

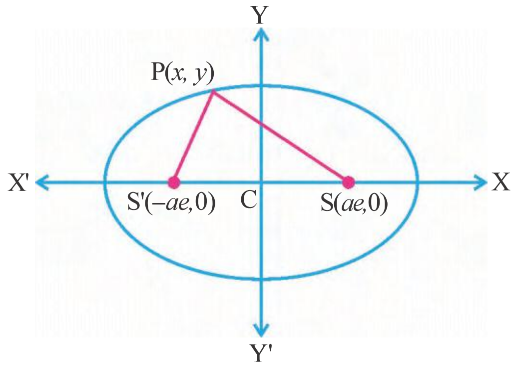


Figure 8.21

Since by (i) $a > c$, $e = \frac{c}{a} < 1$. Hence $a^2(1 - e^2) > 0$.

Thus there exists a positive real number b such that $b^2 = a^2(1 - e^2)$.

Thus, we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This is an ellipse with length of major axis equal to $2a$.

This property is often used as a definition of an ellipse.

An important application of ellipse :

If a source of light (or sound or in general any wave) is placed at one focus S of an elliptic mirror, then after reflection from the mirror, light will reach the other focus S'.

This property of ellipses was used by ancient Indian architects in construction of whispering gallaries. Some whispering galaries are found at Bijapur in Karnataka and Golkonda Fort in Hydrabad. In the design of telescopes this property of an ellipse is also used.

Further, in medical science, this property of ellipses is used in lithotripper which is used to break stones in kidney or bladder. Here, the lithotripper is placed at one focus of an ellipse and ultra-high frequency, shock-waves are produced at the other focus. The reflected waves break the kidney stone.

Example 23 : Find parametric equations of the ellipse $3x^2 + 5y^2 = 15$.

Solution : Dividing given equation by 15, we get

$$\frac{x^2}{5} + \frac{y^2}{3} = 1$$

Thus we get $a = \sqrt{5}$, $b = \sqrt{3}$ and hence parametric equations of the ellipse are $x = \sqrt{5}\,cos\theta$, $y = \sqrt{3}\,sin\theta$. $\theta \in (-\pi, \pi]$

Example 24 : Find the coordinates of foci, the equations of directrices and eccentricity of the ellipse, $x = 2cos\theta$, $y = 5sin\theta$.

Solution : Here $a = 2$, $b = 5$. Since $b > a$ major axis of the ellipse is along Y-axis.

(1) Eccentricity : We have $a^2 = b^2(1 - e^2)$

$\therefore \quad 4 = 25(1 - e^2)$

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$\therefore \quad \frac{4}{25} = 1 - e^2$

$\therefore \quad e^2 = 1 - \frac{4}{25} = \frac{21}{25}$. Thus $e = \frac{\sqrt{21}}{5}$

(2) The coordinates of Foci : $(0, \pm be) = \left(0, \pm 5\frac{\sqrt{21}}{5}\right) = (0, \pm\sqrt{21})$

(3) The equations of Directrices : $y = \pm\frac{b}{e} = \pm 5 \times \frac{5}{\sqrt{21}} = \pm\frac{25}{\sqrt{21}}$

Exercise 8.5

1. Obtain parametric equations of the following ellipses :

(1) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (2) $\frac{x^2}{16} + \frac{y^2}{12} = 1$

(3) $3x^2 + 4y^2 - 12 = 0$ (4) $\frac{x^2}{16} + \frac{y^2}{7} = 1$

(5) $x^2 + 2y^2 - 18 = 0$

2. Find eccentricity and foci of the following ellipses :

(1) $x = 2cos\theta, y = 3sin\theta, \quad \theta \in (-\pi, \pi]$

(2) $3x = 5cos\theta, 5y = 7sin\theta, \quad \theta \in (-\pi, \pi]$

(3) $x = 4cos\theta, y = 3sin\theta, \quad \theta \in (-\pi, \pi]$

3. If the sum of distances of a variable point P from points S(1, 0) and S'(−1, 0) is constant and equal to 8, then find the set of points.

*

Hyperbola : Hyperbola is an important curve used in military sciences. For example, source of a fired bullet can be determined by properties of a hyperbola and intensity of sound.

A conic with eccentricity $e > 1$ is called a hyperbola.

Standard Equation of a Hyperbola :

Suppose S is the focus, line l is the directrix and e is the eccentricity of a hyperbola. Let Z be the foot of the perpendicular on l drawn from S. Now let A and A' divide $\overline{SZ}$ from S in the ratio $e : 1$ and $-e : 1$ respectively. Since $\frac{SA}{AZ} = e$ and $\frac{SA'}{A'Z} = e$, A and A' are on the hyperbola.

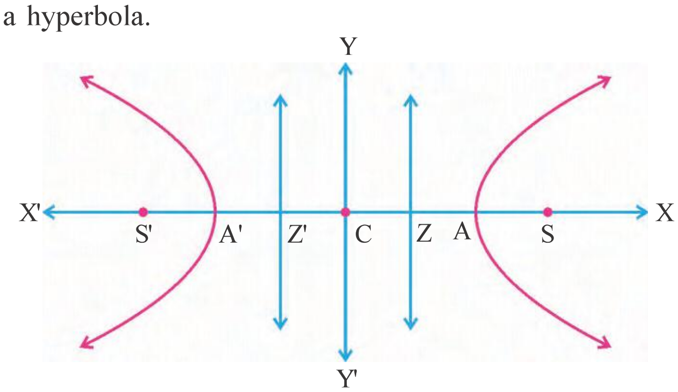

Figure 8.22

Let AA' = $2a$ and C is the mid-point of $\overline{AA'}$. Also CA = CA' = a.

Let C be the origin and take $\overrightarrow{CA}$ as the positive direction of X-axis. Then A = $(a, 0)$ and A' = $(-a, 0)$. Let the coordinates of S and Z be $(p, 0)$ and $(q, 0)$ respectively. Since A and A' divide $\overline{SZ}$ in the ratio e and $-e$,

$$\frac{eq + p}{e + 1} = a \text{ and } \frac{-eq + p}{-e + 1} = -a$$

$\therefore \quad eq + p = ae + a$ and $-eq + p = ae - a$

$\therefore \quad p = ae$ and $q = \frac{a}{e}$

$\therefore$ The coordinates of the focus S are $(ae, 0)$ and the equation of the directrix l is $x = \frac{a}{e}$.

Suppose $P(x, y)$ is a point on the hyperbola and M is the foot of the perpendicular on directrix l drawn from P. Thus coordinates of M are $\left(\frac{a}{e}, y\right)$.

Now, $\frac{SP}{PM} = e \Leftrightarrow SP^2 = e^2 PM^2$

$$\Leftrightarrow (x - ae)^2 + y^2 = (ex - a)^2$$

$$\Leftrightarrow x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$

$$\Leftrightarrow (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$$

$$\Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Here $a^2 > 0$ and $e > 1$. Hence $e^2 - 1 > 0$

$\therefore \quad a^2(e^2 - 1) > 0$. Thus there exists a real number b such that $a^2(e^2 - 1) = b^2$.

$\therefore$ **$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of a hyperbola.**

Some conclusions can be drawn from the standard equation, they are discussed below :

(1) Symmetry :

Hyperbola is symmetric about both the axes and also symmetric about the origin. Also, origin is centre and hence hyperbola is also a central conic.

(2) Intersection with axes :

To obtain intersection of a hyperbola with axes, we put $y = 0$ in the equation of the hyperbola.

We get, $\frac{x^2}{a^2} = 1 \Rightarrow x = \pm a$

So the hyperbola intersects X-axis in the points $A(a, 0)$ and $A'(-a, 0)$. A and A' are called the vertices of the hyperbola.

Putting $x = 0$ in the equation of hyperbola we get $y^2 = -b^2$. As $b \neq 0$, for no real value of y, $y^2 = -b^2$. Thus hyperbola does not intersect Y-axis. In analogy with ellipse the points $B(0, b)$ and $B'(0, -b)$ are also called vertices of the hyperbola, here we note that these points are not on the hyperbola. In the case of a hyperbola $\overline{AA'}$ and $\overline{BB'}$ are called **Transverse axis** and **Conjugate axis** respectively.

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, does not intersect Y-axis but it lies on both sides of the Y-axis. Two parts of the hyperbola have no point in common and they are called **branches** of the hyperbola.

(3) A second pair of focus and directrix :

The equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$

$\therefore \quad (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$

$\therefore \quad x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$

$\therefore \quad (x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$

Let S'$(-ae\ ,\ 0)$ and line l' : $x + \frac{a}{e} = 0$

Let M' be the foot of perpendicular drawn from P$(x,\ y)$ to l'.

$\therefore \quad (\text{S'P})^2 = e^2(\text{P'M})^2$

$\therefore$ The second directrix of the hyperbola is $x + \frac{a}{e} = 0$.

Thus, for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, there are two foci $(\pm ae,\ 0)$ and corresponding directrices are $x \mp \frac{a}{e} = 0$.

(4) Chords, Focal chords and Latera-recta :

A line segment joining two points of a hyperbola is called a **chord** of the hyperbola. If a chord passes through a focus, then it is called a **focal chord** of the hyperbola. A focal chord perpendicular to the transverse axis of the hyperbola is called a **latus-rectum** of the hyperbola.

(5) Length of a latus-rectum :

Consider a latus-rectum $\overline{\text{L}_1\text{L}_4}$ passing through a focus S$(ae,\ 0)$, as shown in the figure 8.23. The equation of the latus-rectum $\overleftrightarrow{\text{L}_1\text{L}_4}$ is $x = ae$. Thus x-coordinates of L_1 and L_4 both are ae. Using $x = ae$ in the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\frac{(ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{y^2}{b^2} = e^2 - 1$$

$$\therefore \quad y^2 = b^2(e^2 - 1)$$

$$= b^2 \cdot \frac{b^2}{a^2}$$

$$= \frac{b^4}{a^2}$$

$$\therefore \quad y = \pm \frac{b^2}{a}$$

$$\therefore \quad \text{L}_1\left(ae,\ \frac{b^2}{a}\right) \text{ and } \text{L}_4\left(ae,\ -\frac{b^2}{a}\right)$$

$$\therefore \quad \text{L}_1\text{L}_4 = \frac{2b^2}{a}$$

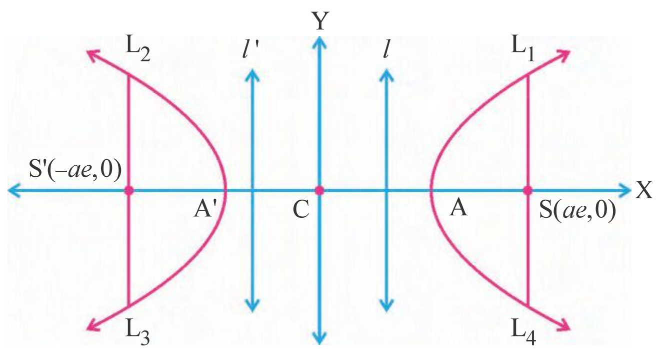

Figure 8.23

(6) Another form of the equation of a hyperbola :

In analogy with ellipse, we can consider hyperbola with transverse axis along Y-axis. The equation would be

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

This hyperbola is said to be **conjugate hyperbola** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Parametric equations of hyperbola :

Comparing the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the trigonometric identity,

$sec^2\theta - tan^2\theta = 1$.

Now for a given point (x, y) on the hyperbola, we choose θ such that $-\pi < \theta \le \pi$; $\theta \ne \frac{\pi}{2}, -\frac{\pi}{2}$ such that $x = a\,sec\theta$, $y = b\,tan\theta$.

Conversely, for any $\theta \in (-\pi, \pi] - \left\{\frac{\pi}{2}, -\frac{\pi}{2}\right\}$, if we take $x = a\,sec\theta$, $y = b\,tan\theta$, then the point (x, y) is on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Here θ is a parameter. In analogy with earlier situations the point $(a\,sec\theta, b\,tan\theta)$ is referred to as θ-point of the hyperbola. Similarly parametric equations of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $x = a\,tan\theta$, $y = b\,sec\theta$, $\theta \in (-\pi, \pi] - \left\{\frac{\pi}{2}, -\frac{\pi}{2}\right\}$.

Rectangular Hyperbola :

If $a^2 = b^2$ for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then it is called a **rectangular hyperbola**. Thus the standard equation of a rectangular hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \quad \text{or} \quad x^2 - y^2 = a^2$$

Eccentricity : For a hyperbola, eccentricity is given by $b^2 = a^2(e^2 - 1)$.

For a rectangular hyperbola, we have $a^2 = b^2$.

$\therefore \quad a^2 = a^2(e^2 - 1)$

$\therefore \quad e^2 = 2$

$\therefore \quad e = \sqrt{2}$ **(as $e > 1$)**

θ-point : A θ-point on a rectangular hyperbola is $(a\,sec\theta, a\,tan\theta)$.

Length of a latus-rectum : Length of the latus-rectum of a hyperbola is given by $\frac{2b^2}{a}$. Here $b^2 = a^2$. Hence length of the latus-rectum of a rectangular hyperbola is $2a$.

Properties of a hyperbola :

If S and S' are foci of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and P is any point on the hyperbola then $|SP - S'P|$ is constant.

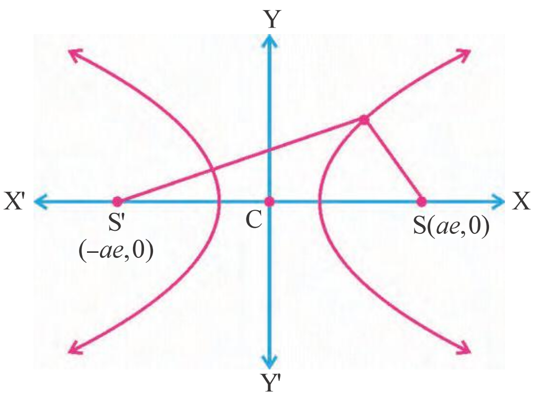

Figure 8.24

Proof : The foci are $S(ae, 0)$ and $S(-ae, 0)$.

Now, $SP = ePM$

$= e\left|x - \frac{a}{e}\right|$

Here $\overline{PM}$ is perpendicular to the directrix $x = \frac{a}{e}$ from $P(x, y)$.

$\therefore \quad SP = ePM = e\left|x - \frac{a}{e}\right| = |ex - a|$

$\therefore \quad SP = |ex - a|$. Similarly $S'P = |ex + a|$

$\therefore$ $(SP - S'P)^2 = SP^2 + S'P^2 - 2SP \cdot S'P$

$= (ex - a)^2 + (ex + a)^2 - 2\,|\,e^2x^2 - a^2\,|$

$= (ex - a)^2 + (ex + a)^2 - 2(e^2x^2 - a^2)$ $\quad$ **$(e^2 > 1, x^2 \geq a^2 \Rightarrow e^2x^2 > a^2)$**

$= 4a^2$

$\therefore$ $|\,SP - S'P\,| = 2a$

Note : The converse of above is also true. Thus we have an equivalent definition, "hyperbola is the set of points (in a plane), the difference of whose distance from two fixed points in the plane is constant."

Using this definition also the equation of a hyperbola can be derived.

Suppose S and S' are two fixed points and let P be a point in the plane so that $|\,SP - S'P\,| = 2a$.

Let $(c, 0)$ and $(-c, 0)$ be the coordinates of S and S' respectively and mid-point C of $\overline{SS'}$ be the origin.

$$\therefore \left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a$$

$$\therefore \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$$

$$\therefore \sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a$$

$$\therefore (x + c)^2 + y^2 = (x - c)^2 + y^2 \pm 4a\sqrt{(x-c)^2+y^2} + 4a^2$$

$$\therefore cx - a^2 = \pm\, a\sqrt{(x-c)^2+y^2}$$

$$\therefore \left(\frac{c}{a}x - a\right)^2 = (x - c)^2 + y^2$$

Taking $\frac{c}{a} = e$, $c = ae$

$$\therefore (ex - a)^2 = (x - ae)^2 + y^2$$

$$\therefore (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2 \qquad \textbf{(i)}$$

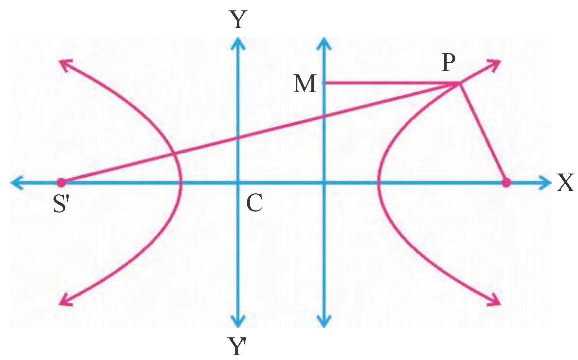

Figure 8.25

Further, $S = (c, 0) = (ae, 0)$

Suppose $l : x - \frac{a}{e} = 0$ is a line, then from (i)

$$\therefore (SP)^2 = e^2(PM)^2$$

$$\therefore \frac{SP}{PM} = e$$

Also $|\,SP - S'P\,| = 2a < SS' = 2c$ $\qquad \left(P \notin \overleftrightarrow{SS'} - \overline{SS'}\right)$

$$\therefore \frac{c}{a} > 1$$

$$\therefore e > 1$$

$\therefore$ The point set of P is a hyperbola with eccentricity $e > 1$.

Example 25 : Obtain the equation of the hyperbola whose focus is (0, 1), the equation of the directrix is $x + 3 = 0$ and eccentricity is $\sqrt{2}$.

Solution : $SP^2 = e^2 PM^2$

$\therefore \quad x^2 + (y - 1)^2 = 2(x + 3)^2$

$\therefore \quad x^2 + y^2 - 2y + 1 = 2(x^2 + 6x + 9)$

$\therefore \quad x^2 - y^2 + 12x + 2y + 17 = 0$ is the equation of the required hyperbola.

Example 26 : By shifting origin to $(-1, -2)$, show that $(x + 1)^2 - (y + 2)^2 = 16$ represents a hyperbola. Find its eccentricity, coordinates of foci and equation of directrices.

Solution : In the standard notations taking $x = x' - 1$, $y = y' - 2$,

$(x')^2 - (y')^2 = 16$

This equation represents a rectangular hyperbola with $a = b = 4$ and $e = \sqrt{2}$.

$\therefore$ The coordinates of foci are $(\pm 4\sqrt{2}, 0)$ and the corresponding equations of directrices are $x' \mp 2\sqrt{2} = 0$. **(in $x' - y'$ system)**

$\therefore$ In original coordinates system, coordinates of foci are $(\pm 4\sqrt{2} - 1, -2)$ and

The equations of directrices are $x + 1 \pm 2\sqrt{2} = 0$.

Example 27 : Point P is a variable point such that difference of its distances from fixed points S and S', which are 12 units apart, is constant 8. Find the point set of P.

Solution : $| SP - S'P | = 2a = 8$

$\therefore \quad a = 4$, $SS' = 2c = 12$. Hence $c = 6$

$e = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

Now, $b^2 = a^2(e^2 - 1) = 16\left(\frac{9}{4} - 1\right) = 36 - 16 = 20$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{20} = 1$.

Example 28 : For the following hyperbola, find the coordinates of foci, the equations of directrices, eccentricity, length of the latus-rectum and length of transverse and conjugate axes :

(1) $x^2 - 16y^2 = 16$ (2) $\frac{x^2}{25} - \frac{y^2}{24} = 1$

(3) $\frac{y^2}{25} - \frac{x^2}{9} = 1$ (4) $x^2 - y^2 = 4$

Solution :

(1) This equation can be written as $\frac{x^2}{16} - \frac{y^2}{1} = 1$.

$\therefore \quad a = 4, b = 1$

as $b^2 = a^2(e^2 - 1)$, $\quad 1 = 16(e^2 - 1)$

$\therefore \quad e^2 - 1 = \frac{1}{16}$ or $e^2 = \frac{17}{16}$

$\therefore \quad e = \frac{\sqrt{17}}{4}$

Foci are $(\pm ae, 0) = \left(\pm 4\left(\frac{\sqrt{17}}{4}\right), 0\right) = (\pm\sqrt{17}, 0)$.

Directrices are $x = \pm \frac{a}{e}$ i.e. $x = \pm 4\left(\frac{4}{\sqrt{17}}\right)$.

$\therefore \quad x = \pm \frac{16}{\sqrt{17}}$ are equations of directrices.

Length of the latus-rectum $= \frac{2b^2}{a} = \frac{2}{4} = \frac{1}{2}$

Length of the transverse axis $= 2a = 8$

Length of the conjugate axis $= 2b = 2$

(2) Here $a^2 = 25$, $b^2 = 24$

$\therefore \quad b^2 = a^2(e^2 - 1)$

$\therefore \quad 24 = 25(e^2 - 1)$

$\therefore \quad e^2 - 1 = \frac{24}{25}$

$\therefore \quad e^2 = \frac{49}{25}$

$\therefore \quad e = \frac{7}{5}$

Foci : $(\pm ae, 0) = \left(\pm 5\left(\frac{7}{5}\right), 0\right) = (\pm 7, 0)$

Directrices : $x = \pm \frac{a}{e} = \pm \frac{5}{\left(\frac{7}{5}\right)} = \pm \frac{25}{7}$

$\therefore$ The equations of directrices are $x = \pm \frac{25}{7}$.

Length of the latus-rectum $= \frac{2b^2}{a} = \frac{2(24)}{5} = \frac{48}{5}$

Length of the transverse axis $= 2a = 10$

Length of the conjugate axis $= 2b = 2\sqrt{24} = 4\sqrt{6}$

(3) In this hyperbola, directrices are parallel to X-axis. Here $a^2 = 9$, $b^2 = 25$

For eccentricity, we have

$\therefore \quad a^2 = b^2(e^2 - 1)$

$\therefore \quad 9 = 25(e^2 - 1)$

$\therefore \quad e^2 = 1 + \frac{9}{25} = \frac{34}{25}$

$\therefore \quad e = \frac{\sqrt{34}}{5}$

Foci : $(0, \pm be) = \left(0, \pm 5\left(\frac{\sqrt{34}}{5}\right)\right) = (0, \pm\sqrt{34})$

Directrices : $y = \pm \frac{b}{e} = \pm 5\left(\frac{5}{\sqrt{34}}\right) = \pm \frac{25}{\sqrt{34}}$

$\therefore \quad y = \pm \frac{25}{\sqrt{34}}$ are equations of directrices of the ellipse.

Length of the latus-rectum $= \frac{2a^2}{b} = \frac{2 \cdot 9}{5} = \frac{18}{5}$

Length of the transverse axis $= 2b = 10$

Length of the conjugate axis $= 2a = 6$

(4) This equation can be written $\frac{x^2}{4} - \frac{y^2}{4} = 1$. This is a rectangular hyperbola. $a^2 = b^2 = 4$

Eccentricity : $e = \sqrt{2}$, the coordinates of foci $(\pm 2\sqrt{2}, 0)$, the equations of directrices : $x = \pm\sqrt{2}$

Length of the latus-rectum $= 2a = 4$

Length of the transverse axis $= 2a = 4$

Length of the conjugate axis $= 2b = 4$

Example 29 : Find the equation of the hyperbola from the following conditions :

(1) Foci $(\pm 7, 0)$, vertices $(\pm 5, 0)$

(2) Foci $(0, \pm 3)$, eccentricity = 2

(3) Distance between foci 16 (foci on X-axis), eccentricity = $\sqrt{2}$

Solution : (1) Here, foci are $(\pm ae, 0) = (\pm 7, 0)$

$\therefore \quad ae = 7$ **(i)**

Now vertices are $(\pm 5, 0)$.

$\therefore \quad a = 5$ **(ii)**

$\therefore \quad ae = 5e = 7$

$\therefore \quad e = \frac{7}{5}$

Now $b^2 = a^2(e^2 - 1) = 25\left(\frac{49}{25} - 1\right) = 24$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{24} = 1$.

(2) Foci $(0, \pm 3)$. Foci are on Y-axis. Thus directrices are parallel to X-axis.

Given that $e = 2$

$\therefore \quad be = 3$

$\therefore \quad 2b = 3$

$\therefore \quad b = \frac{3}{2}$

Now $a^2 = b^2(e^2 - 1)$

$\therefore \quad a^2 = \frac{9}{4}(4 - 1) = \frac{9}{4}(3) = \frac{27}{4}$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{\frac{9}{4}} - \frac{x^2}{\frac{27}{4}} = 1$

$\therefore \quad \frac{4y^2}{9} - \frac{4x^2}{27} = 1$

(3) Distance between foci = $2ae = 16$. Thus $ae = 8$ **(i)**

$e = \sqrt{2}$

$\therefore \quad a\sqrt{2} = 8$

$a = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

Now $b^2 = a^2(e^2 - 1) = (4\sqrt{2})^2(2 - 1) = 32$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{(4\sqrt{2})^2} - \frac{y^2}{32} = 1$ or $x^2 - y^2 = 32$

Exercise 8.6

1. Find the coordinates of foci, the equations of directrices, length of the latus-rectum, lengths of transverse and conjugate axes of the following hyperbolas :

(1) $\frac{x^2}{100} - \frac{y^2}{25} = 1$ (2) $x^2 - y^2 = 64$ (3) $2x^2 - 3y^2 = 5$

(4) $9y^2 - 16x^2 = 144$ (5) $\frac{y^2}{25} - \frac{x^2}{39} = 1$

2. Find the equation of the hyperbola for the following situations. Also write their parametric equations :

(1) Eccentricity $e = \frac{4}{3}$, Vertices $(0, \pm 7)$

(2) Foci $(\pm \sqrt{13}, 0)$, Eccentricity $\frac{\sqrt{13}}{3}$

(3) Foci $(\pm 3\sqrt{5}, 0)$, Length of the latus-rectum = 8

(4) Foci $(0, \pm 8)$, Eccentricity $\sqrt{2}$

(5) Distance between foci (on Y-axis) = 10, Eccentricity $\frac{5}{4}$

3. If the eccentricities of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are e_1 and e_2 respectively, then prove that $e_1^2 + e_2^2 = e_1^2 e_2^2$.

4. Find the equation of the hyperbola for which distance from one vertex to two foci are 9 and 1.

5. Write parametric equations of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

*

Miscellaneous Problems :

Example 30 : The two supporting pillars of a suspension bridge in the shape of a parabola are 30 *m* high and 200 *m* apart. The height of the bridge above its centre is 5 *m*. There is a pillar of height 11.25 *m*. Find its distance from the centre.

Solution : As shown in the figure 8.26 CAB is the suspension bridge in the shape of a parabola. The centre of parabola is vertex, which is at height 5 *m*. Taking A as origin, $\overleftrightarrow{OA}$ as Y-axis, the equation of the parabola is $x^2 = 4ay$. Now the coordinates of O are $(0, -5)$, thus by shifting the origin at O, the equation of the parabola is,

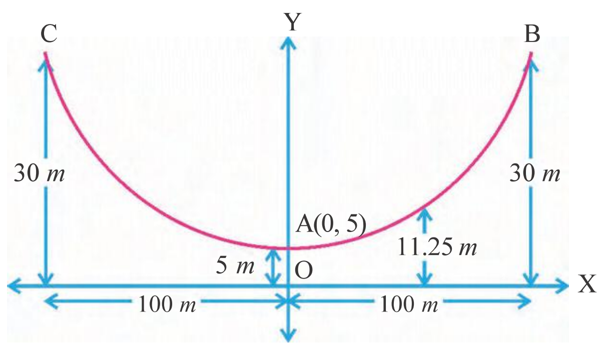

Figure 8.26

$$(x')^2 = 4a(y' - 5) \quad \text{(i)}$$

For the supports C and B, we are given that coordinates are $(-100, 30)$ and $(100, 30)$ respectively. Using these in (i), we get

$$(100)^2 = 4a(30 - 5)$$

$$\therefore \quad 10000 = 100a$$

$$\therefore \quad a = 100$$

Thus, (i) gives $x^2 = 400(y - 5)$ **(ii)**

Further to find the distance of supports at height 11.25, we substitute $y = 11.25$ in (ii).

$$x^2 = 400(11.25 - 5) = 400(6.25) = 2500$$

$$\therefore \quad x = \pm 50$$

Hence there are two supports on each side of the centre at distance 50 *m* from the centre having heights 11.25 *m*.

Example 31 : A 12 *m* long rod slides in such a way that its ends stay on the two axes. Find the point-set of the point on the rod 3 *m* away from its end-point on the X-axis.

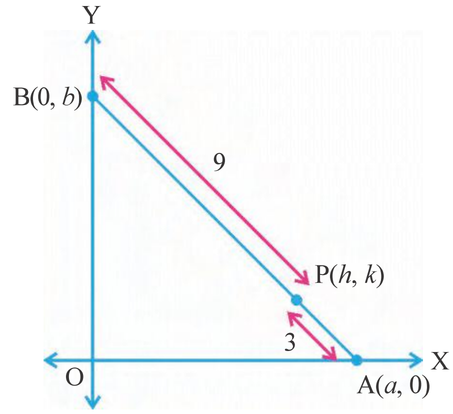

Figure 8.27

Solution : The end-points of the rod are A(a, 0) and B(0, b) and the point on the rod 3 *m* away from A is P(h, k).

Thus, AP = 3 *m*, PB = 9 *m*

$\therefore$ P divides $\overline{AB}$ from A's side in the ratio 1 : 3.

$\therefore$ $h = \frac{3a}{4}$ and $k = \frac{b}{4}$

$\therefore$ $a = \frac{4h}{3}$ and $b = 4k$

Now, in the right ΔAOB, $OA^2 + OB^2 = AB^2$. So $a^2 + b^2 = 144$

$\therefore$ $\frac{16h^2}{9} + \frac{16k^2}{1} = 144$

$\therefore$ $\frac{h^2}{81} + \frac{k^2}{9} = 1$

$\therefore$ Point-set of P is $\frac{x^2}{81} + \frac{y^2}{9} = 1$. It is an ellipse.

Example 32 : The orbit of the earth around the sun is an ellipse. The sun is at one of the foci of this ellipse. If the length of the major axis of this ellipse is 300 million *km* and the eccentricity is 0.0167, find the minimum and maximum distance of the earth from the sun.

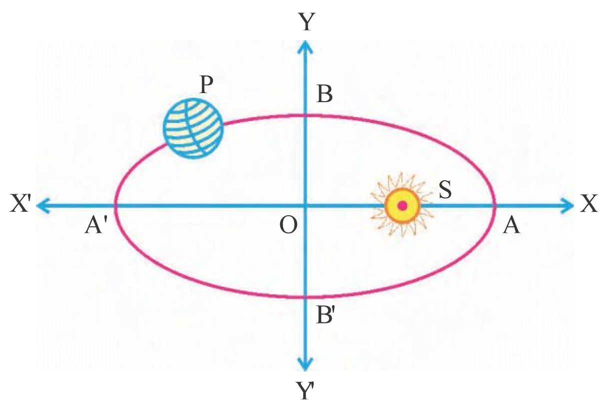

Figure 8.28

Solution : Take the focus of the orbit at S (where the sun is) and take a point P on elliptical orbit. Then the focal distance of P is

$SP = a(1 - e\cos\theta)$.

Now, $2a = 3 \times 10^8$ *km*

$\therefore$ $a = 1.5 \times 10^8$ *km*

$\therefore$ $SP = 1.5 \times 10^8$ *km* $(1 - 0.0167\cos\theta)$

When the earth-sun distance is minimum, the earth is on the major axis at its end. So $\theta = 0$ and $\cos\theta = 1$. Hence minimum distance of sun from earth is

1.5×10^8 *km* $(1 - 0.0167\cos\theta)$

$= 1.5 \times 10^8 (1 - 0.0167)$

$=$ 147,495,000 *km*

Earth is at its maximum distance when it is at the other end of the major axis and away from S. The maximum distance is

$1.5 \times 10^8 (1 - 0.0167(-1))$ *km* $= 1.5 \times 10^8 (1 + 0.0167)$ *km*

$=$ 152,505,000 *km*

Exercise 8

1. Find the equation of the circle having (1, 2), (2, −3) as extremities of a diameter.
2. Find the equation of the circle which passes through the points (4, 0), (−4, 0) and (0, 8).
3. Find the equation of the circle concentric with $x^2 + y^2 - 4x - 6y - 5 = 0$ and touching X-axis.
4. Find the focus and the length of the latus-rectum of the parabola $y^2 = x$.
5. Find the standard equation of the ellipse whose foci are on X-axis and 8 units apart from each other and eccentricity is $\frac{1}{3}$.
6. Obtain the standard equation of hyperbola having directrix parallel to X-axis.
7. Using definition, find the equation of parabola having focus at (−4, 0) and directrix $x = 2$.
8. A cross-section of a parabolic reflector is shown. The diameter of opening at the focus is 10 *cm*. Find the equation of the porabola. Find diameter of the opening $\overline{PQ}$ at 11 *cm* from the vertex. (See figure 8.29)

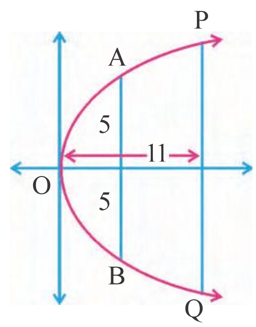

Figure 8.29

9. A parabolic reflector is 24 *cm* in diameter and 6 *cm* deep. Find coordinates of the focus.
10. An arch is in the form a semi-ellipse. It is 10 *m* wide and 4 *m* high at the centre. Find the height of the arc at a point 2 *m* from one end.
11. A toy train moves such that sum of its distances from two signals is always constant and equal to 10 *m* and the distance between the signals is 8 *m*. Find the path traced by the train.
12. Select the proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The equation of the circle whose extremities of a diameter are centres of the circles, $x^2 + y^2 + 6x - 14y = 1$ and $x^2 + y^2 - 4x + 10y = 2$ is ... ☐

(a) $x^2 + y^2 + x - 2y - 41 = 0$ (b) $x^2 + y^2 + x + 2y - 41 = 0$

(c) $x^2 + y^2 + x + 2y + 41 = 0$ (d) $x^2 + y^2 - x - 2y - 41 = 0$

(2) If one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + 5 = 0$ has coordinates (−3, 2), then the coordinates of the other end are ... ☐

(a) (5, 3) (b) (6, 2) (c) (1, −8) (d) (11, 2)

(3) If a circle has centre on X-axis, radius 5 and it passes through the point (2, 3), then the equation of the circle is ... ☐

(a) $x^2 + y^2 - 12x + 11 = 0$ (b) $x^2 + y^2 - 12y + 11 = 0$

(c) $x^2 + y^2 - 12x - 11 = 0$ (d) $x^2 + y^2 - 4x + 12y = 0$

(4) The equation of circle, with centre at (4, 5) and passing through the centre of the circle $x^2 + y^2 + 4x - 6y = 12$ is ... ☐

(a) $x^2 + y^2 + 8x - 10y + 1 = 0$ (b) $x^2 + y^2 - 8x - 10y + 1 = 0$

(c) $x^2 + y^2 - 8x + 10y - 1 = 0$ (d) $x^2 + y^2 - 8x - 10y - 1 = 0$

(5) Area of the circle centred at (1, 2) and passing through the point (4, 6) is ...

(a) 30π sq units (b) 5π sq units (c) 15π sq units (d) 25π sq units

(6) Coordinates of the centre of the circle passing through the points (0, 0), (a, 0), (0, b) are ...

(a) $\left(\frac{b}{2}, \frac{a}{2}\right)$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (c) (b, a) (d) (a, b)

(7) The parametric equations of the parabola $x^2 = 4ay$ are ....

(a) $x = at^2, y = at^2$ (b) $x = 2at, y = 2at$ (c) $x = 2at, y = at^2$ (d) $x = 2at^2, y = at$

(8) The line $2x - 3y + 8 = 0$ intersects the parabola $y^2 = 8x$ in P and Q. The mid-point of $\overline{PQ}$ is ...

(a) (2, 4) (b) (8, 8) (c) (5, 6) (d) (6, 5)

(9) The eccentricity of the ellipse whose latus-rectum is half of the minor axis is ...

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\sqrt{2}$

(10) The eccentricity of the ellipse whose minor axis is equal to the distance between foci is ...

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{3}}$

(11) The eccentricity of the ellipse $9x^2 + 25y^2 = 225$ is ...

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) 0

(12) Length of the latus-rectum of the ellipse $4x^2 + 9y^2 = 1$ is ...

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{2}{9}$ (d) $\frac{2}{3}$

(13) ...... is a focus of the ellipse $9x^2 + 4y^2 = 36$.

(a) $(\sqrt{5}, 0)$ (b) $(0, \sqrt{5})$ (c) $(3\sqrt{5}, 0)$ (d) $(0, 3\sqrt{5})$

(14) Length of the major axis of the ellipse $25x^2 + 9y^2 = 1$ is ...

(a) $\frac{2}{5}$ (b) $\frac{2}{3}$ (c) $\frac{1}{5}$ (d) $\frac{1}{9}$

(15) The foci of the hyperbola $9x^2 - 16y^2 = 144$ are ...

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

(16) The length of the latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$ is ...

(a) $\frac{32}{3}$ (b) $\frac{16}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

(17) The eccentricity of the hyperbola $16y^2 - 9x^2 = 144$ is ...

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{5}{4}$ (d) $\frac{4}{5}$

(18) The eccentricity of the hyperbola $x^2 - 4y^2 = 1$ is ...

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{5}}$

(19) If the parabola $y^2 = 4ax$ passes through the point $(2, -6)$, then the length of the latus-rectum is ...

(a) 9 (b) 16 (c) 18 (d) 8

(20) The length of the latus-rectum of the ellipse $5x^2 + 9y^2 = 45$ is ...

(a) $\frac{5\sqrt{5}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{10}{3}$

*

Summary

We have studied following points in this chapter :

1. Standard equation of a circle : $x^2 + y^2 = r^2$

   General equation of a circle : $(x - h)^2 + (y - k)^2 = r^2$

2. Centre of the circle : $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ if $g^2 + f^2 - c > 0$ and does not represent a circle if $g^2 + f^2 - c \le 0$.

3. The equation of a parabola $y^2 = 4ax$, Parametric equations $x = at^2$, $y = 2at$, $t \in R$, Latus-rectum $4\,|\,a\,|$.

4. A property of a parabola : for a focal chord $t_1t_2 = -1$

5. Standard equation of the ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **$(a > b)$**

   Foci $(\pm ae, 0)$, the equations of the directrices $x \pm \frac{a}{e} = 0$

   Parametric equations $x = a\cos\theta$, $y = b\sin\theta$, $0 \in [0, 2\pi)$, length of the latus-rectum $\frac{2b^2}{a}$, major axis $2a$, minor axis $2b$.

6. A property of an ellipse : $SP + S'P = 2a$

7. Standard equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

   Foci $(\pm ae, 0)$, Equations of directrices $x \mp \frac{a}{e} = 0$

   Parametric equations $x = a\sec\theta$, $y = b\tan\theta$, $\theta \in R - \left\{(2k - 1)\frac{\pi}{2} \;\middle|\; k \in Z\right\}$, length of latus-rectum $\frac{2b^2}{a}$.

8. A property of hyperbola : $|\,SP - S'P\,| = 2a$

APPENDIX

Intersection of a Double Cone and a Plane

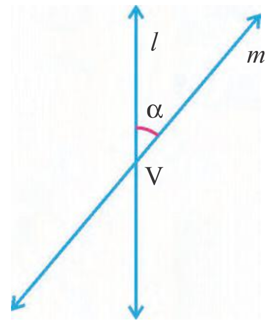

Figure A.1

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and let the measure of the angle made by m with l be α $(0 < \alpha < \frac{\pi}{2})$, as shown in figure A.1. Suppose the line m is rotated around the line l in such a way that the angle α remains constant. Then the surface generated is called a **right circular cone**. The point of intersection V separates the cone in two parts. Hence it is called a double napped cone or a double cone. For simplicity we will refer this as a cone. Since the lines l and m are of infinite extent, the cone is extending indefinitely in both directions (figure A.2). The point V is called the **vertex**. The line l is the **axis** of the cone and the rotating line m is called a **generator** of the cone, and two parts of the cone are called **napes**. We note that looking at a given cone we cannot observe the line m actually. Any of the line on the surface of the cone can be taken as the generator.

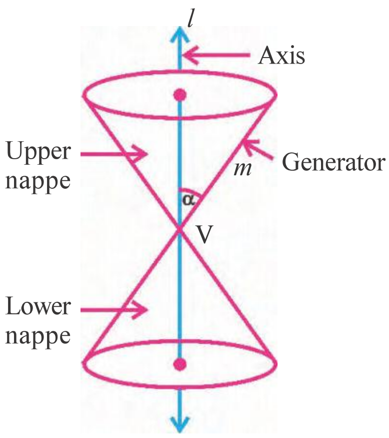

Figure A.2

Now we consider the intersection of a plane with a cone, the section so obtained is called a **conic section**. Thus, conic sections are the curves obtained by intersecting a **right circular cone by a plane and hence the name conics.**

There are many possibilities when we consider intersection of a cone with a plane depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β $(0 < \beta < \frac{\pi}{2})$ be the angle made by the plane with the vertical axis of the cone (figure A.3). There are two possibilities : (1) the plane passes through the vertex; or, (2) the plane does not pass through the vertex. Accordingly the intersection takes place at vertex or at any other part of the napes above or below the vertex.

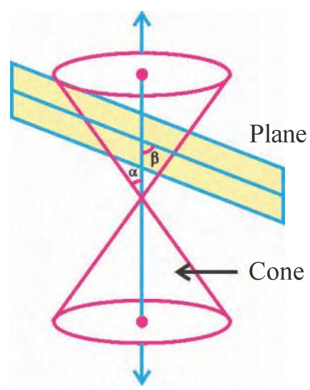

Figure A.3

Various situations of intersection are discussed below; in each case above two possibilities are discussed separately.

Let the angle made by the plane with the axis of the cone be right angle, i.e. $\beta = \frac{\pi}{2}$. If the plane passes through the vertex, then the intersection is the vertex itself (figure A.4 (a)); and if the

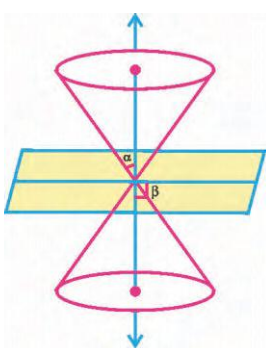

Figure A.4(a)

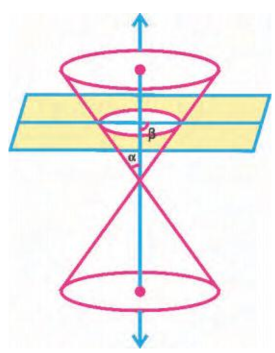

Figure A.4(b)

plane does not pass through the vertex, then the intersection is a circle, either in the upper nape of the cone or the lower nape of the cone depending on the position of the plane as shown in the figure A.4(b). In the first case we got the intersection as a point. Thus it is a degenerate case of the circle.

Suppose $\alpha < \beta < \frac{\pi}{2}$ again. If the plane is passing through the vertex, then the intersection is the vertex itself. If it is not the case, then the intersection is an ellipse (figure A.5). Here also, the first case is degenerate ellipse – a point. (Try to visualize this!).

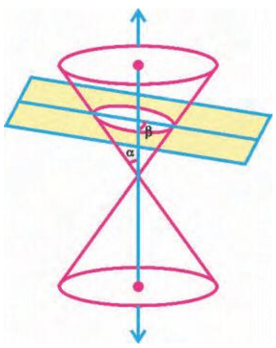

Figure A.5(a)

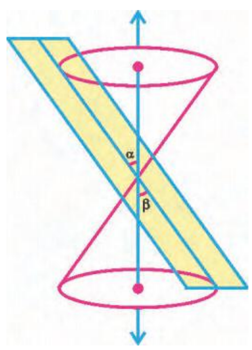

Figure A.5(b)

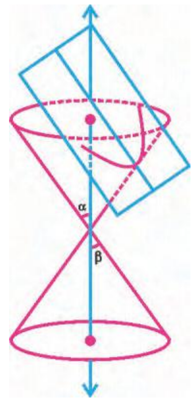

Figure A.5(c)

Now, consider the case, when $\alpha = \beta$. In this case the intersecting plane is parallel to a generator. If the plane passes through the vertex, then the intersection is a straight line. It can be seen that the line of intersection is a generator of the cone. If the vertex is not on the plane, then the intersection is a parabola as shown in figure A.5(c). The intersection being a straight line is actually degenerate parabola, i.e. as if the parabola is opened up straight to get the line.

Finally, consider the case $\beta < \alpha$. In this case the plane intersects both the napes. This did not happen in earlier cases. The intersection is a hyperbola and it has two branches as shown in the Figure A.6. Here the degeneracy occurs in a particular case. In this case the plane passes through the vertex and the intersection is a pair of lines.

In this section we have seen that, circle, ellipse, parabola and hyperbola are various conics, with point, line or a pair of lines as degenerate cases. This discussion about conics is useful for the practical consideration.

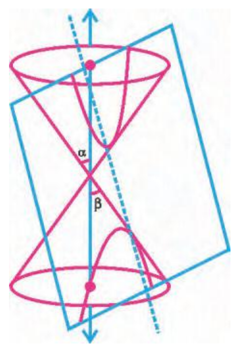

Figure A.6

— ❖ —

Some of Bhaskara's contributions to mathematics include the following :

- A proof of the Pythagorean theorem by calculating the same area in two different ways and then cancelling out terms to get $a^2 + b^2 = c^2$.
- In Lilavati, solutions of quadratic, cubic and quartic indeterminate equations are explained.
- Solutions of indeterminate quadratic equations (of the type $ax^2 + b = y^2$).
- A cyclic Chakravala method for solving indeterminate equations of the form $ax^2 + bx + c = y$. The solution to this equation was traditionally attributed to William Brouncker in 1657, though his method was more difficult than the Chakravala method.
- The first general method for finding the solutions of the problem $x^2 - ny^2 = 1$ (so-called "Pell's equation") was given by Bhaskara II.

Chapter 9

THREE DIMENSIONAL GEOMETRY

As far as the laws of mathematics refer to reality they are not certain and as far as they are certain they do not refer to reality.

– Albert Einstein

9.1 Introduction

Earlier the concepts of plane coordinate geometry were initiated by French mathematician **René Descartes** and simultaneously also by **Fermat** in the beginning of 17th century. It was later systematized by **Bernoulli** and **Euler** in the 18th century. In the 19th century, it was further extended to highter dimensions and found interesting applications in the last century only.

In this chapter, we will discuss some basic concepts of quantities called vectors useful in mathematics and sciences. Also the study of coordinate geometry in plane will be extended to three dimensions, i.e. we will discuss coordinate geometry in the space. This is useful in studying solid objects and things in the space around us. We will use vectors as a tool to discuss three dimensional geometry.

9.2 Vectors

Some physical quantities require magnitude and direction both to completely specify position and application. Such quantities are called **vectors**. Velocity is a vector, as its complete description requires both magnitude as well as direction. Otherwise the meaning is incomplete. We already know about the representation of complex numbers in the Argand plane. In a polar representation of a complex number $z = r(\cos\theta + i\sin\theta)$, there are two important parameters r and θ. Here r is its magnitude and by θ, we can decide its direction. Thus, every complex number is a vector as it has both magnitude and direction. Suppose Dev walks 300 m towards East and then he walks 400 m towards North. Hence to know his final position from original position, we should know direction and magnitude both. This is also a primary illustration of a vector.

In mathematics also we can think of quantities that have both magnitude and direction. For instance, we are familiar with the set R^2 of ordered points of real numbers. Also it is known that there is a one-one correspondence between R^2 and the points in a plane. Taking O(0, 0) as the origin, we can associate magnitude and direction with any element other than O, say (1, −2) of R^2. Suppose the point P represents (1, −2) in the plane. Then with (1, −2), we can associate the magnitude of $\overline{OP}$ (that is length $OP = \sqrt{(1)^2 + (-2)^2}$ and the direction of $\overrightarrow{OP}$. Thus (1, −2) can be regarded as a vector. Similarly, it is possible to regard elements of set of ordered real triplets of R^3.

Having considered elements of R^2 or R^3 as vectors, we can think of the collection R^2 or R^3 of vectors as vector spaces.

9.3 Vectors in R^2 and R^3

Taking R^2 and R^3 as sets of ordered pairs and triplets of real numbers respectively, an element in R^2 or R^3 is denoted by a letter with an overhead bar, say $\bar{x}$. Thus, $\bar{x} = (x_1, x_2, x_3)$ in R^3 and $\bar{x} = (x_1, x_2)$ in R^2.

We first define the notion of equality in R^2 and R^3 as follows :

In R^2, $(x_1, x_2) = (y_1, y_2)$ if $x_1 = y_1$ and $x_2 = y_2$.

In R^3, $(x_1, x_2, x_3) = (y_1, y_2, y_3)$ if $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

Thus (1, 2) and (2, 1) are distinct elements in R^2.

In the further discussion, we shall study R^3 in detail. All these results would be essentially true for R^2 also.

Definition : Let $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ be two elements of R^3. Their addition is defined by $\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. Thus if $\bar{z} = (z_1, z_2, z_3) = \bar{x} + \bar{y}$, then $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, $z_3 = x_3 + y_3$.

Clearly, for $\bar{x} \in R^3$, $\bar{y} \in R^3$ we have $\bar{x} + \bar{y} \in R^3$ i.e. the addition defined above has closure property. $\bar{x} + \bar{y}$ is called the sum of $\bar{x}$ and $\bar{y}$.

Definition : Let $\bar{x} = (x_1, x_2, x_3)$ and $k \in R$. We define multiplication of $\bar{x}$ by k as $k\bar{x} = (kx_1, kx_2, kx_3)$.

Obviously, for $k \in R$ and $\bar{x} \in R^3$, $k\bar{x} \in R^3$.

Some obvious results :

For any $\bar{x}, \bar{y}, \bar{z} \in R^3$ and $k, l \in R$

(i) $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ (Commutative law)

(ii) $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$ (Associative law)

(iii) If $\bar{0} = (0, 0, 0)$, then $\bar{x} + \bar{0} = \bar{x}$ (Existence of identity)

Identity element is unique.

(iv) For each $\bar{x} \in R^3$, $\exists\, \bar{y} \in R^3$ such that $\bar{x} + \bar{y} = \bar{0}$ (Existence of inverse)

(It can be proved that if $\bar{x} = (x_1, x_2, x_3)$, then $\bar{y} = (-x_1, -x_2, -x_3)$ so that $\bar{x} + \bar{y} = \bar{0}$.

$\bar{y}$ is called an additive inverse of $\bar{x}$ and for every $\bar{x}$ there correspond a unique $\bar{y}$. Additive inverse of $\bar{x}$ is denoted by $-\bar{x}$.

$\therefore \quad -\bar{x} = (-x_1, -x_2, -x_3)$

(v) $k(\bar{x} + \bar{y}) = k\bar{x} + k\bar{y}$

(vi) $(k + l)\bar{x} = k\bar{x} + l\bar{x}$

(vii) $(kl)\bar{x} = k(l\bar{x})$

(viii) $1\bar{x} = \bar{x}$

The set R^3 with all above properties is called a vector space over R. There are other sets also which are vector spaces. **Mathematically, elements of a vector space are called vectors. Thus any element of R^3 is called a vector. R^2 is also a vector space over R.**

The sum defined above in R^3 (or R^2) is called a vector sum. When R^3 (or R^2) is considered as a vector space over R, the elements of R are called scalars. Thus a real number is a scalar in this context. Accordingly for $k \in R$, $\overline{x} \in R^3$, $k\overline{x}$ is called the multiplication of vector $\overline{x}$ by a scalar k. The product $k\overline{x}$ is a vector. $\overline{0} = (0, 0, 0)$ is called the zero vector.

9.4 Magnitude of a Vector

If $\overline{x} = (x_1, x_2, x_3)$, then the **magnitude of $\overline{x}$,** is defined as $\sqrt{x_1^2 + x_2^2 + x_3^2}$ and it is denoted by $|\overline{x}|$. Thus, $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

In a similar manner for a vector $\overline{x}$ in R^2, magnitude is defined. If $\overline{x} = (x_1, x_2)$, then $|\overline{x}| = \sqrt{x_1^2 + x_2^2}$.

The following are obvious results :

(1) $|\overline{x}| \geq 0$ because $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$

(2) $|\overline{x}| = 0 \Leftrightarrow \overline{x} = \overline{0}$

(3) $|k\overline{x}| = |(kx_1, kx_2, kx_3)|$

$$= \sqrt{k^2x_1^2 + k^2x_2^2 + k^2x_3^2}$$

$$= \sqrt{k^2\left(x_1^2 + x_2^2 + x_3^2\right)}$$

$$= \sqrt{k^2}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

$= |k||\overline{x}|$; Here $\sqrt{k^2} = |k|$ is the magnitude of the real number k and

$|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the magnitude of vector $\overline{x}$.

$\therefore$ $|k\overline{x}| = |k||\overline{x}|$

Definition : A vector $\overline{x}$ is said to be unit vector, if $|\overline{x}| = 1$.

Some examples of unit vectors in R^2 are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, (1, 0), (0, −1), $(\sin\alpha, \cos\alpha)$, $\alpha \in R$. In R^3, some such examples are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, (1, 0, 0), $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\theta, \alpha \in R$, $(\cos\theta \sin\alpha, \cos\theta \cos\alpha, \sin\theta)$ is also a unit vector.

Example 1 : If $\overline{u} = (3, -1, 4)$, $\overline{v} = (1, -2, -3)$, find $3\overline{u} + \overline{v}$.

Solution : $3\overline{u} + \overline{v} = 3(3, -1, 4) + (1, -2, -3)$

$= (9, -3, 12) + (1, -2, -3)$

$= (9 + 1, -3 - 2, 12 - 3)$

$= (10, -5, 9)$

Example 2 : Find $\overline{x} - 2\overline{y}$, where $\overline{x} = (1, -1, 3)$, $\overline{y} = (1, 1, 1)$.

Solution : $\overline{x} - 2\overline{y} = \overline{x} + (-2)\overline{y}$

$= (1, -1, 3) + (-2)(1, 1, 1)$

$= (1, -1, 3) + (-2, -2, -2)$

$= (1 - 2, -1 - 2, 3 - 2)$

$= (-1, -3, 1)$

Example 3 : For vectors $\overline{x}, \overline{y}, \overline{z}$ in R^3, show that, $\overline{x} + \overline{y} = \overline{x} + \overline{z} \Rightarrow \overline{y} = \overline{z}$.

Solution : Let $\overline{x} = (x_1, x_2, x_3)$, $\overline{y} = (y_1, y_2, y_3)$ and $\overline{z} = (z_1, z_2, z_3)$.

$\overline{x} + \overline{y} = \overline{x} + \overline{z}$

$\therefore \quad (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1, x_2, x_3) + (z_1, z_2, z_3)$

$\therefore \quad (x_1 + y_1, x_2 + y_2, x_3 + y_3) = (x_1 + z_1, x_2 + z_2, x_3 + z_3)$

$\therefore \quad x_1 + y_1 = x_1 + z_1, x_2 + y_2 = x_2 + z_2, x_3 + y_3 = x_3 + z_3$

$\therefore \quad y_1 = z_1, y_2 = z_2, y_3 = z_3$

$\therefore \quad (y_1, y_2, y_3) = (z_1, z_2, z_3)$

$\therefore \quad \overline{y} = \overline{z}$

Another method :

$\overline{x} + \overline{y} = \overline{x} + \overline{z}$

$\therefore \quad (-\overline{x}) + (\overline{x} + \overline{y}) = (-\overline{x}) + \overline{x} + \overline{z}$ **($-\overline{x}$ exists uniquely)**

$\therefore \quad (-\overline{x} + \overline{x}) + \overline{y} = (-\overline{x} + \overline{x}) + \overline{z}$

$\therefore \quad \overline{0} + \overline{y} = \overline{0} + \overline{z}$

$\therefore \quad \overline{y} = \overline{z}$

Example 4 : Solve : $x(3, 1) + y(4, 2) = (1, 0)$

Solution : $x(3, 1) + y(4, 2) = (1, 0)$

$\therefore \quad (3x, x) + (4y, 2y) = (1, 0)$

$\therefore \quad (3x + 4y, x + 2y) = (1, 0)$

$\therefore \quad 3x + 4y = 1, x + 2y = 0$

$\therefore \quad x = 1, \quad y = -\frac{1}{2}$

Exercise 9.1

1. Find :

(1) $x_1(1, 0) + x_2(0, 1)$; $(x_1, x_2 \in R)$

(2) $x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$; $(x, y, z \in R)$

(3) $2(1, 2, 1) + 3(1, -2, 0)$

(4) $2(1, -1, -1) - 2(-1, 1, 1)$

(5) $-2(1, 2, 3) + (1, 0, -1)$

(6) $3(1, -1, 0) - (2, 2, 2)$

2. Solve the following equations to find x and y :

(1) $x(3, 2) + y(1, -1) = (2, 3)$

(2) $x(1, 1) + y(1, -1) = (0, 0)$

(3) $y(1, 2) = x(3, 1) + (1, 3)$

(4) $x(1, 0) + y(0, 1) = \overline{0}$

3. Find magnitude of the following vectors :

(1) $(1, 1, 1)$ (2) $(1, -1, -1)$ (3) $(3, -4, 0)$

(4) $(-1, -2, -3)$ (5) $(2, 3, -5)$ (6) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

4. Verify $|\overline{x} + \overline{y}| \le |\overline{x}| + |\overline{y}|$ for the following vector $\overline{x}$ and $\overline{y}$.

(1) $\overline{x} = (1, -1, 2)$, $\overline{y} = (1, 2, 4)$ (2) $\overline{x} = \left(\frac{-3}{2}, 9, -9\right)$, $\overline{y} = (-1, 6, -6)$

5. If $\overline{u} = (2, 3)$ and $\overline{v} = (2k, k + 2)$ are equal then, find k.

6. If $\overline{u} = \left(\frac{-1}{2}, \frac{3}{5}, 0\right)$ and $\overline{v} = \left(\frac{1}{6}, \frac{-2}{3}, 0\right)$, find $3\overline{u} - 2\overline{v}$.

*

9.5 Direction of a Vector

As discussed earlier vectors in physics are specified with magnitude and direction both. Now we shall associate a **direction** with every non-zero vector. We will restrict our discussion about direction to define equality of directions of two non-zero vectors, two non-zero vectors with opposite directions and two non-zero vectors with different directions. This discussion will help in giving geometric meaning to the vectors in R^2 and R^3.

Suppose $\overline{x}$ and $\overline{y}$ are two non-zero vectors in R^2 or R^3. $\overline{x}$ and $\overline{y}$ are said to have the same direction, if $\overline{y} = k\overline{x}$ for some real number $k > 0$. If $k < 0$ and $\overline{y} = k\overline{x}$, then $\overline{x}$ and $\overline{y}$ are said to have opposite directions. Further, if $\overline{x}$ and $\overline{y}$ have neither same nor opposite directions, then they have different directions. If directions of $\overline{x}$ and $\overline{y}$ are equal, then they are called equi-directed vectors. If $\overline{x}$ and $\overline{y}$ have opposite directions then they are called vectors of opposite directions.

Thus, $(1, -1, 1)$ and $(2, -2, 2)$ have same direction, because

$(2, -2, 2) = 2(1, -1, 1)$ and $2 > 0$

Also $(-1, 1, -1) = (-1)(1, -1, 1)$. So $(1, -1, 1)$ and $(-1, 1, -1)$ have opposite directions.

The vectors $(1, -1, 1)$ and $(2, 0, 2)$ have different directions, because there is no $k \in R$ such that $(1, -1, 1) = k(2, 0, 2)$.

The direction determined by a non-zero vector (x_1, x_2, x_3) is denoted by $<x_1, x_2, x_3>$. The direction opposite to $<x_1, x_2, x_3>$ is denoted by $-<x_1, x_2, x_3>$.

If $k > 0$ then $<kx_1, kx_2, kx_3> = <x_1, x_2, x_3>$ and if $k < 0$ then $<kx_1, kx_2, kx_3> = <-x_1, -x_2, -x_3>$. We note that, we can not write $(kx_1, kx_2, kx_3) = (x_1, x_2, x_3)$ unless $k = 1$.

9.6 Magnitude and Direction of a Vector and Unit Vector

Theorem 1 : Non-zero vectors $\overline{x}$ and $\overline{y}$ are equal if and only if $|\overline{x}| = |\overline{y}|$ and $\overline{x}$ and $\overline{y}$ have the same direction.

Proof : Suppose $\overline{x} = \overline{y}$

$\therefore$ $(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\therefore$ $x_1 = y_1, x_2 = y_2, x_3 = y_3$

$\therefore$ $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{y_1^2 + y_2^2 + y_3^2} = |\overline{y}|$

Also since $\overline{x} = \overline{y}$, $\overline{x} = k\overline{y}$ with $k = 1 > 0$

$\therefore$ $\overline{x}$ and $\overline{y}$ have the same direction,

i.e. $<x_1, x_2, x_3> = <y_1, y_2, y_3>$

Thus, $\overline{x} = \overline{y} \Rightarrow |\overline{x}| = |\overline{y}|$ and $\overline{x}$ and $\overline{y}$ have the same direction.

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Conversely, suppose $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$, $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have the same direction.

As $\bar{x}$ and $\bar{y}$ have the same direction, so $\bar{y} = k\bar{x}$ for some $k > 0$.

Now, $|\bar{y}| = |k\bar{x}| = |k|\,|\bar{x}|$

But we are given that $|\bar{x}| = |\bar{y}|$ So, $|\bar{x}| = |k|\,|\bar{x}|$

As $\bar{x} \neq \bar{0}$, $|k| = 1$

$\therefore$ $k = \pm 1$ But $k > 0$

$\therefore$ $k = 1$

$\therefore$ $\bar{y} = k\bar{x} = 1\bar{x} = \bar{x}$

$\therefore$ $|\bar{x}| = |\bar{y}|$ and $\bar{x}$, $\bar{y}$ have the same direction $\Rightarrow \bar{x} = \bar{y}$

This theorem is in confirmity with the definition of a vector generally given in physics.

Theorem 2 : If $\bar{x} \neq \bar{0}$, then there is a unique unit vector in the direction of $\bar{x}$.

Proof : As $\bar{x} \neq \bar{0}$, so $|\bar{x}| \neq 0$.

Let $\bar{y} = \frac{\bar{x}}{|\bar{x}|} = k\bar{x}$; where $k = \frac{1}{|\bar{x}|} > 0$

$\therefore$ $|\bar{y}| = |k\bar{x}| = |k|\,|\bar{x}| = \left|\frac{1}{|\bar{x}|}\right| |\bar{x}| = \frac{1}{|\bar{x}|}\,|\bar{x}| = 1$ $\qquad (\,\big|\,|\bar{x}|\,\big| = |\bar{x}|\,)$

$\therefore$ $\bar{y}$ is a unit vector and as $\bar{y} = k\bar{x}$ with $k > 0$. $\bar{y}$ is in the same direction as $\bar{x}$ has.

To prove uniqueness of unit vector $\bar{y}$, suppose $\bar{z}$ is also a unit vector in the same direction as $\bar{x}$ has. Then, $|\bar{y}| = |\bar{z}| = 1$ and $\bar{y}$ and $\bar{z}$ are in the same direction (the direction of $\bar{x}$).

$\therefore$ By theorem 1, $\bar{y} = \bar{z}$

Thus, there is a unique unit vector in the direction of every non-zero vector.

To find the unit vector in the direction of $\bar{x} = (2, 1, 2)$, we note that

$$|\bar{x}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = 3$$

So, $\bar{y} = \frac{\bar{x}}{|\bar{x}|} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, is the required vector.

9.7 Three Dimensional Coordinate Geometry

Our study of geometry so far was confined to a plane. Many times we need to study objects which are not in a plane. In fact in actual life, the concept of plane is inadequate. For example, consider the position of a ball thrown in space at different points of time or when a kite is flying in the sky. Its position from time to time changes in the space. Recall that to locate the position of a point in a plane; we need two intersecting mutually perpendicular lines in the plane. These lines are called the **coordinate axes labelled as X-axis and Y-axis**; and the absolute values of coordinates of the point are distances measured perpendicular to the axes. These are called the coordinates of the point with respect to the axes. Thus using these lines, we can associate a unique ordered pair of two real numbers to every point in the plane. Also for each given ordered pair of real numbers, a unique point in the plane can be found of which the given pair are the coordinates. Thus there is a one-to-one correspondence between points in a plane and the set R^2.

If we were to locate the position of a point in the spaces, then two real numbers are not sufficient. For example, to locate the central tip of a ceiling fan in a room, we will require the perpendicular distances of the point to be located from two perpendicular walls of the room and the height of the point from the floor of the room. Therefore, we need three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. In general, a point in the space can be located by describing its **perpendicular distances** from three mutually perpendicular planes. Its position can be determined using these distances. These mutually perpendicular planes are called **coordinate planes**. In analogy with coordinates of a point in XY-plane, here also a coordinate of a point in space can be positive or negative. So, a point in space has three coordinates. Also, for a given triplet of real numbers, we can find a point in the space for which the given triplet represents coordinates. Here we note that there is one-one correspondence between R^3 and points in the space. In this Chapter, we shall study the basic concepts of geometry in **three dimensional space.**

9.8 Coordinate Axes and Coordinate Planes in Three Dimensional Space

In the case of plane, two mutually perpendicular lines are taken as reference lines. While assigning coordinates to a point in the space three mutually perpendicular planes are taken as reference. Consider three planes intersecting at a point O such that these three planes are mutually perpendicular (figure 9.1). Among these three planes any two planes intersect along the lines X'OX, Y'OY and Z'OZ, called the X-axis, Y-axis and Z-axis, respectively. We may note that these lines are mutually perpendicular to each other. Since these lines are mutually perpendicular, they constitute the **rectangular coordinate system**. We will refer to these three mutually perpendicular lines drawn passing through the point O as **coordinate axes** or simply **axes** (figure 9.2).

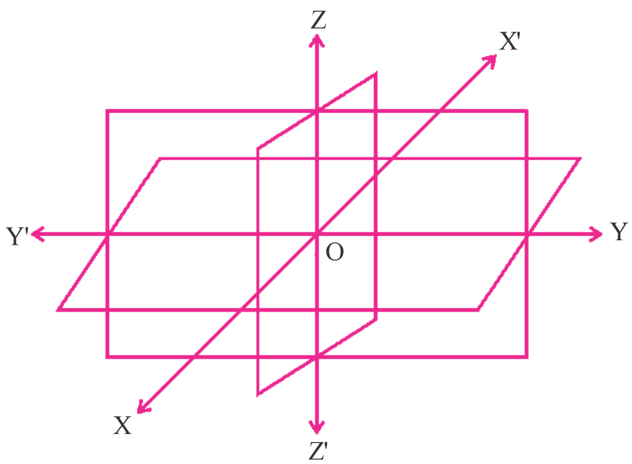

Figure 9.1

The point O is called the origin of the coordinate system. The planes XOY, YOZ and ZOX, called, respectively the **XY-plane, YZ-plane** and the **ZX-plane**, are known as the three **coordinate planes**. We will take the XOY plane as the plane of the paper and the line passing through O perpendicular to the plane as the line ZOZ'. If the plane of the paper is considered as horizontal, then the line Z'OZ will be vertical. In the case of plane we have seen that the coordinate axes divide the plane into four parts called quadrants, in the same manner the three coordinate planes divide the space into eight parts known as octants. These octants could be named as XOYZ, X'OYZ, X'OY'Z, XOY'Z, XOYZ', X'OYZ', X'OY'Z' and XOY'Z' and denoted by octant I, II, III, ..., VIII, respectively.

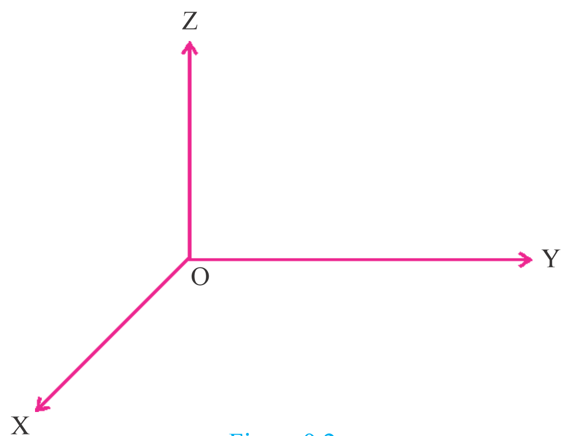

Figure 9.2

Note : The coordinate system discussed here is one of the methods for assigning coordinates to a point in the space. This is called Cartesian coordinate system, named after French mathematician **René Des Cartes**. There are other popular coordinate systems also.

Coordinates of a Point in the Space

Following the method of assigning coordinates to a point in the plane with the help of coordinate axes and the origin, we will now discuss how to associate three coordinates to a given point in the space. Also we will see how a given triplet of real numbers can be associated with a point in the space.

Through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the X-axis, Y-axis and Z-axis in the points A, B and C, respectively as shown in the figure 9.3. Let A(x, 0, 0), B(0, y, 0) and C(0, 0, z). Then, the point P will have the coordinates x, y and z and we write P(x, y, z). Conversely, given real numbers x, y and z, we locate the three points A(x, 0, 0), B(0, y, 0) and C(0, 0, z) on X-axis, Y-axis and Z-axis respectively. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of intersection of these three planes, namely ADPF, BDPE and CEPF is obviously the point P, which corresponds to the ordered triplet (x, y, z). We observe that if P(x, y, z) is any point in the space, then $|x|$, $|y|$ and $|z|$ are perpendicular distances from YZ, ZX and XY planes, respectively. Thus, there is a one to one correspondence between the points in the space and ordered triplets (x, y, z) of real numbers. Thus, the space is identified with the set R^3 of ordered triplets.

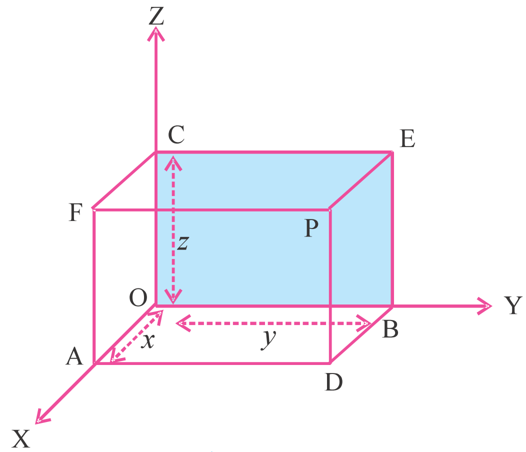

Figure 9.3

Note : The coordinates of the origin O are (0, 0, 0). The coordinates of any point on the X-axis will be (x, 0, 0) and the coordinates of any point in the YZ-plane will be as (0, y, z). Similar remarks apply to the other coordinate axes and other coordinate planes.

Remark : The combination of positive and negative coordinates of a point determines the octant in which the point lies. The following table shows this fact :

Table 9.1

Octants → Coordinates ↓	I OXYZ	II OX'YZ	III OX'Y'Z	IV OXY'Z	V OXYZ'	VI OX'YZ'	VII OX'Y'Z'	VIII OXY'Z'
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Example 5 : Let coordinates of the vertex A of a cuboid be (1, 3, 2) as shown in the figure 9.4. $\overline{AB}$ is perpendicular to Z-axis. Find z-coordinate of the vertex B. If the side $\overline{AB}$ measures 3, then find y-coordinate of B.

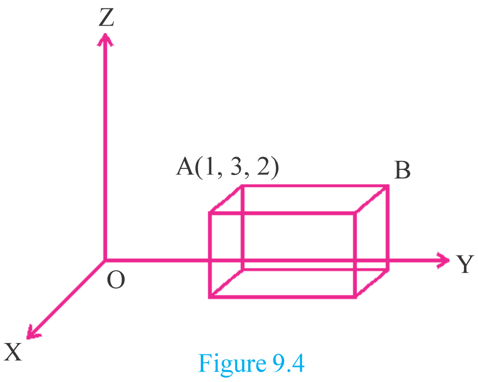

Figure 9.4

Solution : Vertices A and B are on the same heights and hence their z-coordinates are equal and hence z-coordinate of B is 2.

Now, side $\overline{AB}$ is parallel to Y-axis.

Thus y-coordinate of B = y-coordinate of A + 3 = 3 + 3 = 6

Exercise 9.2

1. Fill in the blank in the column, in the following table, by writing the name of the octant of the point in first column :

Point	Octant
(1, 2, 3)	
(1, −2, −4)	
($\sqrt{2}$, 2, −1)	
(−1, −2, 0)	
(−1, −1, −1)	

2. Ram starts walking from a point (−1, 2, 0). He walks 1 unit along $\overrightarrow{OX}$ and then moves in the $\overrightarrow{OY'}$ direction and walks further 2 units. What will be Ram's final position ?

*

9.9 Geometric Representation of Vector

Suppose P is a point in the coordinate plane other than the origin. The line segment $\overline{OP}$ with the direction from O to P, i.e. the direction of the $\overrightarrow{OP}$ will be denoted by $\overrightarrow{\mathbf{OP}}$. Thus, $\overrightarrow{\mathbf{OP}}$ is a directed line segment with the same direction as the ray OP.

We know that any point P in the coordinate plane can be identified with an ordered pair of real numbers, say (x_1, x_2) and conversely, corresponding to any ordered pair of real numbers (x_1, x_2), there exists a point in the plane. We say that the coordinates of the point are (x_1, x_2). In this manner the plane is identified with the set R^2 of ordered pairs of real numbers. Thus we will use R^2 and plane interchangeably.

Position Vector : Let P be a point other than origin in the coordinate plane having coordinates (x_1, x_2). The directed line segment $\overrightarrow{\mathbf{OP}}$ is called the position vector of the point P with respect to the origin O. The coordinates x_1 and x_2 of the point P are taken as components of the position vector $\overrightarrow{\mathbf{OP}}$. For simplicity (x_1, x_2) will be called the position vector of the point P.

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The position vector of the origin has components 0 and 0. Using the definitions of addition of two vectors and multiplication by a scalar it is easy to define addition of two position vectors and multiplication of a position vector by a scalar.

Now we consider a line segment $\overline{AB}$. It is possible to associate direction with this line segment in analogy with the concept of a vector. The direction of the line segment $\overline{AB}$ is same as the direction of the ray from the point A towards the point B. Thus we define directed line segment $\overrightarrow{AB}$ whose length is AB and direction is the same as the direction of the ray $\overrightarrow{AB}$. Using this we define the position vector of point B with respect to point A as the directed line segment $\overrightarrow{AB}$. Here position vector of a point with respect to itself is zero vector.

Look at the following diagram :

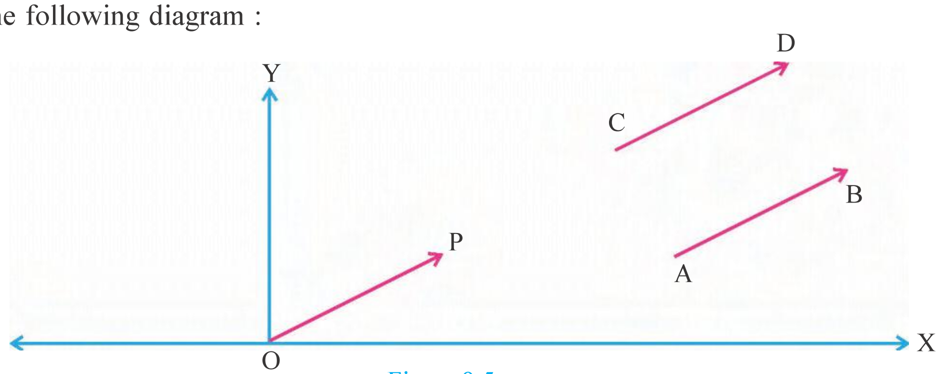

Figure 9.5

We define equality of two directed line segments in analogy with equality of two vectors. Thus, $\overrightarrow{AB} = \overrightarrow{CD}$, if AB = CD and $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction. For every $\overrightarrow{AB}$ there is a directed line segment $\overrightarrow{OP}$, such that $\overrightarrow{AB} = \overrightarrow{OP}$. In the figure, it can be observed that $\overrightarrow{AB} = \overrightarrow{OP}$ and also $\overrightarrow{CD} = \overrightarrow{OP}$. In fact, in the plane there are infinitely many directed line segment that are equal (as directed line segments) but distinct as line segments. For every directed line segment $\overrightarrow{AB}$ there is a position vector $\overrightarrow{OP}$ such that $\overrightarrow{AB} = \overrightarrow{OP}$. Thus, $\overrightarrow{OP}$ represents the class of all directed line segments that are equal to $\overrightarrow{AB}$. The position vectors like $\overrightarrow{OP}$ are called **bound vectors** because one of their end-points namely, O is fixed, whereas the other directed line segments equivalent to $\overrightarrow{OP}$ (like $\overrightarrow{AB}$) are called **free vectors** as both their end-points can be chosen arbitrarily, without changing the vector.

Now look at the figure 9.6.

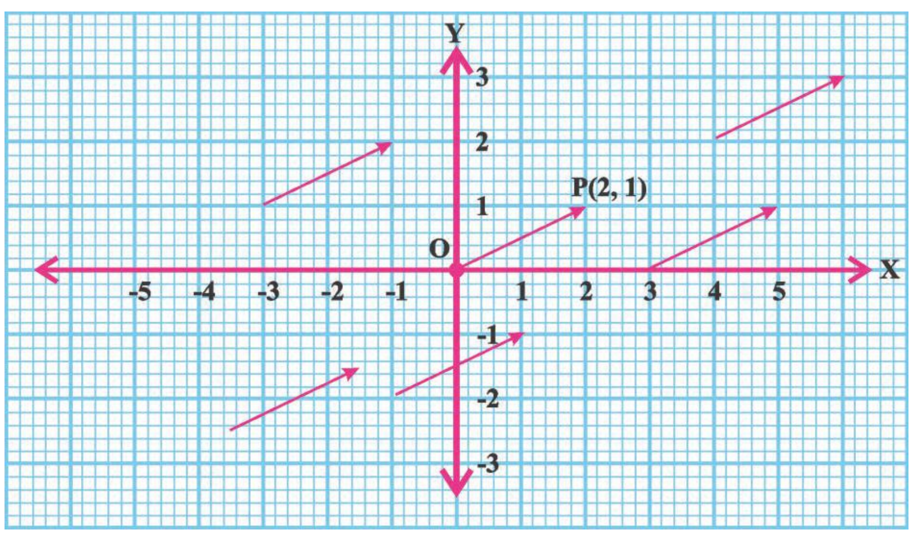

Figure 9.6

Here all the segments are directed in the same way and the end-point of each is obtained by moving horizontally 2 unit towards right and then 1 units vertically upwards (like moving a knight on the chess board) from the initial point. This means each is equal to the position vector (2, 1). In other words the vector (2, 1) represents all the vectors in the figure 9.6. Thus, for every free vector, there exists a bound vector equal to the given vector.

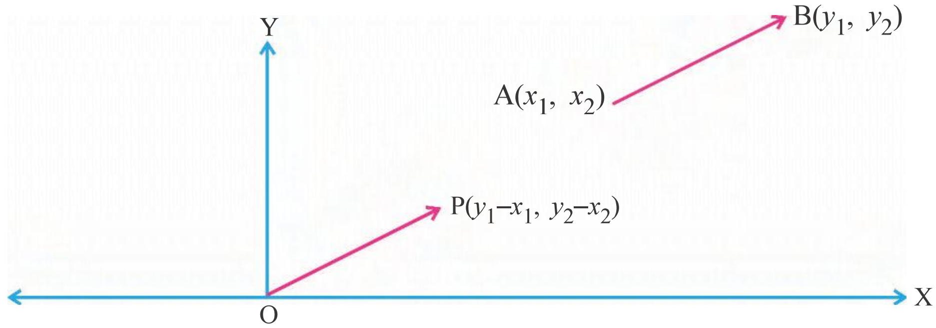

Figure 9.7

Suppose $A(x_1, x_2)$, $B(y_1, y_2)$ and $P(y_1 - x_1, y_2 - x_2)$ are points as shown in the figure 9.7. We have direction of $\overrightarrow{AB}$ = direction of $\overrightarrow{OP}$ and AB = OP = $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. Thus free vector $\overrightarrow{AB}$ is equal to the bound vector $\overrightarrow{OP}$. Also,

$\overrightarrow{AB} = \overrightarrow{OP}$ **(they have the same direction and the same magnitude)**

$= (y_1 - x_1, y_2 - x_2)$

$= (y_1, y_2) - (x_1, x_2)$

= **Position vector of B − Position vector of A**

In a similar manner, we can define position vector of point in the space. Also we define free vectors and bound vectors in the space analogously. Suppose $A(x_1, x_2, x_3)$, $B(y_1, y_2, y_3)$ and $P(y_1 - x_1, y_2 - x_2, y_3 - x_3)$ are points in the space. Then we, write the free vector $\overrightarrow{AB}$ as,

$\overrightarrow{AB} = \overrightarrow{OP} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$

$= (y_1, y_2, y_3) - (x_1, x_2, x_3)$

= **Position vector of B − Position vector of A**

Also, corresponding to this free vector $\overrightarrow{AB}$ there is a bound vector $\overrightarrow{OP}$ such that

$\overrightarrow{AB} = \overrightarrow{OP}$

This is how, we represent a vector in space geometrically.

Example 5 : In each of the following pairs of vectors, determine whether the two vectors have the same or opposite directions or different directions :

(1) (1, 1, 1), (2, 2, 2)

(2) (1, −1, 2), (0.5, −0.5, 1)

(3) (1, −1, 0), (0, 1, −1)

(4) (3, 6, −9), (−1, −2, 3)

(5) (1, 0, 0), (0, 1, 0)

(6) (2, 5, 7), (−2, 5, −7)

Solution : (1) (2, 2, 2) = 2(1, 1, 1). Here $k = 2 > 0$

∴ The vectors have the same direction.

(2) $(0.5, -0.5, 1) = (0.5)(1, -1, 2)$,

Here $k = 0.5 > 0$

$\therefore$ The vectors have the same direction.

(3) If possible, let $(0, 1, -1) = k(1, -1, 0)$, where $k \in \mathrm{R} - \{0\}$.

$\therefore$ $0 = k$, $1 = -k$, $-1 = 0$

$\therefore$ $k = 0$, $k = -1$ and $-1 = 0$ which is not possible.

Thus there is no such k. Hence the vectors have different directions.

(4) $(3, 6, -9) = -3(-1, -2, 3)$. Here $k = -3 < 0$

$\therefore$ The vectors have opposite directions.

(5) As we did in (3) above, for no $k \in \mathrm{R}$,

$(1, 0, 0) = k(0, 1, 0)$

$\therefore$ The vectors have different directions.

(6) If possible, suppose, for some $k \in \mathrm{R} - \{0\}$.

$(2, 5, 7) = k(-2, 5, -7)$ then

$2 = -2k$, $5 = 5k$, $7 = -7k$

$\therefore$ $k = -1$, $k = 1$, $k = -1$

This is not possible as, the first equation is satisfied for $k = -1$, but second one is not satisfied. Thus, the vectors have different directions.

Note : (1) Suppose $\overline{x}$ and $\overline{y}$ are non-zero vectors and $x_i \neq 0$, $y_i \neq 0$ $(i = 1, 2, 3)$

If $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = k$ then according to $k > 0$ or $k < 0$, $\overline{x}$ and $\overline{y}$ have the same direction or opposite directions. If $\frac{y_1}{x_1} \neq \frac{y_2}{x_2}$ or $\frac{y_2}{x_2} \neq \frac{y_3}{x_3}$ or $\frac{y_3}{x_3} \neq \frac{y_1}{x_1}$, then their directions are different.

(2) If $x_1 = 0 = y_1$ and $\frac{y_2}{x_2} = \frac{y_3}{x_3} = k > 0$, then $\overline{x}$ and $\overline{y}$ have the same direction and if $k < 0$ then $\overline{x}$ and $\overline{y}$ have opposite directions.

$\frac{y_2}{x_2} \neq \frac{y_3}{x_3}$, then their direction are different. Similar results are true, if $x_2 = 0 = y_2$ or $x_3 = 0 = y_3$.

(3) Finally, if $x_1 = x_2 = y_1 = y_2 = 0$, then for $\frac{y_3}{x_3} > 0$, the directions are same and for $\frac{y_3}{x_3} < 0$ the directions are opposite.

We note again that $\overline{0} = (0, 0, 0)$ has no direction.

Example 6 : Find unit vector along the vector $\overline{u} = (6, -7, 6)$.

Solution : Here $|\overline{u}| = \sqrt{6^2 + (-7)^2 + 6^2} = \sqrt{121} = 11$

$\therefore$ The unit vector in the direction of $\overline{u}$ is, $\frac{\overline{u}}{|\overline{u}|} = \left(\frac{6}{11}, \frac{-7}{11}, \frac{6}{11}\right)$.

Example 7 : Find the unit vector in the direction opposite to the direction of $\overline{x} - 2\overline{y}$, given that, $\overline{x} = (4, 7, -2)$, $\overline{y} = (1, 2, 2)$.

Solution : $\overline{x} - 2\overline{y} = (4, 7, -2) - 2(1, 2, 2) = (2, 3, -6) = \overline{z}$ (say)

Now $|\overline{z}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$

$\therefore$ The unit vector in the direction opposite to direction of $\overline{z}$ is,

$$-\frac{\overline{z}}{|\overline{z}|} = \left(-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right).$$

Example 8 : For the pairs of points A, B given below find vector $\overrightarrow{AB}$.

(1) A(1, −1), B(1, 2) (2) A(1, −1, 1), B(1, 1, −1)

(3) A(1, 2, 3), B(4, 5, 6) (4) A(1, −2, 1), B(−1, 1, 1)

Solution : $\overrightarrow{AB}$ = **Position vector of B − Position vector of A**

(1) $\overrightarrow{AB} = (1, 2) - (1, -1) = (0, 3)$

(2) $\overrightarrow{AB} = (1, 1, -1) - (1, -1, 1) = (0, 2, -2)$

(3) $\overrightarrow{AB} = (4, 5, 6) - (1, 2, 3) = (3, 3, 3)$

(4) $\overrightarrow{AB} = (-1, 1, 1) - (1, -2, 1) = (-2, 3, 0)$

Exercise 9.3

1. For the following pairs of vectors, determine whether the two vectors have the same or opposite directions or different directions :

(1) (2, −5, 3), (0.4, −1, 0.6) (2) (1, 2, 4), (3, 4, 6)

(3) (2, 4, −6), (−1, −2, 3) (4) (1, 0, 1), (0, 1, 1)

2. Find the unit vector in the direction of the following vectors :

(1) $\overline{x} = (3, -4)$ (2) $\overline{y} = (-3, -4)$ (3) $\overline{x} = (1, 3, 5)$

(4) $\overline{y} = \left(1, \frac{1}{2}, \frac{1}{3}\right)$ (5) $\overline{y} = (1, 0, 0)$ (6) $\overline{y} = (-5, 12)$

3. If $\overline{x} = (x_1, x_2)$ and $\overline{x} = \alpha(1, 2) + \beta(2, 1)$, find α, β.

*

9.10 Distance Formula

Let $\overline{r}_1$ and $\overline{r}_2$ be the position vectors of points A and B respectively and let $\overline{r}_1 = (x_1, y_1, z_1)$ and $\overline{r}_2 = (x_2, y_2, z_2)$. We know that,

$$\overrightarrow{AB} = \text{Position vector of B} - \text{Position vector of A}$$

$$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$$

$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

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This is called distance formula, it gives distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in R^3.

Note : In XY-plane z-coordinates of a point is zero. Hence setting $z_1 = z_2 = 0$ in the distance formula we get the distance formula in plane which was studied in std. 10.

Example 9 : Find the distance between points (1, −1, 2) and (−2, 1, 8).

Solution : Taking P(1, −1, 2) and Q(−2, 1, 8) we have

$$PQ = \sqrt{(1-(-2))^2+(-1-1)^2+(2-8)^2} = \sqrt{3^2+(-2)^2+(-6)^2} = \sqrt{49} = 7$$

Thus the distance between two given points is 7.

Example 10 : Using distance formula, show that the points P(4, −3, −1), Q(5, −7, 6) and R(3, 1, −8) are collinear.

Solution : We have,

$$PQ = \sqrt{(4-5)^2+(-3+7)^2+(-1-6)^2} = \sqrt{1+16+49} = \sqrt{66}$$

$$QR = \sqrt{(5-3)^2+(-7-1)^2+(6+8)^2} = \sqrt{4+64+196} = 2\sqrt{66}$$

$$PR = \sqrt{(4-3)^2+(-3-1)^2+(-1+8)^2} = \sqrt{1+16+49} = \sqrt{66}$$

Thus, PQ + PR = QR and hence Q–P–R.

∴ The given points are collinear.

Example 11 : If A(1, 2, 4), B(1, 2, 0) and C(1, 5, 0), show that ΔABC is a right angled triangle.

Solution : $AB^2 = (1-1)^2 + (2-2)^2 + (4-0)^2 = 16$. So AB = 4

$BC^2 = (1-1)^2 + (2-5)^2 + (0-0)^2 = 9$. So BC = 3

$AC^2 = (1-1)^2 + (5-2)^2 + (0-4)^2 = 25$. So AC = 5

∴ A, B, C are non-collinear and form a triangle.

Also $AC^2 = AB^2 + BC^2$ and hence ΔABC is a right angled triangle with right angle at B.

Example 12 : Find coordinates of points on X-axis at distance $3\sqrt{3}$ from the point A(2, −1, 1).

Solution : A point on X-axis is P(x, 0, 0). Now AP = $3\sqrt{3}$

$$\sqrt{(x-2)^2+(0+1)^2+(0-1)^2} = 3\sqrt{3}$$

∴ $x^2 - 4x + 4 + 1 + 1 = 27$

∴ $x^2 - 4x + 4 = 25$

∴ $(x - 2)^2 = 5$

∴ $x - 2 = \pm 5$

∴ $x = 7$ or $x = -3$

Thus, there are two such points namely P(7, 0, 0) and P(−3, 0, 0).

Example 13 : Find the equation of the set of points which are equidistant from the points (2, −1, 1) and (1, 3, 1).

Solution : Let (x, y, z) be the coordinates of the points equidistant from the given points (2, −1, 1) and (1, 3, 1).

$$(x - 2)^2 + (y + 1)^2 + (z - 1)^2 = (x - 1)^2 + (y - 3)^2 + (z - 1)^2$$

$\therefore \quad x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 2z + 1$

$\qquad = x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 2z + 1$

$\therefore \quad -4x + 2y + 5 = -2x - 6y + 10$

$\therefore \quad 2x - 8y + 5 = 0$

This is the equation of the required set.

Note : In the plane this type of set is called the **perpendicular bisector line** of the given segment. In space this is called the **perpendicular bisector plane** of the given segment. It is a plane perpendicular to the segment and passes through the mid-point of the segment.

Exercise 9.4

1. Find the distance between the following pairs of points :

(1) (1, −1, 3), (1, −1, 3) (2) (1, 2, 3), (3, 4, 5)

(3) (2, −3, 18), (0, 1, 14) (4) $(1, \sqrt{2}, -1), (3, 3\sqrt{2}, 1)$

(5) (1, −2, 5014), (4, 2, 5014) (6) (1, 1, 0), (0, 1, 0)

2. Using distance formula, determine whether the following points are collinear or not :

(1) P(1, 3, 2), Q(1, 2, 1), R(2, 3, 1) (2) A(0, 1, 0), B(0, −1, 0), C(0, 2, 0)

(3) L(1, 2, 3), M(−3, −1, 1), A(−3, 2, 7) (4) V(1, 2, 3), A(2, 3, 1), H(3, 1, 2)

3. Given that A(0, 7, 10), B(−1, 6, 6), C(−4, 9, 6), determine the type of ΔABC.

4. Find the points on Z-axis which are at a distance $\sqrt{14}$ from the point (−2, 1, 3).

5. Find the equation of the set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (−1, 2, 7) respectively, $k \in R$.

6. Show that O(0, 0, 0), A(2, −3, 6), B(0, −7, 0) are vertices of an isosceles triangle.

*

9.11 Section Formula

We have studied section formula for a line segment joining two points in R^2. Now using vectors we will derive section formula for a line segment joining two points in R^3.

Let $\bar{r}_1 = (x_1, y_1, z_1)$ and $\bar{r}_2 = (x_2, y_2, z_2)$ be the position vectors of two points A and B in the space respectively. Suppose $P \in \overleftrightarrow{AB}$ ($P \neq A$, $P \neq B$). As the points A, B and P are on the same line, the directions of $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are same or opposite. Thus, we have

$$\overrightarrow{AP} = k\overrightarrow{PB}, \text{ where } k \neq 0 \qquad \text{(i)}$$

$\therefore \quad |\overrightarrow{AP}| = |k|\,|\overrightarrow{PB}|$ or $AP = |k|\,PB$

$\therefore \quad \frac{AP}{PB} = |k|$

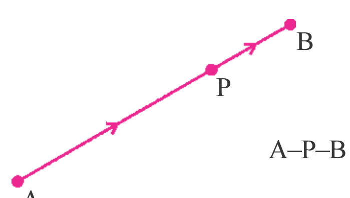

Figure 9.8

Let the position vector of P be $\bar{r} = (x, y, z)$.

Now let P divide $\overline{AB}$ from the side of A in the ratio λ.

(i) If $\lambda > 0$ and A–P–B and $\frac{AP}{PB} = \lambda$, we say that P divides $\overline{AB}$ internally from the side of A in the ratio λ. (figure 9.8)

$\therefore \quad \frac{AP}{PB} = |k| = \lambda$

Further, as $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction, $k > 0$

So, $|k| = k$.

Since, $|k| = \lambda$, $k = \lambda$

$\therefore \quad \overrightarrow{AP} = \lambda \overrightarrow{PB}$. **(using (i))**

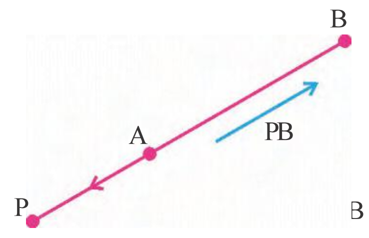

Figure 9.9

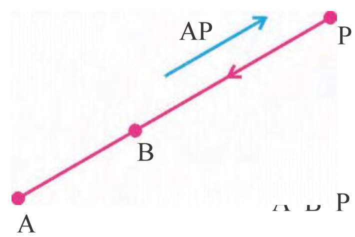

Figure 9.10

(ii) If $\lambda < 0$ and P–A–B or A–B–P and $\frac{AP}{PB} = -\lambda$, we say P divides $\overline{AB}$ externally from the side of A in the ratio λ. As shown in the figures 9.9 and 9.10, it is clear that $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have opposite directions, so $k < 0$.

$\therefore \quad |k| = -k$

$\therefore \quad \frac{AP}{PB} = |k| = -k$ and $\frac{AP}{PB} = -\lambda$

Hence $k = \lambda$

$\overrightarrow{AP} = \lambda \overrightarrow{PB}$ **$(\overrightarrow{AP} = k \overrightarrow{PB})$**

Thus, in each case $\overrightarrow{AP} = \lambda \overrightarrow{PB}$

$\therefore \quad \bar{r} - \bar{r}_1 = \lambda(\bar{r}_2 - \bar{r})$

$\therefore \quad \bar{r} - \bar{r}_1 = \lambda\bar{r}_2 - \lambda\bar{r}$

$\therefore \quad (1 + \lambda)\,\bar{r} = \lambda\bar{r}_2 + \bar{r}_1$

Note that by the definition of division, $\lambda \neq -1$.

$\therefore \quad \bar{r} = \frac{1}{\lambda + 1}(\lambda\bar{r}_2 + \bar{r}_1)$

$$(x, y, z) = \frac{1}{\lambda+1}\left(\lambda(x_2, y_2, z_2) + (x_1, y_1, z_1)\right)$$

$$= \frac{1}{(\lambda+1)}(\lambda x_2 + x_1, \lambda y_2 + y_1, \lambda z_2 + z_1)$$

$$\therefore \quad (x, y, z) = \left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right)$$

This is called **section formula.** It gives coordinates of the point which divides line segment $\overline{AB}$ in the ratio λ from the side of point $A(x_1, y_1, z_1)$.

If the ratio λ is $m : n$, then above formula gives,

$$\bar{r} = \frac{1}{\frac{m}{n}+1}\left(\frac{m}{n}\bar{r}_2 + \bar{r}_1\right) = \frac{1}{m+n}(m\bar{r}_2 + n\bar{r}_1); \quad m + n \neq 0$$

$$\therefore \quad (x, y, z) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

9.12 Some Applications of Section Formula

(i) Coordinates of mid-points : If P is the mid-point of $\overline{AB}$, then AP = PB and A–P–B.

$\therefore \quad \frac{AP}{PB} = \lambda = 1$

$\therefore$ Let position vector of P be $\bar{r}$.

If $\bar{r}_1 = (x_1, y_1, z_1)$ and $\bar{r}_2 = (x_2, y_2, z_2)$ and $\bar{r} = (x, y, z)$, then section formula gives

$$(x, y, z) = \frac{1}{2}((x_1, y_1, z_1) + (x_2, y_2, z_2)) \qquad (\lambda = 1)$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$\therefore$ The position vector of the mid-point of $\overline{AB}$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

(ii) Centroid of a Triangle : Let ABC be a triangle in R^3. Suppose position vectors of A, B and C are $\bar{r}_1 = (x_1, y_1, z_1)$, $\bar{r}_2 = (x_2, y_2, z_2)$ and $\bar{r}_3 = (x_3, y_3, z_3)$ respectively.

As shown in the figure 9.11, D is mid-point of $\overline{BC}$. Hence its position vector is $\frac{\bar{r}_2 + \bar{r}_3}{2}$.

Let G be the point dividing $\overline{AD}$ in the ratio 2 : 1 from the side of A. The position vector of G is

$$\frac{1}{2+1}\left(2 \cdot \frac{1}{2}(\bar{r}_2 + \bar{r}_3) + \bar{r}_1\right) = \frac{1}{3}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3)$$

Symmetry of this result shows that G is on all the three medians. Thus, the medians of a triangle are concurrent in G.

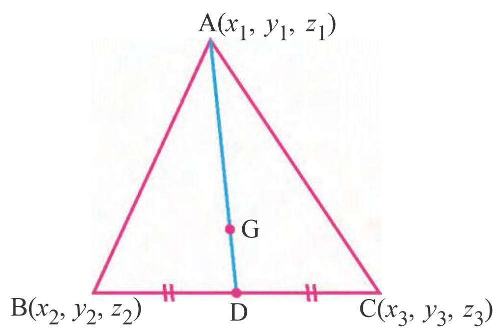

Figure 9.11

Thus, G is the centroid of ΔABC and its position vector is $\frac{1}{3}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3)$. So the coordinates of G are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

Example 14 : Find the coordinates of the point which divides the segment joining the points A(2, 3, –1) and B(1, –3, 5) from A in the ratio (i) 3 : 5 internally, (ii) 3 : 5 externally.

Solution : (i) Let P(*x*, *y*, *z*) divides $\overline{AB}$ from A in the ratio 3 : 5 internally. Thus $m = 3$, $n = 5$. Now by section formula,

$$x = \frac{3(1) + 5(2)}{3+5} = \frac{3+10}{8} = \frac{13}{8}$$

$$y = \frac{3(-3) + 5(3)}{3+5} = \frac{-9+15}{8} = \frac{6}{8} = \frac{3}{4}$$

$$z = \frac{3(5) + 5(-1)}{3+5} = \frac{15-5}{8} = \frac{10}{8} = \frac{5}{4}$$

Thus, the point $\left(\frac{13}{8}, \frac{3}{4}, \frac{5}{4}\right)$ divides $\overline{AB}$ in the ratio 3 : 5 internally from A.

(ii) Here the division is external. Thus $m = 3$, $n = -5$. Hence the coordinates of required point are

$$x = \frac{3(1) - 5(2)}{3 - 5} = \frac{3 - 10}{-2} = \frac{-7}{-2} = \frac{7}{2}$$

$$y = \frac{3(-3) - 5(3)}{3 - 5} = \frac{-9 - 15}{-2} = 12$$

$$z = \frac{3(5) - 5(-1)}{3 - 5} = \frac{15 + 5}{-2} = -10$$

Thus, the coordinates of the point which divides $\overline{AB}$ from A in the ratio 3 : 5 externally are $\left(\frac{7}{2}, 12, -10\right)$.

Example 15 : Use section formula to examine collinearity of the points (1, –3, 3), (3, 7, 1), (1, 1, 1).

Solution : If A(1, –3, 3), B(3, 7, 1) and C(1, 1, 1) are collinear, then one of them divides the line segment joining the other two in some ratio say $k : 1$. Suppose B divides $\overline{AC}$ in some ratio k.

$$3 = \frac{k(1) + 1}{k + 1} = \frac{k + 1}{k + 1} = 1$$

This is not true. Hence the points are not collinear.

Example 16 : Show that the triangle with vertices (–1, 6, 6), (–4, 9, 6) and (0, 7, 10) is a right angled triangle. Further verify that the mid-point of its hypotenuse is equidistant from all vertices.

Solution : Let A(–1, 6, 6), B(–4, 9, 6) and C(0, 7, 10).

Now, $AB^2 = (-4 + 1)^2 + (9 - 6)^2 + (6 - 6)^2 = 9 + 9 = 18$

$BC^2 = (0 + 4)^2 + (7 - 9)^2 + (10 - 6)^2 = 16 + 4 + 16 = 36$

$AC^2 = (0 + 1)^2 + (7 - 6)^2 + (10 - 6)^2 = 1 + 1 + 16 = 18$

$\therefore \quad AB^2 + AC^2 = BC^2$

Thus, ΔABC is a right angled triangle and $\overline{BC}$ is its hypotenuse.

Let $M(x, y, z)$ be the mid-point of $\overline{BC}$. Then

$$(x, y, z) = \left(\frac{0 - 4}{2}, \frac{7 + 9}{2}, \frac{10 + 6}{2}\right) = (-2, 8, 8).$$

Now as, M is the mid-point of $\overline{BC}$ and $BC = \sqrt{36} = 6$

$BM = CM = 3$

Further $AM = \sqrt{(-2 + 1)^2 + (8 - 6)^2 + (8 - 6)^2} = \sqrt{1 + 4 + 4} = 3$

Thus, AM = BM = CM, i.e. M is equidistant from all the vertices of ΔABC.

Miscellaneous Problems :

In a plane if four points are given, then they form a **quadrilateral** provided any three of them are non-collinear. Using distance formula and section formula, the type of the quadrilateral can be determined. In the case of four points in the space, they may form a quadrilateral if all these points are coplanar. Thus, before determining the type of a quadrilateral, we must make sure that the points are coplanar. Following examples are based on this.

Example 17 : Determine whether the points A(0, 0, 0), B(1, 0, 0), C(0, 1, 0), D(0, 0, 1) are vertices of a quadrilateral or not. If they form a quadrilateral, then determine its type.

Solution : $\overrightarrow{AC} = (0, 1, 0)$, $\overrightarrow{BD} = (-1, 0, 1)$.

$\overrightarrow{AC}$ and $\overrightarrow{BD}$ have different directions. Thus, $\overleftrightarrow{AC} \nparallel \overleftrightarrow{BD}$.

Now let us examine if they intersect in a point.

If they intersect in a point, it may happen the point of intersection is A or B or C or D.

$\overrightarrow{AC} = (0, 1, 0)$, $\overrightarrow{AD} = (0, 0, 1)$ **(i)**

$\overrightarrow{AC}$ and $\overrightarrow{AD}$ have different directions.

$\therefore$ A, C, D cannot be collinear.

$\overrightarrow{BC} = (-1, 1, 0)$, $\overrightarrow{BD} = (-1, 0, 1)$ **(ii)**

$\therefore$ B, C and D cannot be collinear.

Similarly from (i) and (ii) A, B, C or A, B, D are not collinear.

Now suppose, if possible $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in a point P other than A or B or C or D.

Four distinct points A, B, C, D lie in a plane if either $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect in a point or $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$. If possible, suppose they intersect in a point P(*x*, *y*, *z*). Thus $P \in \overleftrightarrow{AC}$ and $P \in \overleftrightarrow{BD}$. Let P divide $\overline{AC}$ from the side of A in the ratio λ and it divide $\overline{BD}$ from the side of B in the ratio μ($\lambda \in R - \{0, -1\}$, $\mu \in R - \{0, -1\}$). By section formula,

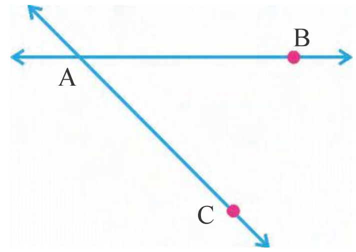

Figure 9.12(i)

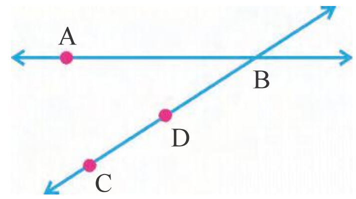

Figure 9.12(ii)

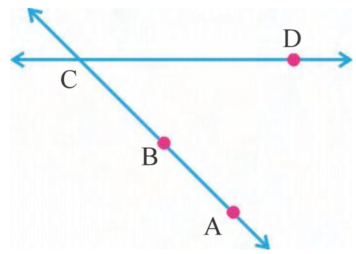

Figure 9.12(iii)

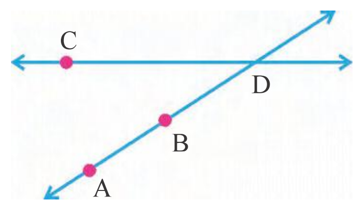

Figure 9.12(iv)

$$P \in \overleftrightarrow{AC} \Rightarrow \left.\begin{aligned} x &= \frac{\lambda(0)+0}{\lambda+1} = 0 \\ y &= \frac{\lambda(1)+0}{\lambda+1} = \frac{\lambda}{\lambda+1} \\ z &= \frac{\lambda(0)+0}{\lambda+1} = 0 \end{aligned}\right\} \quad \textbf{(iii)}$$

$$\text{and } P \in \overleftrightarrow{BD} \Rightarrow \left.\begin{aligned} x &= \frac{\mu(0)+1}{\mu+1} = \frac{1}{\mu+1} \\ y &= \frac{\mu(0)+0}{\mu+1} = 0 \\ z &= \frac{\mu(1)+0}{\mu+1} = \frac{\mu}{\mu+1} \end{aligned}\right\} \quad \textbf{(iv)}$$

Thus, from (iii) and (iv) $x = 0 = \frac{1}{\mu+1}$ which is not possible. Thus, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ neither intersect nor are parallel. Thus the points A, B, C and D are not coplanar. Hence given points are not vertices of a quadrilateral.

Note : Four non-coplanar points in the space form a geometrical figure called a tetrahedron (figure 9.12(v)). A tetrahedron has four triangular faces and six edges.

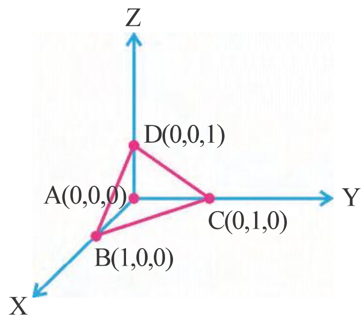

Figure 9.12(v)

Example 18 : Examine coplanarity of the point P(1, 1, 1), Q(−2, 4, 1), R(−1, 5, 5) and S(2, 2, 5). Also determine the type of quadrilateral formed by them, if any.

Solution : The mid-point of $\overline{PR}$ = M(0, 3, 3)

The mid-point of $\overline{QS}$ = M(0, 3, 3)

$\therefore$ $\overleftrightarrow{PR}$ and $\overleftrightarrow{QS}$ intersect in M.

$\therefore$ P, Q, R, S are coplanar.

Now, $\overrightarrow{PQ} = (-2, 4, 1) - (1, 1, 1) = (-3, 3, 0)$

$\overrightarrow{QR} = (-1, 5, 5) - (-2, 4, 1) = (1, 1, 4)$

$\overrightarrow{SR} = (-1, 5, 5) - (2, 2, 5) = (-3, 3, 0)$

$\overrightarrow{PS} = (2, 2, 5) - (1, 1, 1) = (1, 1, 4)$

Thus, $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ have same directions. $\overrightarrow{QR}$ and $\overrightarrow{PS}$ have same directions.

Further,

$$PQ = \sqrt{(-3)^2 + (3)^2 + 0} = \sqrt{18} = RS$$

$$QR = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = PS$$

Also as seen above diagonals $\overline{PR}$ and $\overline{QS}$ bisect each other and also

$$PR = \sqrt{(1+1)^2 + (1-5)^2 + (1-5)^2} = \sqrt{4+16+16} = 6$$

$$QS = \sqrt{(-2-2)^2 + (4-2)^2 + (1-5)^2} = \sqrt{16+4+16} = 6$$

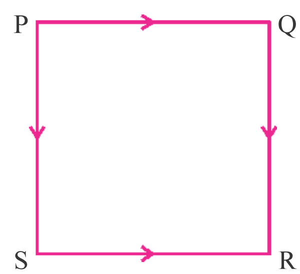

Figure 9.13

Thus for the parallelogram PQRS, all four sides are of equal length and diagonals have equal length. Thus, $\square$ PQRS is a square.

So for collinearity for three points was checked using distance formula and also using section formula. Suppose three distinct points A, B and C are given. Then they are collinear only if one of the following is true.

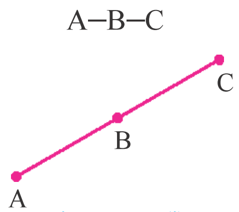

Figure 9.14(i)

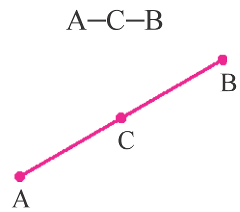

Figure 9.14(ii)

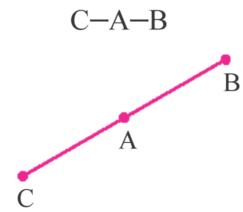

Figure 9.14(iii)

In all three cases $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same or opposite directions. Hence three points A, B and C are collinear only if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same or opposite directions. The following examples are based on this fact.

Example 19 : Using directions examine if following points are collinear :

(1) A(0, 2), B(2, 4), C(–2, 0) (2) P(1, –1, 0), Q(–3, 1, 2), R(–1, 0, 1)

(3) A(1, 2, 3), P(5, 2, 2), S(2, 3, 1) (4) L(0, 0), M(1, 0), N(0, 1)

Solution : (1) $\overrightarrow{AB} = (2, 4) - (0, 2) = (2, 2)$

$\overrightarrow{BC} = (-2, 0) - (2, 4) = (-4, -4)$

Obviously $\overrightarrow{BC} = (-2)\overrightarrow{AB}$

Hence $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have opposite directions. Thus A, B and C are collinear. ($\overleftrightarrow{AB}$ ⫲ $\overleftrightarrow{BC}$)

(2) $\overrightarrow{PQ} = (-3, 1, 2) - (1, -1, 0) = (-4, 2, 2)$

$\overrightarrow{QR} = (-1, 0, 1) - (-3, 1, 2) = (2, -1, -1)$

Here $\overrightarrow{PQ} = (-2)\overrightarrow{QR}$. So, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have opposite directions. Thus, P, Q, R are collinear.

($\overleftrightarrow{PQ}$ ⫲ $\overleftrightarrow{QR}$)

(3) $\overrightarrow{AP} = (5, 2, 2) - (1, 2, 3) = (4, 0, -1)$

$\overrightarrow{PS} = (2, 3, 1) - (5, 2, 2) = (-3, 1, -1)$

If possible suppose for a non-zero, $k \in$ R

$\overrightarrow{AP} = k(\overrightarrow{PS})$

$\therefore$ $(4, 0, -1) = k(-3, 1, -1)$

$\therefore$ $4 = -3k$, $0 = k$, $-1 = -k$

For any $k \in$ R all there are not satisfied. So, $\overrightarrow{AP}$ and $\overrightarrow{PS}$ have different directions. Hence A, P and S are not collinear.

(4) $\overrightarrow{LM} = (1, 0) - (0, 0) = (1, 0)$

$\overrightarrow{MN} = (0, 1) - (1, 0) = (-1, 1)$

If possible suppose for some $k \in$ R $-$ {0},

$\overrightarrow{LM} = k(\overrightarrow{MN})$

$\therefore$ $(1, 0) = k(-1, 1)$

$\therefore$ $1 = -k$, $k = 0$ which is not possible.

So, $\overrightarrow{LM}$ and $\overrightarrow{MN}$ have different directions. Hence given points are non-collinear.

Example 20 : Prove that A(1, 2, 3), B(−1, −2, −1), C(2, 3, 2) and D(4, 7, 6) forms a parallelogram.

Solution : Mid-point of $\overline{AC} = \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$. Mid-point of $\overline{BD} = \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

$\therefore$ $\overline{AC}$ and $\overline{BD}$ bisect each other and they intersect at the mid-point. Hence $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are coplanar.

$\therefore$ A, B, C, D form a quadrilateral in a plane and its diagonals bisect each other.

$\therefore$ □ ABCD is a parallelogram.

Alternate Method :

$\overrightarrow{AB} = (-2, -4, -4)$, $\overrightarrow{BC} = (3, 5, 3)$, $\overrightarrow{DC} = (-2, -4, -4)$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are in the same direction.

$\therefore$ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or A, B, C, D are collinear. But $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are in different direction,

$\therefore$ $C \notin \overleftrightarrow{AB}$

$\therefore$ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Similarly, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ $\quad$ ($\overrightarrow{AD} = (3, 5, 3)$)

$\therefore$ A, B, C, D are coplanar and □ ABCD is a parallelogram.

Note : Solution given below is not proper :

$AB = \sqrt{4+16+16} = 6$, $CD = \sqrt{4+16+16} = 6$, $AD = \sqrt{9+25+9} = \sqrt{43} = BC$

$\therefore$ Opposite sides of □ ABCD congruent. Hence □ ABCD is a parallelogram.

If A, B, C, D are coplanar, then this decision is correct. So it is necessary to prove A, B, C, D are coplanar. See the example given below :

Example 21 : Prove that for O(0, 0, 0), A(1, 1, 0), B(1, 0, 1), C(0, 1, 1), OA = AB = BC = AC = OB = OC, but O, A, B, C do not form a parallelogram.

Solution : $\overrightarrow{OA} = (1, 1, 0)$, $\overrightarrow{OB} = (1, 0, 1)$, $\overrightarrow{OC} = (0, 1, 1)$

$\overrightarrow{AB} = (0, -1, 1)$, $\overrightarrow{BC} = (-1, 1, 0)$, $\overrightarrow{AC} = (-1,$

$\therefore$ $OA = OB = OC = AB = BC = AC = \sqrt{2}$

But any two of above vectors are not in the same or in the opposite directions.

$\therefore$ O, A, B, C do not form a parallolegram.

That these points are non-coplanar can be proved. Points O, A, B, C form a tetrahedron.

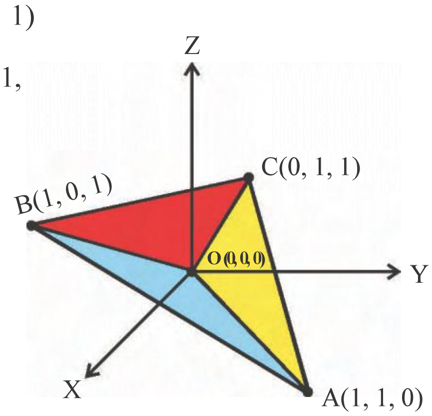

Figure 9.15

Exercise 9.5

1. Find the points of trisection of the segment $\overline{AB}$, where A(1, 3, −2), B(2, 4, −1).
2. Using section formula, check the collinearity of points :

(1) P(1, −1, 1), Q(1, 0, 3), R(2, 0, 0) $\qquad$ (2) A(5, 6, −1), B(1, −1, 3), C(1, 1, 1)

(3) L(2, −3, 4), M(−1, 2, 1), $N\left(-\frac{1}{4}, \frac{3}{4}, \frac{5}{4}\right)$ $\qquad$ (4) O(0, 0, 0), A(1, 1, 1), B(2, 2, 2)

(5) L(1, 2, 3), M(−1, −2, −3), N(1, −2, 3)

Exercise 9

1. Show that the points A(−2, −3, −1), B(2, 1, 1), C(−3, −2, −2) and D(−7, −6, −4) form a parallelogram. Is it a rectangle ?

2. Determine the type of ΔABC, given that A(0, 1, 2), B(2, −1, 3), C(1, −3, 1).

3. Find the equation of the set of points at the same distance from the points (1, 2, 3) and (3, 2, −1).

4. Find the lengths of medians and coordinates of the centroid in each of the following triangles :

(1) A(1, 0, 1), B(1, 2, 0), C(1, 1, 2)

(2) P(1, 2, 3), Q(−1, 1, 0), R(0, 0, 3)

(3) L(−1, −2, −3), M(1, 2, 3), N(1, 2, 1)

5. Let P(1, 2, −3), Q(3, 0, 1) and R(−1, 1, 4) be the mid-points of the sides of ΔABC. Find the centroid of ΔABC.

6. Using vectors examine the collinearity of the points given below. If they are collinear, then in which ratio and from which side one point divides segment joining other two ?

(1) A(5, 4, 6), B(1, −1, 3), C(4, 3, 2)

(2) A(2, 3, 4), B(−4, 1, −10), C(−1, 2, −3)

(3) A(1, 2, 3), B(0, 4, 1), C(−1, −1, −1)

(4) L(3, 2, −4), M(5, 4, −6), N(9, 8, −10)

(5) P(2, 3, 4), Q(3, 4, 5), R(1, 2, 3)

7. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The magnitude of sum of vectors $(1, -\sqrt{2})$, $(2, \sqrt{2})$ is ...

(a) −3 (b) 3 (c) 9 (d) −9

(2) Given that the points A(1, 0, 1), B(2, −1, 3) and C(3, −2, 5) are collinear, then the ratio in which C divides $\overline{AB}$ from side of A is ...

(a) 2 : 1 (b) −1 : 2 (c) 1 : 2 (d) −2 : 1

(3) The centroid of the triangle whose vertices are P(1, −2, 1), Q(2, 3, −1), R(1, −1, −1) is ...

(a) (1, 2, 1) (b) $\left(\frac{4}{3}, 0, -\frac{1}{3}\right)$ (c) $\left(\frac{3}{2}, \frac{1}{2}, 0\right)$ (d) $\left(-\frac{4}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$

(4) If the position vectors of A and B are respectively (1, 1, 0) and (0, 1, 1) then $\overrightarrow{AB}$ = ...

(a) (0, 0, 0) (b) (1, 0, −1) (c) (−1, 0, 1) (d) (1, 2, 1)

(5) The direction of (1, 1, 2) and (2, 1, 0) is ......

(a) same (b) opposite (c) different (d) not defined

(6) $<2, 2, 2>$ = ...

(a) $-<-4, -4, -4>$ (b) $<1, 1, -1>$ (c) $<-1, 1, -1>$ (d) $<0, 0, 0>$

(7) $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$ = ...

(a) $<1, 1, -1>$ (b) $<\cos\theta \cos\alpha, \cos\theta \sin\alpha, \sin\theta>$

(c) $<5, 5, 5>$ (d) $<3, 3, -3>$

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(8) Unit vector in the direction of (2, 2, −1) is ......

(a) $\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ (b) $\left(\frac{-2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ (c) (2, 2, 1) (d) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(9) Unit vector in the direction of (1, 0, 0) is ......

(a) (0, 1, 0) (b) (0, 0, 1) (c) (−1, 0, 0) (d) (1, 0, 0)

(10) If the centroid of ΔABC is (0, 0, 0), where A(1, 1, 1), B(2, 1, 2), C(x, y, z) then (x, y, z) = ...

(a) (3, 2, 3) (b) (0, 0, 0) (c) (−3, −2, −3) (d) (1, −1, 1)

(11) If A(1, 1, 2), B(2, 1, 2), C(2, 2, 1) then A, B, C are ......

(a) vertices of a triangle (b) collinear

(c) on axes (d) non-coplanar

(12) If A(1, 2, 1), B(2, 3, 2), C(2, 1, 3), D(3, 2, 4), then directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are ......

(a) same (b) perpendicular to each other

(c) different (d) not defined

(13) If A(1, 2, 1), B(2, 3, 2), C(2, 1, 3), D(3, 2, 4) then ......

(a) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (b) $\overleftrightarrow{AB} = \overleftrightarrow{CD}$

(c) $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is singleton (d) C $\in \overleftrightarrow{AB}$

(14) Vector (0, 0, 0) ....

(a) has no direction (b) has no magnitude

(c) is in the direction of (1, 1, 1) (d) is in opposite direction of (−1, −1, −1)

(15) P(2, 3, 1) and Q(7, 15, 1) then $|\overrightarrow{PQ}|$ = ...

(a) 5 (b) 12 (c) 13 (d) 17

(16) A vector which is in the directions of (3, 6, 2) and has magnitude 4 is ....

(a) $\left(\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right)$ (b) (12, 24, 8) (c) $\left(\frac{12}{7}, \frac{24}{7}, \frac{8}{7}\right)$ (d) (−12, −24, −8)

(17) A unit vector which is in the opposite direction of (2, −2, 1) is ....

(a) $\left(\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ (b) (−2, 2, −1) (c) $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(18) ($cos\alpha$, $sin\alpha$) and ($cos(\pi + \alpha)$, $sin(\pi + \alpha)$) ($\alpha \in$ R) have directions ....

(a) same (b) opposite (c) different (d) same as (1, 0)

(19) If $\overline{x}$ is a non-zero vector and $k > 0$, $k \neq 1$, then $\frac{-k\overline{x}}{|\overline{x}|}$ is ....

(a) unit vector in the direction of $\overline{x}$

(b) in the direction of $\overline{x}$ having magnitude k

(c) in the opposite direction of $\overline{x}$ having magnitude k

(d) unit vector in the opposite direction of $\overline{x}$

(20) If $\overline{x}$ is a non-zero vector and $k < 0$, $k \neq -1$, then $\frac{k\overline{x}}{|\overline{x}|}$ is ....

(a) unit vector in the direction of $\overline{x}$

(b) unit vector in the opposite direction of $\overline{x}$

(c) in the opposite direction of $\overline{x}$ having magnitude $|k|$

(d) in the direction of $\overline{x}$ having magnitude $|k|$

*

Summary

We studied following points in this chapter :

1. Set of ordered pairs and ordered triplets of real numbers R^2 and R^3 respectively form a vector space over R.

2. Magnitude of a vector $\overline{x} = (x_1, x_2, x_3)$ is $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and if $\overline{x} = (x_1, x_2)$, then $|\overline{x}| = \sqrt{x_1^2 + x_2^2}$.

3. $|\overline{x}| = 0 \Leftrightarrow \overline{x} = \overline{0}$ and $|k\overline{x}| = |k||\overline{x}|$

4. For two non-zero vectors $\overline{x}$ and $\overline{y}$ if $\overline{x} = k\overline{y}$, then $\overline{x}$, $\overline{y}$ have the same direction, if $k > 0$. They have opposite directions, if $k < 0$.

5. For two points A and B **(in R^3 or R^3)**

$\overrightarrow{AB}$ = Position vector of B − Position vector of A

6. Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

7. If $\overline{r}_1$ and $\overline{r}_2$ are position vectors of points A and B respectively and point P divides $\overline{AB}$ in the ratio λ from the side of A, then position vector of P is $\frac{1}{\lambda+1}(\lambda\overline{r}_2 + \overline{r}_1)$.

8. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ then position vector of the centroid is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Bhaskara II

- Solutions of Diophantine equations of the second order, such as $61x^2 + 1 = y^2$. This very equation was posed as a problem in 1657 by the French mathematician Pierre de Fermat, but its solution was unknown in Europe until the time of Euler in the 18th century.
- Solved quadratic equations with more than one unknown and found negative and irrational solutions.
- Preliminary concept of infinitesimal calculus, along with notable contributions towards integral calculus.
- Conceived differential calculus, after discovering the derivative and differential coefficient.
- Stated Rolle's theorem, a special case of one of the most important theorems in analysis, the mean value theorem. Traces of the general mean value theorem are also found in his works.
- Calculated the derivatives of trigonometric functions and formulae.
- In *Siddhanta Shiromani*, Bhaskara developed spherical trigonometry along with a number of other trigonometric results.

Bhaskara II gave the formula : $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$

Bhaskaracharya studied Pell's equation $px^2 + 1 = y^2$ for $p = 8, 11, 32, 61$ and 67. When $p = 61$, he found the solutions $x = 226153980$, $y = 1776319049$. When $p = 67$ he found the solutions $x = 5967$, $y = 48842$. He studied many Diophantine problems.

The topics covered in Lilavati, thirteen chapters of the book are : definitions; arithmetical terms; interest; arithmetical and geometrical progressions; plane geometry; solid geometry; the shadow of the gnomon; the kuttaka; combinations.

Chapter 10

LIMITS

If people do not believe that mathematics is simple, it is only because they do not realise how complicated life is.

– John Louis Von Neumann

10.1 Introduction and History

Now we start with the study of calculus. Whatever we have studied so far is known as pre-calculus. Calculus is a Latin word meaning a small stone used for counting. Calculus is the study of change in the way that geometry is the study of shape and algebra is the study of operations and their applications to solving equations. Calculus has widespread applications in science, economics and engineering.

The ancient period saw some of the ideas that led to integral calculus. Calculations of volumes and areas by integral calculus can be found in the Egyptian ***Moscow Papyrus*** (1820 B.C.). But the formulae are mere instructions and some of them are wrong. From the age of Greek mathematics **Eudoxus** (408-335 B.C.) used the method of exhaustion which prefigures the concept of the limit to calculate areas and volumes. **Archimedes** (287-212 B.C.) developed the idea further. The method of exhaustion was reinvented by **Lie Hui** in China in the third century A.D. to find the area of a circle.

Brahmagupta's Yuktibhasha is considered to be the first book on calculus. **Bhaskar's** work on calculus precedes much before the time of **Leibnitz** and **Newton**. **Bhaskara-2** used principles of differential calculus in problems on Astrotomy. There is a strong evidence that **Bhaskar** was a pioneer on some principles of differential calculus. He stated **Rolle's** Mean value theorem. In his book ***Siddhanta Shiromani***, we find elementary concept of mathematical analysis and infinitesimal calculus.

These ideas were systematized into calculus by **Gottfried Wilhelm Leibnitz**. He independently invented calculus along the same time as **Newton**. **Leibnitz** and **Newton** are both credited with the invention of calculus. **Newton** derived his results first but **Leibnitz** published them first. Both arrived at the results independently. But **Leibnitz** started with integral calculus and **Newton** started with differentiation. The name calculus was given by Leibnitz. In the 19th century calculus was put on a much rigorous footing by **Cauchy, Riemann** and **Weierstrass**. The modern ε-δ definition of limit is due to **Weierstrass**.

The modern notion of the limit of a function dates back to **Bolzano**. He introduced ε-δ technique in 1817 for continuous functions. **Cauchy** discussed limits in his ***cours de' analyse*** in 1821. But he gave only a verbal definition. **Weierstrass** introduced modern ε-δ definition which is studied today. He also gave notations *lim* and lim $x \to x_0$. The modern notation $\lim\limits_{x \to x_0}$ is due to **Hardy** given in his book ***'A Course of Pure Mathematics'*** in 1908.

10.2 Intuitive Idea of Limit

Now we turn to the main idea of calculus namely limits. Before giving definition, we will get intuitive idea of limits. We understand that the discussion that follows only gives some intuitive idea of limits and the examples solved only suggest ideas leading to the concept of limits.

Limit of a function is the **'ultimate'** value of the function, if it exists, when variable changes continuouly in the domain and goes nearer and nearer to a specific value. Let us be more specific. Limit of $f(x) = 3x + 2$ when x approaches 1 is written as $\lim\limits_{x \to 1} (3x + 2)$ and let us see how we 'find' it. Let us tabulate some values of x and $f(x)$ as follows :

x	0.9	0.99	0.999	0.9999	1.1	1.01	1.001	1.0001
$f(x)$	4.7	4.97	4.997	4.9997	5.3	5.03	5.003	5.0003

We observe that as $x \to 1$ through values less than 1, $f(x)$ approaches 5. This we express by saying that limit of $f(x)$ is 5 as x approaches 1 from left and we write $\lim\limits_{x \to 1-} f(x) = 5$ in notation. Similarly the limit of $f(x)$ as x approaches 1 from right is 5 or $\lim\limits_{x \to 1+} f(x) = 5$. Incidentally $f(1) = 3 + 2 = 5$. But this is not necessary.

If $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ exist and are equal, we say $\lim\limits_{x \to a} f(x)$ exists and is equal to either of $\lim\limits_{x \to a-} f(x)$ or $\lim\limits_{x \to a+} f(x)$.

Let us understand by a graph.

See that as $x \to 1-$, y-coordinate approaches 5 and so is the case with $x \to 1+$. Note that in discussing this limit, $f(1) = 5$ has no bearing on the limit.

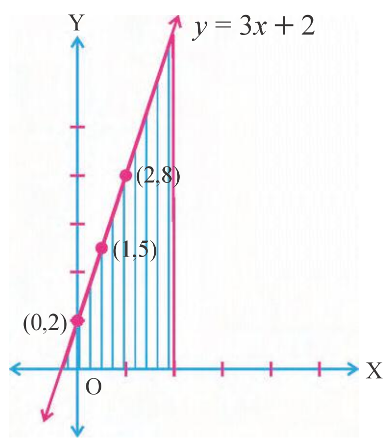

Figure 10.1

Example 1 to 13 are for understanding of concept of limit only. They are not meant to be asked in the examination.

Example 1 : Verify $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1} = 2$ by tabulation. $\left(x \neq \frac{1}{2}\right)$

x	0.49	0.499	0.4999	0.51	0.501	0.5001
$f(x)$	1.98	1.998	1.9998	2.02	2.002	2.0002

See that

$$f(x) = \frac{4x^2 - 1}{2x - 1}$$

$$= 2x + 1 \text{ as } x \neq \frac{1}{2}.$$

Hence we can see that

$$\lim_{x \to \frac{1}{2}} f(x) = 2.$$

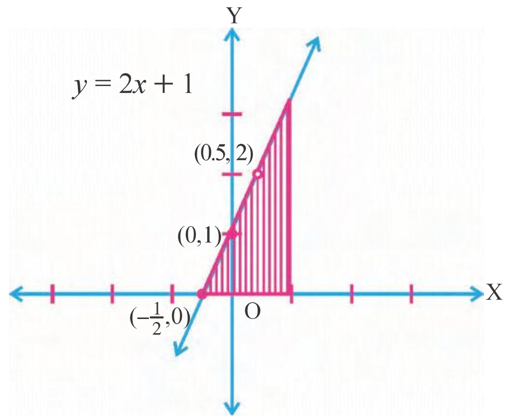

Figure 10.2

Explanation : As x approaches $\frac{1}{2}$ from left or from right $f(x)$ approaches 2. Here the graph does not contain the point corresponding to $x = \frac{1}{2}$ namely $\left(\frac{1}{2}, 2\right)$. All the while 'ultimate' value of $f(x)$, as approaches 1, is 2.

Example 2 : Find $\lim_{x \to 0} |x|$.

Solution : We know $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Hence,

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	0.1	0.01	0.001	0.1	0.01	0.001

We can guess that $\lim_{x \to 0} f(x) = 0$.

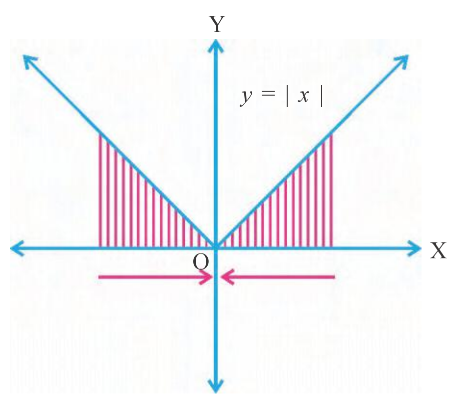

Figure 10.3

See that $f(0) = 0$

Example 3 : Prove that $\lim_{x \to 2} [x]$ does not exist.

Solution : $f(x) = [x] = \begin{cases} 1 & \text{if} \quad 1 \leq x < 2 \\ 2 & \text{if} \quad 2 \leq x < 3 \end{cases}$

x	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
$f(x)$	1	1	1	1	2	2	2	2

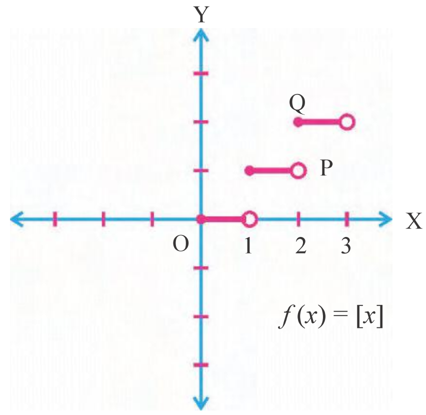

Figure 10.4

So, $\lim\limits_{x \to 2-} f(x) = 1$ and $\lim\limits_{x \to 2+} f(x) = 2$

$\therefore \lim\limits_{x \to 2} f(x)$ does not exist.

Explanation : Observe that there is a 'gap' between P and Q. Left limit and right limit do not coincide.

Example 4 : What can you say about $\lim\limits_{x \to 0} \frac{|x|}{x}$? $(x \neq 0)$

Solution : Here $f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$

f is not defined for $x = 0$.

x	-0.1	-0.01	-0.001	0.1	0.01	0.001
$f(x)$	-1	-1	-1	1	1	1

Obviously, $\lim\limits_{x \to 0+} f(x) = 1$, $\lim\limits_{x \to 0-} f(x) = -1$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist.

Note : In the example 1, $f\left(\frac{1}{2}\right)$ is not defined but $\lim\limits_{x \to \frac{1}{2}} f(x)$ exists.

In the example 2, $f(0)$ is defined and $\lim\limits_{x \to 0} f(x) = f(0)$.

In the example 3, $\lim\limits_{x \to 2-} f(x) \neq \lim\limits_{x \to 2+}$, $f(x)$ but $f(2)$ exists. But limit does not exist.

In the example 4, $\lim\limits_{x \to 0+} f(x) \neq \lim\limits_{x \to 0-} f(x)$ and $f(0)$ does not exist. Limit does not exist.

So we have enough ground to conclude that existence or value of $\lim\limits_{x \to a} f(x)$ is not affected by its value at a, namely $f(a)$.

Example 5 : Find $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x + 3 & x < 0 \\ 3 - x & x \geq 0 \end{cases}$

Solution : Here for $x < 0$, $f(x) = x + 3$ and for $x > 0$, $f(x) = 3 - x$.

$\therefore$ The table of values will be as follows :

x	-0.1	-0.01	-0.001	0.1	0.01	0.001
$f(x)$	2.9	2.99	2.999	2.9	2.99	2.999

$\therefore \quad \lim_{x \to 3-} f(x) = \lim_{x \to 3+} f(x) = 3$

and also $f(0) = 3 - 0 = 3$.

Explanation : (0, 3) is on the graph. As $x \to 0-$, point A move towards C and as $x \to 0+$, point B moves towards C. Hence $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ coincide.

Also $f(0) = 3$. All the three coincide.

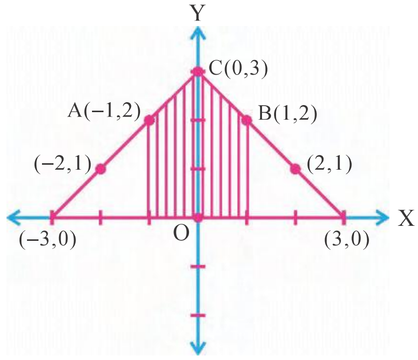

Figure 10.5

Example 6 : Find $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x + 3 & x > 1 \\ 10 & x = 1 \\ x + 5 & x < 1 \end{cases}$

Solution :

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	5.9	5.99	5.999	4.1	4.01	4.001

$x < 1$ $\qquad$ $x > 1$

Thus, $\lim_{x \to 1-} f(x)$ seems to be 6 and $\lim_{x \to 1+} f(x)$ appears to be 4. Thus $\lim_{x \to 1} f(x)$ does not exist.

Also $f(1) = 10$. All the three are distinct.

Explanation :

Hence, $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$

and the two are different.

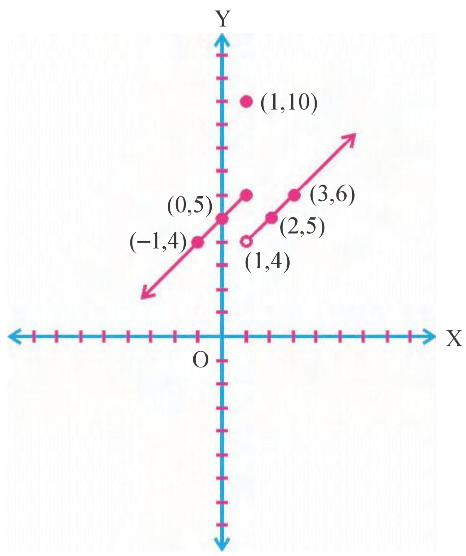

Figure 10.6

Example 7 : Find $\lim_{x \to 1} (x^2 - x)$.

Solution :

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	−0.09	−0.0099	−0.000999	0.11	0.0101	0.000101

Thus, $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 0$, $f(1) = 1^2 - 1 = 0$

$\therefore \quad \lim_{x \to 1} f(x) = 0 = f(1)$

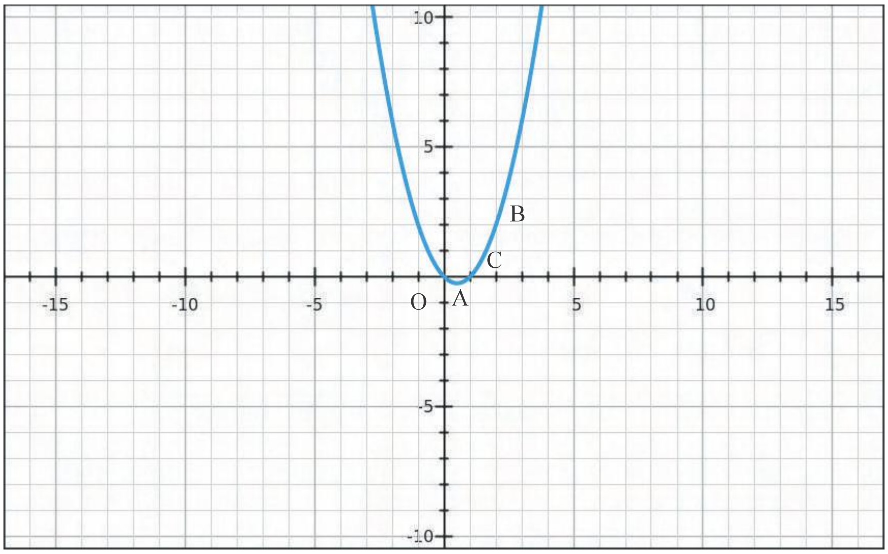

Figure 10.7

Explanation : As $x \to 1-$, A approaches C and $x \to 1+$, B approaches C.

$\therefore \quad \lim_{x \to 1} f(x) = 0$

Example 8 : $f : R \to R$. $f(x) = 5$, Find $\lim_{x \to 10} f(x)$.

Solution :

x	9.9	9.99	9.999	10.1	10.01	10.001
$f(x)$	5	5	5	5	5	5

$\therefore \quad \lim_{x \to 10} f(x) = 5$

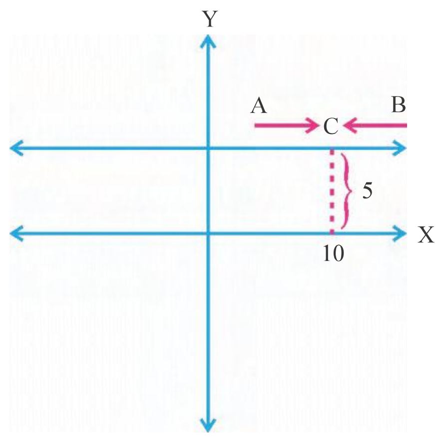

Figure 10.8

Explanation : As $x \to 10-$, A approaches C and as $x \to 10+$, B approaches C.

C is (10, 5).

$\lim_{x \to 10} f(x) = 5$

Example 9 : Find $\lim_{x \to \frac{\pi}{2}} cosx$.

Solution :

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} - 0.001$	$\frac{\pi}{2} + 0.1$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.001$
$f(x)$	0.099833	0.009999833	0.0009999998	−0.099833	−0.009999833	−0.0009999998

Obviously $\lim_{x \to \frac{\pi}{2}} cosx = 0$

Explanation : Look at the graph of *cosx*.

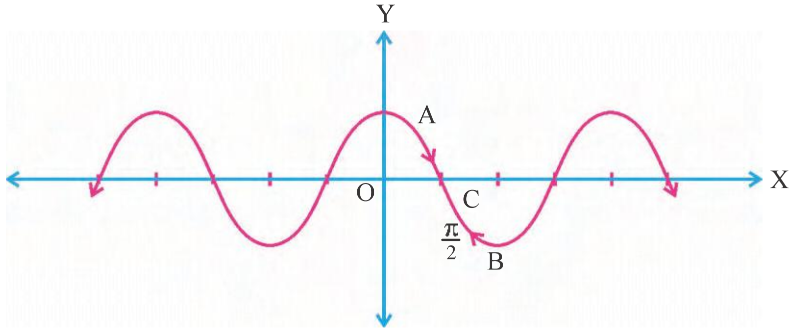

Figure 10.9

As before A approaches C and B tends to C as $x \to \frac{\pi}{2}-$ and $x \to \frac{\pi}{2}+$ respectively.

$\therefore \quad \lim_{x \to \frac{\pi}{2}} cosx = 0$

Example 10 : Verify $\lim_{x \to 0} \frac{sinx}{x} = 1. \qquad (x \neq 0)$

Solution :

x	−0.7	−0.2	−0.05	1.4	0.3	0.03	0.01
$f(x)$	0.92031	0.993347	0.999583	0.97275	0.98506	0.99985	0.999983

Explanation : Note that $\frac{sinx}{x}$ is an even function i.e. $\frac{sin(-x)}{-x} = \frac{-sinx}{-x} = \frac{sinx}{x}$

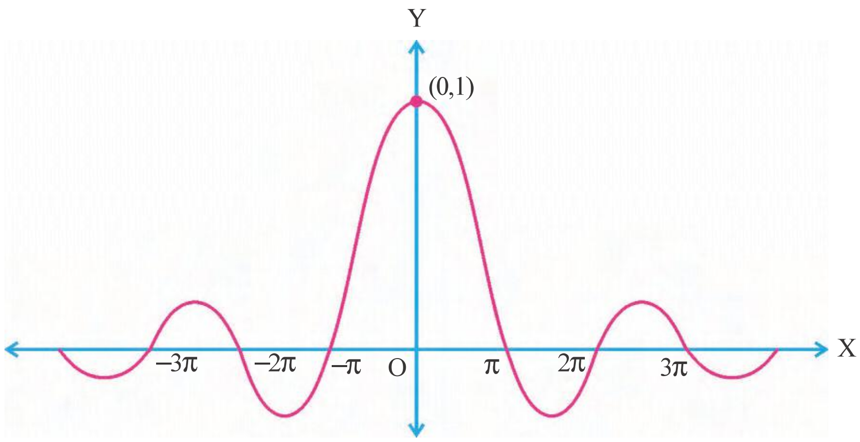

Figure 10.10

So we need consider $x > 0$ only. It is apparent that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. This is reflected in the graph given also. In fact we will prove this in this chapter later on.

Example 11 : Find $\lim_{x \to 0} (x + \cos x)$.

Solution :

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	0.895004165	0.98995	0.9989995	1.095004165	1.009995	1.0009995

Explanation : From the graph as well the table we infer that $\lim_{x \to 0} (x + \cos x) = 1$.

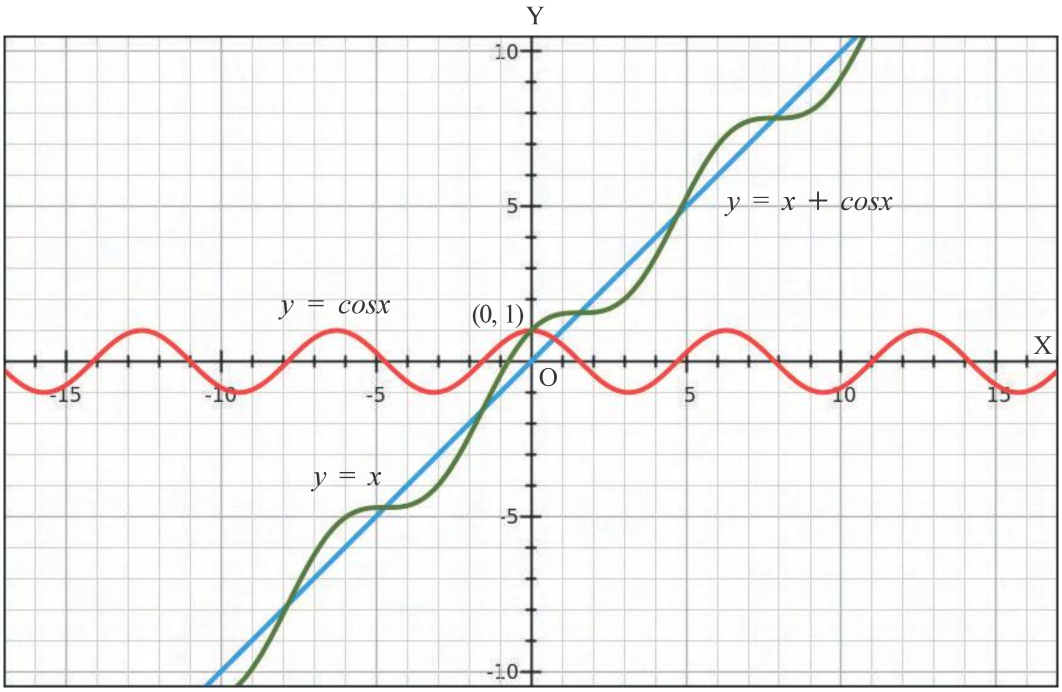

Figure 10.11

See $\lim_{x \to 0} x = 0$, $\lim_{x \to 0} \cos x = 1$.

$\therefore \quad \lim_{x \to 0} (x + \cos x) = \lim_{x \to 0} x + \lim_{x \to 0} \cos x$

Example 12 : Discuss existence of $\lim_{x \to 0} \frac{1}{x}$.

Solution :

x	−0.1	−0.01	−0.001	0.1	0.01	0.001
$f(x)$	−10	−100	−1000	10	100	1000

Explanation : Here we observe that as $x \to 0+$, $\frac{1}{x}$ increases 'unboundedly' and as $x \to 0-$, we say $\frac{1}{x}$ decreases 'unboundedly'. So $\lim_{x \to 0+} \frac{1}{x}$ or $\lim_{x \to 0-} \frac{1}{x}$ do not exist. We say as $x \to 0+$, $\frac{1}{x} \to \infty$ and as $x \to 0-$, $\frac{1}{x} \to -\infty$. $\lim_{x \to 0} \frac{1}{x}$ does not exist.

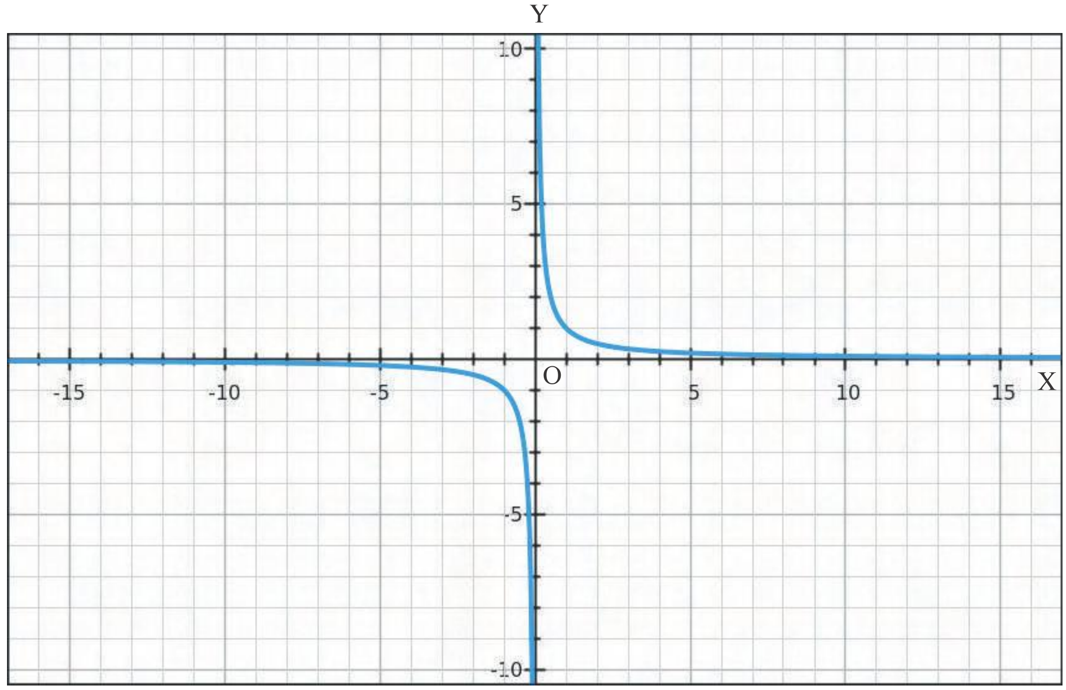

Figure 10.12

It is incorrect to say $\lim_{x \to 0+} \frac{1}{x} = \infty$ or $\lim_{x \to 0-} \frac{1}{x} = -\infty$. Note that ∞ and $-\infty$ are merely symbols or members of the extended real number system. We are dealing with limits in real number system only.

Example 13 : Discuss $\lim_{x \to 0} \frac{1}{x^2}$.

Solution :

x	-0.1	-0.01	-0.001	0.1	0.01	0.001
$f(x)$	10^2	10^4	10^6	10^2	10^4	10^6

Explanation : In this case, whether $x \to 0+$ or $x \to 0-$, $\frac{1}{x^2}$ increases unboundedly or $\frac{1}{x^2} \to \infty$.

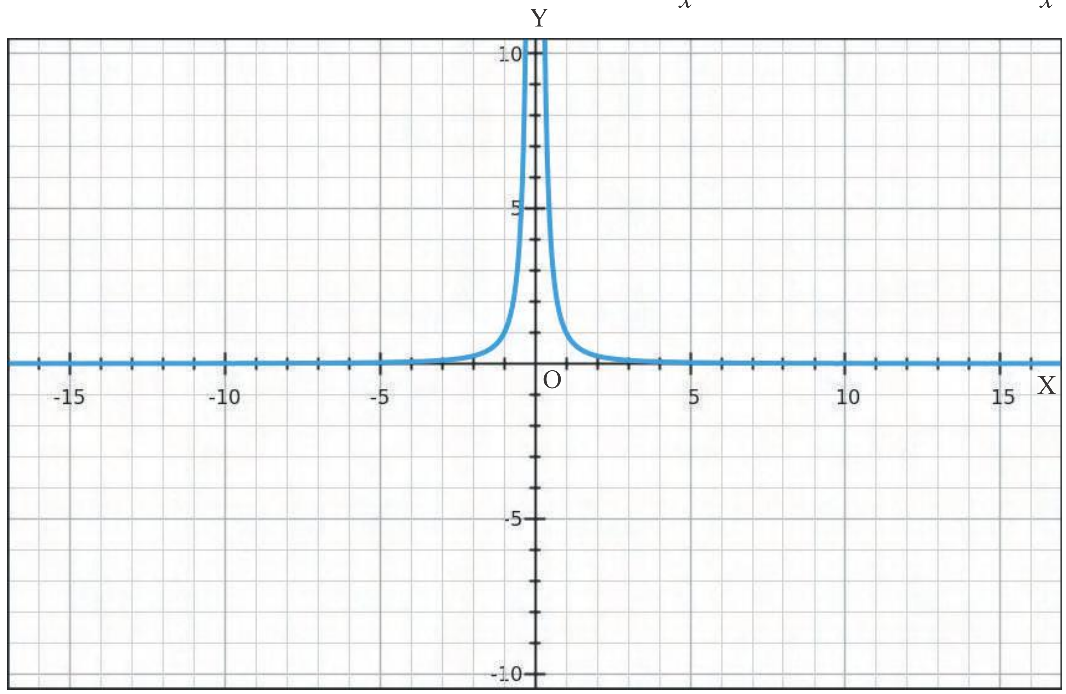

Figure 10.13

Again we do not write $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$. $\lim\limits_{\infty \to 0} \frac{1}{x^2}$ does not exist.

10.3 Formal Definition of Limit

Now we are ready to give formal definition of limit. So far we had inferred certain limits by observing some tabulated values and graphs. But in practice it is not possible even in simple examples and this tabulation may even mislead. Look at the graph of $sin\frac{1}{x}$ (figure 10.14).

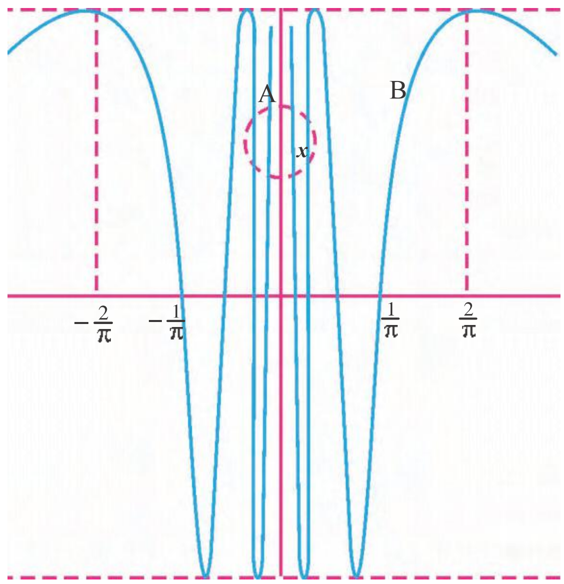

Figure 10.14

Can we infer anything about $\lim\limits_{x \to 0} sin\frac{1}{x}$? When x takes a sequence of values $\frac{1}{k\pi}$, $k \in Z - \{0\}$, $sin\frac{1}{x} = 0$, for $x = \frac{2}{(4m+1)\pi}$, $sin\frac{1}{x} = 1$ and for $x = \frac{2}{(4m+3)\pi}$, $sin\frac{1}{x} = -1$. Other values of x also exist which we have not considered. So it is difficult to guess anything about $\lim\limits_{x \to 0} sin\frac{1}{x}$.

Definition : Limit of a function : Let $f(x)$ be a function defined on a domain containing some interval containing a but a may not be in the domain of f. If for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $a - \delta < x < a + \delta$, $x \neq a \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$, we say $\lim\limits_{x \to a} f(x) = l$ or limit of $f(x)$ as x tends to a is l.

See that $\delta > 0$ is any positive number. Hence $f(x)$ can be brought as near to l as we please. $-\varepsilon < f(x) - l < \varepsilon$ or $|f(x) - l| < \varepsilon$ just by proper selection of δ such that $a - \delta < x < a + \delta$, $x \neq a$ or $-\delta < x - a < \delta$, $x \neq a$ i.e. $|x - a| < \delta$, $x \neq a$.

Thus $f(x)$ can be brought as near to l as we please if we can choose $\delta > 0$ such that x should be brought near to a.

Left limit of a function : If $f(x)$ is a function defined in some interval $(a - h, a)$, $(h > 0)$ and if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x \in (a - \delta, a)$ and $\delta < h$, we say left limit of $f(x)$ is l as $x \to a-$ or $\lim\limits_{x \to a-} f(x) = l$.

Right limit of a function : If $f(x)$ is a function defined in an interval $(a, a + k)$, $(k > 0)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x \in (a, a + \delta)$, $\delta < k$, then we say right limit of $f(x)$ is l as $x \to a+$ or $\lim\limits_{x \to a+} f(x) = l$.

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Notes : (1) Nowhere in the definition, it is required that a be in domain of f. $f(x)$ must be defined 'around' a. f must be defined in an interval containing a except for possibly at $x = a$. f may or may not be defined at $x = a$.

(2) $\varepsilon > 0$ is any given number and $\delta > 0$ is to be found out depending upon f.

Let us understand the definition more closely by some examples.

Example 14 : Prove : $\lim\limits_{x \to 2} (3x + 2) = 8$

Solution : Let $\varepsilon > 0$ be any positive number.

We require $8 - \varepsilon < 3x + 2 < 8 + \varepsilon$ **($l = 8$)**

$$8 - \varepsilon < 3x + 2 < 8 + \varepsilon \Leftrightarrow 6 - \varepsilon < 3x < 6 + \varepsilon$$

$$\Leftrightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}.$$

Comparing with $2 - \delta < x < 2 + \delta$, we are motivated to let $\delta = \frac{\varepsilon}{3}$. **($a = 2$)**

Now let $\delta = \frac{\varepsilon}{3}$

$$\therefore \quad 2 - \delta < x < 2 + \delta, x \neq 2 \Rightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$$

$$\Rightarrow 6 - \varepsilon < 3x < 6 + \varepsilon$$

$$\Rightarrow 8 - \varepsilon < 3x + 2 < 8 + \varepsilon$$

This is what we wanted and $\delta = \frac{\varepsilon}{3}$ exists for every $\varepsilon > 0$ such that

$$2 - \delta < x < 2 + \delta, x \neq 2 \Rightarrow 8 - \varepsilon < 3x + 2 < 8 + \varepsilon$$

$\therefore \quad \lim\limits_{x \to 2} (3x + 2) = 8.$

Example 15 : Prove : $\lim\limits_{x \to a} x = a$

Solution : Let $\varepsilon = \delta$, $\varepsilon > 0$. Then, $a - \delta < x < a + \delta$, $x \neq a \Rightarrow a - \varepsilon < x < a + \varepsilon$

$\therefore \quad \lim\limits_{x \to a} x = a$

Note : It is not obvious that $x \to a$, as $x \to a$, we have proved it using definition.

Example 16 : Prove : $\lim\limits_{x \to a} (mx + c) = ma + c \qquad (m \neq 0)$

Solution : Let $\delta = \frac{\varepsilon}{|m|}$, $\varepsilon > 0$.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow a - \frac{\varepsilon}{|m|} < x < a + \frac{\varepsilon}{|m|}$$

$$\Rightarrow ma - \frac{\varepsilon}{|m|}m < mx < ma + \frac{\varepsilon}{|m|}m \qquad (m > 0)$$

$$\Rightarrow ma - \varepsilon < mx < ma + \varepsilon$$

$$\Rightarrow ma - \varepsilon + c < mx + c < ma + \varepsilon + c$$

Let $l = ma + c$

$\therefore \quad a - \delta < x < a + \delta, x \neq a \Rightarrow l - \varepsilon < mx + c < l + \varepsilon$

$\therefore \quad$ If $m > 0$, $\lim\limits_{x \to a} (mx + c) = ma + c$

Similarly if $m < 0$ we can prove.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow ma + c + \varepsilon > mx + c > ma + c - \varepsilon \qquad (|m| = -m)$$

$$\Rightarrow ma + c - \varepsilon < mx + c < ma + c + \varepsilon$$

$\therefore$ If $m < 0$, $\lim\limits_{x \to a} (mx + c) = ma + c$.

10.4 Algebra of Limits

It is tedious and difficult to find limits using definition. So some working rules are derived. They can be proved but we will not prove them.

Let $\lim\limits_{x \to a} f(x)$ exist and be equal to l and let $\lim\limits_{x \to a} g(x)$ exist and be equal to m.

Then (1) $\lim\limits_{x \to a} (f(x) + g(x))$ exists and

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m$$

(2) $\lim\limits_{x \to a} (f(x)\, g(x))$ exists and

$$\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = lm$$

(3) If $m \neq 0$, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ exists and $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$

Example 17 : Prove if $f(x)$ is a constant function and if $f(x) = c$, then $\lim\limits_{x \to a} f(x) = c$ or in other words $\lim\limits_{x \to a} c = c$.

Deduce $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$, if $\lim\limits_{x \to a} f(x)$ exists.

Solution : Let $f(x) = c$ and $x \in (a - \delta, a + \delta) - \{a\}$. Let $l = c$.

$$a - \delta < x < a + \delta, x \neq a \Rightarrow |f(x) - l| = |c - c| = 0 < \varepsilon \text{ as } 0 < \varepsilon.$$

$\therefore$ $\lim\limits_{x \to a} f(x) = c$ i.e. $\lim\limits_{x \to a} c = c$

If $\lim\limits_{x \to a} f(x)$ exists, then $\lim\limits_{x \to a} cf(x) = \lim\limits_{x \to a} c \lim\limits_{x \to a} f(x)$

$$= c \lim_{x \to a} f(x)$$

Note : Using $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ and

$\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$, we can prove

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

If $c = -1$,

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} (f(x) + (-1)g(x))$$

$$= \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x)$$

$$= \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Theorem 1 : Prove $\lim_{x \to a} x^n = a^n \quad n \in N$

Let $P(n) : \lim_{x \to a} x^n = a^n \quad n \in N$

We have proved $\lim_{x \to a} x = a$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \lim_{x \to a} x^k = a^k$

Let $n = k + 1$

$\therefore \lim_{x \to a} x^{k+1} = \lim_{x \to a} x^k \cdot x$

$= \lim_{x \to a} x^k \lim_{x \to a} x$ **(Product rule for limits)**

$= a^k \cdot a = a^{k+1}$ **(P(k) and P(1))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Theorem 2 : $\lim_{x \to a} (f_1(x) + f_2(x) + ... + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$,

Let $P(n) : \lim_{x \to a} (f_1(x) + f_2(x) + ... + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$

if individual limits $\lim_{x \to a} f_i(x)$ exist **(i = 1, 2, 3, ..., n)**

For $n = 1$ the result is obvious.

Let P(k) be true.

$\therefore \lim_{x \to a} (f_1(x) + f_2(x) + ... + f_k(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_k(x)$

Let $n = k + 1$

$\therefore \lim_{x \to a} (f_1(x) + ... + f_k(x) + f_{k+1}(x))$

$= \lim_{x \to a} (f_1(x) + ... + f_k(x)) + \lim_{x \to a} f_{k+1}(x)$ $\left(\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)\right)$

$= \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_k(x) + \lim_{x \to a} f_{k+1}(x)$ **(P(k))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Limit of a Polynomial :

We know $f(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_0$, $x \in R$ **($c_n \neq 0$, c_0, c_1, ... $c_n \in R$)**

is called a polynomial of degree n.

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$$\lim_{x \to a} f(x) = \lim_{x \to a} (c_n x^n + c_{n-1} x^{n-1} + ... + c_0)$$

$$= \lim_{x \to a} c_n x^n + \lim_{x \to a} c_{n-1} x^{n-1} + ... + \lim_{x \to a} c_0 \quad \textbf{(Lemma 2)}$$

$$\left(\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)\right)$$

$$= \lim_{x \to a} c_n \lim_{x \to a} x^n + \lim_{x \to a} c_{n-1} \lim_{x \to a} x^{n-1} + ... + \lim_{x \to a} c_0$$

$$\left(\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)\right)$$

$$= c_n a^n + c_{n-1} a^{n-1} + ... + c_0 \quad \left(\lim_{x \to a} x^n = a^n, \lim_{x \to a} c_k = c_k\right)$$

$$= f(a)$$

Thus, limit of a polynomial as $x \to a$ is obtained by just substituting $x = a$ in the polynomial.

(This is called **'continuity'** of polynomials.)

Example 18 : Find $\lim_{x \to 2} (2x^3 + 3x^2 - 5x + 1)$.

Solution : $\lim_{x \to 2} (2x^3 + 3x^2 - 5x + 1) = 2 \cdot 2^3 + 3 \cdot 2^2 - 5 \cdot 2 + 1$

$= 16 + 12 - 10 + 1$

$= 19$

Limit of Rational Functions :

If $p(x)$ and $q(x)$ are polynomials defined over a domain in which $q(x) \neq 0$, then $h(x) = \frac{p(x)}{q(x)}$ is called a rational function.

If $p(x)$ and $q(x)$ are polynomials defined in a domain containing a and $q(a) \neq 0$ then

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} = \frac{p(a)}{q(a)} = h(a).$$

In other words rational function $h(x)$ is also a **'continuous'** function.

Example 19 : Find $\lim_{x \to 1} \frac{x^2 + 1}{x^2 + 3x + 4}$.

Solution : Here $x^2 + 3x + 4 \neq 0$ for $x = 1$.

$$\therefore \lim_{x \to 1} \frac{x^2 + 1}{x^2 + 3x + 4} = \frac{2}{8} = \frac{1}{4}$$

Hence in case of a rational function $h(x) = \frac{p(x)}{q(x)}$ if $q(a) \neq 0$, then $\lim_{x \to a} h(x) = h(a)$ is obtained by just substituting $x = a$ in $h(x)$. But what happens if $q(a) = 0$?

By remainder theorem, we know that $x - a$ is a factor of $q(x)$. Now we consider some cases.

Case (1) : $p(x) = (x - a)^k f(x)$

$q(x) = (x - a)^k g(x)$, $f(a) \neq 0$, $g(a) \neq 0$, $k \in N$

Now $\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)}$

$$= \lim_{x \to a} \frac{(x-a)^k f(x)}{(x-a)^k g(x)}$$

$$= \lim_{x \to a} \frac{f(x)}{g(x)} \qquad \text{(while discussing limit } x \neq a\text{)}$$

$$= \frac{f(a)}{g(a)}$$

Thus, if $(x - a)$ occurs to the same index in both numerator and denominator, we can cancel it and have the limit by substituting $x = a$ after cancellation of the factor $(x - a)^k$.

Example 20 : Find $\lim_{x \to 0} \frac{x^3 - 3x^2 + x}{4x^3 - 5x^2 + 3x}$.

Solution : Here $\lim_{x \to 0} \frac{x^3 - 3x^2 + x}{4x^3 - 5x^2 + 3x} = \lim_{x \to 0} \frac{x(x^2 - 3x + 1)}{x(4x^2 - 5x + 3)}$

$$= \lim_{x \to 0} \frac{x^2 - 3x + 1}{4x^2 - 5x + 3}$$

$$= \frac{1}{3}$$

Example 21 : Find $\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 3x + 1}{3x^4 - 5x^3 + 6x^2 - 10x + 6}$.

Solution : $\lim_{x \to 1} \frac{x^4 - 7x^3 + 8x^2 - 3x + 1}{3x^4 - 5x^3 + 6x^2 - 10x + 6} = \lim_{x \to 1} \frac{(x-1)(x^3 - 6x^2 + 2x - 1)}{(x-1)(3x^3 - 2x^2 + 4x - 6)}$

$$= \lim_{x \to 1} \frac{x^3 - 6x^2 + 2x - 1}{3x^3 - 2x^2 + 4x - 6}$$

$$= \frac{-4}{-1} = 4$$

Note : Here $p(1) = q(1) = 0$. Hence $(x - 1)$ is a factor of $p(x)$ and $q(x)$. After factorisation of $p(x)$ and $q(x)$, we remove the factor $(x - 1)$ and substitute $x = 1$.

Example 22 : Find $\lim_{x \to 2} \frac{x^3 - 5x^2 + 8x - 4}{2x^3 - 9x^2 + 12x - 4}$.

Solution : $p(2) = 8 - 20 + 16 - 4 = 0$, $q(2) = 16 - 36 + 24 - 4 = 0$

$\therefore$ $(x - 2)$ is a factor of $p(x)$ and $q(x)$.

$$\lim_{x \to 2} \frac{x^3 - 5x^2 + 8x - 4}{2x^3 - 9x^2 + 12x - 4} = \lim_{x \to 2} \frac{(x-2)^2(x-1)}{(x-2)^2(2x-1)}$$

$$= \lim_{x \to 2} \frac{x-1}{2x-1} = \frac{1}{3}$$

Here $(x - 2)^2$ is a factor of both $p(x)$ and $q(x)$.

Case (2) : Let us see what happens if $(x - a)^k$ and $(x - a)^m$ are factors of $p(x)$ and $q(x)$ respectively where $k \neq m$ and $\frac{p(x)}{(x-a)^k}$ and $\frac{q(x)}{(x-a)^m}$ do not have $x - a$ as a factor.

Now $h(x) = \frac{p(x)}{q(x)} = \frac{(x-a)^k f(x)}{(x-a)^m g(x)} = \frac{(x-a)^{k-m} f(x)}{g(x)}$ if $k > m$.

Here $k - m \in N$

$\therefore$ Also $f(a) \neq 0$, $g(a) \neq 0$.

$\therefore \lim_{x \to a} h(a) = \frac{0 \cdot f(a)}{g(a)} = 0$

Thus, if $(x - a)$ occurs to higher index in $p(x)$, then $\lim_{x \to a} h(x) = 0$.

Case (3) : If $p(x) = (x - a)^k f(x)$, $q(x) = (x - a)^m g(x)$ with $k < m$ and $\frac{p(x)}{(x-a)^k} = f(x)$ and $\frac{q(x)}{(x-a)^m} = g(x)$ are non-zero for $x - a$, we proceed as follows :

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{(x-a)^k f(x)}{(x-a)^m g(x)} = \lim_{x \to a} \frac{f(x)}{(x-a)^{m-k} g(x)}$$

Now $f(a)$ is a real number. $(a - a)^{m - k} g(a) = 0$

$\therefore$ Denominator of $h(x)$ becomes unbounded as $x \to a$ and we say $\lim_{x \to a} h(x)$ does not exist.

Example 23 : Find $\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^2 - 1}$.

Solution : $\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)^3}{(x-1)(x+1)}$

$$= \lim_{x \to 1} \frac{(x-1)^2}{x+1} = \frac{0}{2} = 0$$

Example 24 : Find $\lim_{x \to 0} \frac{x^4 - x^3 + x^2}{x^6 - x^5 + x}$.

Solution : $\lim_{x \to 0} \frac{x^4 - x^3 + x^2}{x^6 - x^5 + x} = \lim_{x \to 0} \frac{x^2(x^2 - x + 1)}{x(x^5 - x^4 + 1)}$

$$= \lim_{x \to 0} \frac{x(x^2 - x + 1)}{x^5 - x^4 + 1} = \frac{0 \cdot 1}{1} = 0$$

An Important Limit :

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad n \in N \ (x \neq a),\ x,\ a \in R$$

We can see that this is a rational function.

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + a^{n-1})}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + ... + a^{n-1} = na^{n-1}$$

Note : This result is true even for $n \in$ R. But then $x \in$ R$^+$, $a \in$ R$^+$, $x \neq a$. We will use this extended result in future.

Example 25 : Find $\lim\limits_{x \to 1} \dfrac{x^{18} - 1}{x^{16} - 1}$.

Solution : $\lim\limits_{x \to 1} \dfrac{x^{18} - 1}{x^{16} - 1} = \lim\limits_{x \to 1} \dfrac{x^{18} - 1}{x - 1} \times \dfrac{x - 1}{x^{16} - 1}$

$$= \frac{\lim\limits_{x \to 1} \frac{x^{18} - 1}{x - 1}}{\lim\limits_{x \to 1} \frac{x^{16} - 1}{x - 1}}$$

$$= \frac{18(1)^{17}}{16(1)^{15}} = \frac{18}{16} = \frac{9}{8}$$

Example 26 : Find $\lim\limits_{x \to -2} \dfrac{x^5 + 32}{x^3 + 8}$.

Solution : $\lim\limits_{x \to -2} \dfrac{x^5 + 32}{x^3 + 8} = \lim\limits_{x \to -2} \dfrac{x^5 - (-2)^5}{x^3 - (-2)^3}$

$$= \frac{\lim\limits_{x \to -2} \frac{x^5 - (-2)^5}{x - (-2)}}{\lim\limits_{x \to -2} \frac{x^3 - (-2)^3}{x - (-2)}}$$

$$= \frac{5(-2)^4}{3(-2)^2} = \frac{5 \cdot 16}{3 \cdot 4} = \frac{20}{3}$$

Example 27 : $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^3 - 3x^2 + 3x - 2}$.

Solution : $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^3 - 3x^2 + 3x - 2} = \dfrac{\lim\limits_{x \to 2} \frac{x^4 - 2^4}{x - 2}}{\lim\limits_{x \to 2} \frac{(x - 2)(x^2 - x + 1)}{x - 2}} = \dfrac{4 \cdot 2^3}{4 - 2 + 1} = \dfrac{32}{3}$

Rule of Substitution or Rule of Limit of a Composite Function :

Suppose $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{y \to b} g(y)$ exists and $\lim\limits_{y \to b} g(y) = l$. Then $\lim\limits_{x \to a} g(f(x)) = l$.

Here $\lim\limits_{x \to a} f(x)$ exists means f is defined in $(a - \delta, a + \delta) - \{a\}$ for some $\delta > 0$ and $y = f(x)$. g is defined in $(b - \delta', b + \delta') - \{b\}$ for some $\delta' > 0$.

Example 28 : Find $\lim\limits_{x \to 0} \dfrac{(x + 2)^5 - 32}{x}$.

Solution : Let $y = f(x) = x + 2$. Then $\lim\limits_{x \to 0} f(x) = 2 = b$.

$$\lim_{y \to 2} g(y) = \lim_{y \to 2} \frac{y^5 - 2^5}{y - 2}$$

$$= 5 \cdot 2^4 = 80$$

$$\therefore \quad \lim_{x \to 0} g(f(x)) = \lim_{x \to 0} \frac{(x+2)^5 - 32}{x} = 80$$

In practice, we just take the substitution $y = x + 2$ and write $y \to 2$ as $x \to 0$ in the example. This is valid for so called **'continuous'** functions.

Another Method :

$$\lim_{x \to 0} \frac{(x+2)^5 - 32}{x} = \lim_{x \to 0} \frac{x^5 + \binom{5}{1}x^4 \cdot 2 + \binom{5}{2}x^3 \cdot 2^2 + \binom{5}{3}x^2 \cdot 2^3 + \binom{5}{4}x \cdot 2^4 + \binom{5}{5}2^5 - 32}{x}$$

$$= \lim_{x \to 0} \left(x^4 + \binom{5}{1}2x^3 + \binom{5}{2}4x^2 + \binom{5}{3}8x + \binom{5}{4}2^4\right)$$

$$= 5 \cdot 16 = 80$$

Example 29 : Find $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

Solution : Let $y = x + h$. Then $y \to x$ as $h \to 0$.

$$\therefore \quad \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{y \to x} \frac{y^{\frac{1}{2}} - x^{\frac{1}{2}}}{y - x} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example 30 : Find $\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x^2 + x + 2} - \sqrt{3x + 2}}$.

Solution : $\lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x^2 + x + 2} - \sqrt{3x + 2}}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}{\left(\sqrt{x^2 + x + 2} - \sqrt{3x + 2}\right)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}{(x^2 + x + 2) - (3x + 2)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}{x^2 - 2x}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}{x(x-2)}$$

$$= \lim_{x \to 2} \frac{(x^2 + 2x + 4)\left(\sqrt{x^2 + x + 2} + \sqrt{3x + 2}\right)}{x}$$

$$= \frac{(12)(\sqrt{8} + \sqrt{8})}{2} = 6(4\sqrt{2}) = 24\sqrt{2}$$

$\left(\lim_{x \to 2} \sqrt{x^2 + x + 2} = \sqrt{\lim_{x \to 2}(x^2 + x + 2)} = \sqrt{8}\right.$ by rule of limit of composite function.$\left.\right)$

Two Important Rules :

(1) If $f(x) < g(x)$, $\forall x$ in the same domain and both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

(2) If $g(x) < f(x) < h(x)$, $\forall x$ in the same domain and if $\lim_{x \to a} g(x)$ and $\lim_{x \to a} h(x)$ exist and are both equal to l, then $\lim_{x \to a} f(x)$ exists and is equal to l.

This is known as **Sandwich Theorem or Squeeze Theorem.**

(We do not prove it.)

Example 31 : Prove $\lim_{x \to 0} x sin\frac{1}{x} = 0$. $(x \ne 0)$

Solution : $-1 \le sin\frac{1}{x} \le 1$

$\therefore \quad -x \le x sin\frac{1}{x} \le x \qquad (x > 0)$

$\lim_{x \to 0+} x = 0, \quad \lim_{x \to 0+} -x = -\lim_{x \to 0+} x = 0$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0+} x sin\frac{1}{x} = 0$

Similarly $\lim_{x \to 0-} x sin\frac{1}{x} = 0$

$\therefore \quad \lim_{x \to 0} x sin\frac{1}{x} = 0$

Note : It is incorrect to argue as follows :

$$\lim_{x \to 0} x sin\frac{1}{x} = \lim_{x \to 0} x \lim_{x \to 0} sin\frac{1}{x}$$

$= 0$ (a number between -1 and 1)

$= 0$

Product rule for limit applies only if both the factors have limits. Here $\lim_{x \to 0} sin\frac{1}{x}$ does not exist. (Look at the graph 10.14)

Example 32 : Prove $\lim_{x \to 0} x^2 sin\frac{1}{x} = 0$. $(x \ne 0)$

Solution : $-1 \le sin\frac{1}{x} \le 1$

$\therefore \quad -x^2 \le x^2 sin\frac{1}{x} \le x^2 \qquad (x^2 > 0)$

$\lim_{x \to 0} x^2 = 0, \quad \lim_{x \to 0} -x^2 = -\lim_{x \to 0} x^2 = 0$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} x^2 sin\frac{1}{x} = 0$

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Note : Tabulating values will not offer rigorous results. Thus we can proceed by definition.

Let $\delta = \sqrt{\varepsilon}$

Since $\varepsilon > 0$, δ exists.

$0 < |x - 0| < \delta \Rightarrow 0 < |x| < \sqrt{\varepsilon}$

$\Rightarrow 0 < |x|^2 < \varepsilon$

Now, $\left|x^2 sin\frac{1}{x} - 0\right| = \left|x^2 sin\frac{1}{x}\right| \le |x|^2 < \varepsilon$ as $\left|sin\frac{1}{x}\right| \le 1$ and $0 < |x - 0| < \delta$

$\therefore \lim\limits_{x \to 0} x^2 sin\frac{1}{x} = 0$

10.5 Trigonometric Limits

We proceed to prove some lemmas.

Lemma 1 : $cosx < \frac{sinx}{x} < 1; \quad 0 < |x| < \frac{\pi}{2}$.

Proof : Let x be the radian measure of $\angle AOP$ such that $0 < x < \frac{\pi}{2}$. Then $P(x) \in \widehat{AB}$. $\odot(O, OA)$ is unit circle.

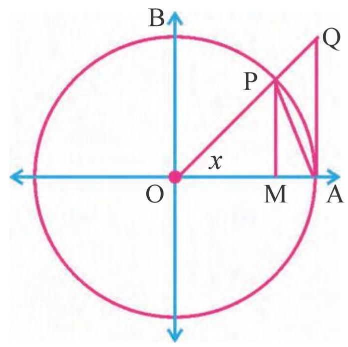

Figure 10.15

Let $\overrightarrow{OP}$ intersect tangent at A at Q.

Let $\overline{PM} \perp$ X-axis and $M \in \overline{OA}$.

Obviously area of ΔOAP < area of sector OAP < area of ΔOAQ **(i)**

Now $PM = sinx$, $AQ = \frac{AQ}{OA} \cdot OA = OA\, tanx = tanx$

$\therefore \frac{1}{2}OA \cdot PM < \frac{1}{2}(OA)^2 x < \frac{1}{2} OA \cdot AQ$ **(from (i) and area of a sector = $\frac{1}{2}r^2\theta$)**

$sinx < x < tanx$ **(OA = 1)**

$\therefore 1 < \frac{x}{sinx} < \frac{1}{cosx}$ **($sinx > 0$)**

$\therefore cosx < \frac{sinx}{x} < 1 \qquad 0 < x < \frac{\pi}{2}$

If $x < 0$, let $x = -y$, $y > 0$

$\therefore cosy < \frac{siny}{y} < 1 \qquad 0 < y < \frac{\pi}{2}$

$\therefore cos(-x) < \frac{sin(-x)}{-x} < 1 \qquad 0 < -x < \frac{\pi}{2}$

$\therefore cosx < \frac{sinx}{x} < 1 \qquad 0 < |x| < \frac{\pi}{2}$ **($|x| = -x$)**

Lemma 2 : $|\sin x| \le |x| \qquad \forall x \in R$

Proof : If $|x| \ge 1$, then $|\sin x| \le 1 \le |x|$ is true.

For $x = 0 \quad |\sin x| = 0 \le 0 = |0|$

Thus, we have to prove the result for $0 < |x| < 1$.

We have, $\frac{\sin x}{x} < 1 \qquad 0 < |x| < \frac{\pi}{2}$

Let $0 < x < 1$

$\therefore \quad 0 < x < 1 < \frac{\pi}{2}$

$\therefore \quad \frac{\sin x}{x} < 1$

$\therefore \quad \sin x < x$ **$(x > 0)$**

$\therefore \quad |\sin x| \le |x|$ as $\sin x > 0$, $x > 0$ for $0 < x < \frac{\pi}{2}$.

Let $-1 < x < 0$. Let $x = -y$

Then $-1 < -y < 0$ or $0 < y < 1$

$\therefore \quad |\sin y| < |y|$

$\therefore \quad |\sin(-x)| < |-x|$

$\therefore \quad |-\sin x| < |-x|$ Hence $|\sin x| < |x|$

$\therefore \quad |\sin x| \le |x| \qquad \forall x \in R$

Lemma 3 : $\lim_{x \to 0} |x| = 0$

Proof : Let $\varepsilon = \delta$. Then, $-\delta < x < \delta \Rightarrow |x| < \delta$

$\Rightarrow |x| < \varepsilon$ **$(\delta = \varepsilon)$**

$\Rightarrow \big||x|\big| < \varepsilon$ **$(\big(\big||x|\big|\big) = |x|)$**

$\Rightarrow \big||x| - 0\big| < \varepsilon$

$\therefore \quad \lim_{x \to 0} |x| = 0$

Lemma 4 : If $\lim_{x \to 0} |f(x)| = 0$, $\lim_{x \to 0} f(x) = 0$

Proof : $-|f(x)| \le f(x) \le |f(x)|$

$\lim_{x \to 0} -|f(x)| = -\lim_{x \to 0} |f(x)| = 0$, $\lim_{x \to 0} |f(x)| = 0$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} f(x) = 0$

Lemma 5 : $\lim_{x \to 0} \sin x = 0$

Proof : $0 \le |\sin x| \le |x| \qquad \forall x \in R$

$\lim_{x \to 0} 0 = 0, \quad \lim_{x \to 0} |x| = 0$ **(Sandwich theorem)**

$\lim_{x \to 0} |\sin x| = 0$

$\therefore \quad \lim_{x \to 0} \sin x = 0$ **(Lemma 4)**

Lemma 6 : $1 - \frac{x^2}{2} \le \cos x \le 1 \qquad \forall x \in R$

Proof : We know $1 - \cos x = 2\sin^2\frac{x}{2}$

$|\sin x| \le |x|$

$\therefore \quad \left|\sin\frac{x}{2}\right| \le \left|\frac{x}{2}\right|$

$\therefore \quad \sin^2\frac{x}{2} \le \frac{x^2}{4}$

$\therefore \quad 1 - \cos x = 2\sin^2\frac{x}{2} \le 2 \times \frac{x^2}{4} = \frac{x^2}{2}$

$\therefore \quad 1 - \frac{x^2}{2} \le \cos x \le 1$

Theorem 3 : $\lim_{x \to 0} \cos x = 1$

Proof : $1 - \frac{x^2}{2} \le \cos x \le 1$

$\lim_{x \to 0} 1 - \frac{x^2}{2} = 1 - 0 = 1$ **(limit of a polynomial)**

$\therefore \quad \lim_{x \to 0} 1 = 1$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} \cos x = 1$

Theorem 4 : $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof : $\cos x < \frac{\sin x}{x} < 1 \qquad 0 < |x| < \frac{\pi}{2}$

$\therefore \quad \lim_{x \to 0} \cos x = 1, \quad \lim_{x \to 0} 1 = 1$

$\therefore$ **By sandwich theorem** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Theorem 5 : $\lim_{x \to a} \sin x = \sin a$

Proof : Let $x - a = h$. Then $x = a + h$

$\therefore$ As $x \to a, \quad h \to 0$

$\therefore \lim_{h \to 0} sinh = 0$

$\lim_{h \to 0} sin(a + h) = \lim_{h \to 0} (sina\ cosh + cosa\ sinh)$

$= sina \lim_{h \to 0} cosh + cosa \lim_{h \to 0} sinh$ **(algebra of limits)**

$= sina \cdot 1 + cosa \cdot 0$ $\left(\lim_{h \to 0} cosh = 1, \lim_{h \to 0} sinh = 0\right)$

$= sina$

$\therefore \lim_{x \to a} sinx = sina$

Theorem 6 : $\lim_{x \to a} cosx = cosa$

Proof : Again let $x = a + h$

$\therefore$ As $x \to a,\ h \to 0$

$\therefore \lim_{h \to 0} cosh = 1$ and $\lim_{h \to 0} sinh = 0$

$\lim_{x \to a} cosx = \lim_{h \to 0} cos(a + h) = \lim_{h \to 0} (cosa\ cosh - sina\ sinh)$

$= cosa \lim_{h \to 0} cosh - sina \lim_{h \to 0} sinh$ **(algebra of limits)**

$= cosa \cdot 1 + sina \cdot 0$

$= cosa$

$\therefore \lim_{x \to a} cosx = cosa$

Theorem 7 : $\lim_{x \to 0} \frac{tanx}{x} = 1$

Proof : $\lim_{x \to 0} \frac{tanx}{x} = \lim_{x \to 0} \frac{sinx}{xcosx}$

$= \frac{\lim_{x \to 0} \frac{sinx}{x}}{\lim_{x \to 0} cosx} = \frac{1}{1} = 1$ $\left(\lim_{x \to 0} \frac{sinx}{x} = 1, \lim_{x \to 0} cosx = 1\right)$

$\therefore \lim_{x \to 0} \frac{tanx}{x} = 1$

Now we will apply these results to examples.

Example 33 : Find $\lim_{x \to 0} \frac{sinax}{sinbx}$, $a, b \neq 0$

Solution : $\lim_{x \to 0} \frac{sinax}{sinbx} = \frac{\lim_{x \to 0} \frac{sinax}{ax} \cdot a}{\lim_{x \to a} \frac{sinbx}{bx} \cdot b}$

$= \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b}$ $\left(\lim_{x \to 0} \frac{sinx}{x} = 1\right)$

Example 34 : Find $\lim_{x \to 0} \frac{1 - cos\ 2x}{1 - cosx}$.

Solution : $\lim_{x \to 0} \frac{1 - cos\ 2x}{1 - cosx} = \lim_{x \to 0} \frac{2sin^2 x}{2sin^2 \frac{x}{2}}$

$$= \lim_{x \to 0} \frac{sinx}{x} \cdot \frac{sinx}{x} \cdot \frac{\frac{x}{2} \cdot 2}{sin\frac{x}{2}} \cdot \frac{\frac{x}{2} \cdot 2}{sin\frac{x}{2}}$$

$$= 1 \cdot 1 \cdot 2 \cdot 2 = 4$$

Another Method : $\lim_{x \to 0} \frac{1 - cos\ 2x}{1 - cosx} = \lim_{x \to 0} \frac{(1 - cos\ 2x)(1 + cos\ 2x)(1 + cosx)}{(1 + cos\ 2x)(1 - cos\ x)(1 + cosx)}$

$$= \lim_{x \to 0} \frac{sin^2 2x(1 + cosx)}{sin^2 x(1 + cos2x)}$$

$$= \lim_{x \to 0} \frac{sin^2 2x}{4x^2} \frac{4x^2}{sin^2 x} \frac{(1 + cosx)}{(1 + cos\ 2x)}$$

$$= 1 \cdot 4 \cdot \frac{(2)}{(2)} = 4$$

Example 35 : Find $\lim_{x \to 0} \frac{sinax + bx}{ax + sinbx}$.

Solution : $\lim_{x \to 0} \frac{sinax + bx}{ax + sinbx} = \lim_{x \to 0} \frac{\frac{sinax}{x} + b}{a + \frac{sinbx}{x}}$

$$= \lim_{x \to 0} \frac{a\frac{sinax}{ax} + b}{a + b\frac{sinbx}{bx}}$$

$$= \frac{a + b}{a + b} = 1$$

Example 36 : Find $\lim_{x \to \frac{\pi}{2}} \frac{tan\ 2x}{\frac{\pi}{2} - x}$.

Solution : Let $\frac{\pi}{2} - x = \alpha$. Then as $x \to \frac{\pi}{2}$, $\alpha \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{tan\ 2x}{\frac{\pi}{2} - x} = \lim_{\alpha \to 0} \frac{tan\ 2\left(\frac{\pi}{2} - \alpha\right)}{\alpha}$$

$$= \lim_{\alpha \to 0} \frac{tan\ (\pi - 2\alpha)}{\alpha} = \lim_{\alpha \to 0} \frac{-tan\ 2\alpha}{2\alpha} \cdot 2 = -2$$

Example 37 : Find $\lim_{x \to 0} \frac{tanx - sinx}{x^3}$.

Solution : $\lim_{x \to 0} \frac{tanx - sinx}{x^3} = \lim_{x \to 0} \frac{\frac{sinx}{cosx} - sinx}{x^3}$

$$= \lim_{x \to 0} \frac{sinx(1 - cosx)}{cosx \cdot x^3}$$

$$= \lim_{x \to 0} \frac{tanx \cdot 2sin\frac{x}{2} \cdot sin\frac{x}{2}}{x \cdot 2 \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot 2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Example 38 : Find $\lim\limits_{x\to 0}\dfrac{cosecx - cotx}{x}$.

Solution : $\lim\limits_{x\to 0}\dfrac{cosecx - cotx}{x} = \lim\limits_{x\to 0}\dfrac{1-cosx}{sinx\cdot x}$

$$= \lim_{x\to 0}\frac{2sin\frac{x}{2}\cdot sin\frac{x}{2}}{2\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot 2}\;\frac{x}{sinx}$$

$$= \frac{2}{4} = \frac{1}{2}$$

or $\lim\limits_{x\to 0}\dfrac{1-cosx}{sinx\cdot x} = \lim\limits_{x\to 0}\dfrac{(1-cosx)(1+cosx)}{(1+cosx)\;xsinx}$

$$= \lim_{x\to 0}\frac{sin^2x}{xsinx\;(1+cosx)}$$

$$= \lim_{x\to 0}\frac{sinx}{x(1+cosx)}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Example 39 : Find $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{sinx - cosx}{\frac{\pi}{4}-x}$.

Solution : $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{sinx - cosx}{\frac{\pi}{4}-x} = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\left(\frac{1}{\sqrt{2}}sinx - \frac{1}{\sqrt{2}}cosx\right)}{\frac{\pi}{4}-x}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left(sinx\;cos\frac{\pi}{4} - sin\frac{\pi}{4}cosx\right)}{\frac{\pi}{4}-x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\,sin\left(x-\frac{\pi}{4}\right)}{-\left(x-\frac{\pi}{4}\right)}$$

$$= \lim_{\alpha\to 0}\frac{\sqrt{2}\,sin\,\alpha}{-\alpha} \qquad \left(\text{take } \alpha = x - \frac{\pi}{4},\ \alpha\to 0\right)$$

$$= -\sqrt{2}$$

Miscellaneous Problems :

Example 40 : Find $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$.

Solution : $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right) = \lim\limits_{x\to 1}\dfrac{x+1-2}{x^2-1}$

$$= \lim_{x\to 1}\frac{(x-1)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1}\frac{1}{x+1}$$

$$= \frac{1}{2}$$

Example 41 : Find $\lim_{x \to -2} \dfrac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2}$.

Solution : $\lim_{x \to -2} \dfrac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2} = \lim_{x \to -2} \dfrac{(x+2)(x^2 - x + 6)}{(x+2)(x^2 - 2x + 1)}$

$$= \lim_{x \to -2} \frac{x^2 - x + 6}{x^2 - 2x + 1}$$

$$= \frac{4 + 2 + 6}{4 + 4 + 1}$$

$$= \frac{12}{9} = \frac{4}{3}$$

Example 42 : Find $\lim_{x \to 0} \dfrac{\sqrt{x^2 + x + 1} - \sqrt{x + 1}}{x^2}$.

Solution : $\lim_{x \to 0} \dfrac{\left(\sqrt{x^2 + x + 1} - \sqrt{x+1}\right)\left(\sqrt{x^2 + x + 1} + \sqrt{x+1}\right)}{x^2\left(\sqrt{x^2 + x + 1} + \sqrt{x+1}\right)}$

$$= \lim_{x \to 0} \frac{x^2 + x + 1 - x - 1}{x^2\left(\sqrt{x^2 + x + 1} + \sqrt{x+1}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + x + 1} + \sqrt{x+1}}$$

$$= \frac{1}{2}$$

Example 43 : Find $\lim_{x \to \frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right)$.

Solution : $\lim_{x \to \frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim_{x \to \frac{\pi}{2}} \dfrac{x\sin x - \frac{\pi}{2}}{\cos x}$

$$= \lim_{\alpha \to 0} \frac{\left(\frac{\pi}{2} - \alpha\right)\cos\alpha - \frac{\pi}{2}}{\sin\alpha} \qquad \left(\frac{\pi}{2} - x = \alpha, \alpha \to 0\right)$$

$$= \lim_{\alpha \to 0} \frac{\frac{\pi}{2}(\cos\alpha - 1)}{\sin\alpha} - \frac{\alpha\cos\alpha}{\sin\alpha}$$

$$= \lim_{\alpha \to 0} \left(\frac{-\frac{\pi}{2}\left(2\sin^2\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} - \frac{\alpha}{\tan\alpha}\right)$$

$$= \lim_{\alpha \to 0} \left(-\frac{\pi}{2}\tan\frac{\alpha}{2} - \frac{\alpha}{\tan\alpha}\right)$$

$$= -1$$

Example 44 : Find $\lim_{x \to 1} (1 - x)\tan\frac{\pi x}{2}$.

Solution : Let $1 - x = \alpha$, $\alpha \to 0$ as $x \to 1$.

$$\therefore \lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2} = \lim_{\alpha \to 0} \alpha \tan \frac{\pi}{2}(1 - \alpha)$$

$$= \lim_{\alpha \to 0} \alpha \tan\left(\frac{\pi}{2} - \frac{\pi\alpha}{2}\right)$$

$$= \lim_{\alpha \to 0} \alpha \cot \frac{\pi\alpha}{2}$$

$$= \lim_{\alpha \to 0} \frac{\frac{\pi}{2}\alpha}{\frac{\pi}{2} \tan \frac{\pi\alpha}{2}}$$

$$= \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

Example 45 : Find $\lim_{x \to 1} \left(\frac{m}{1 - x^m} - \frac{n}{1 - x^n}\right)$; $(m,\ n \in \mathrm{N})$.

Solution : $\lim_{x \to 1} \left(\frac{m}{1 - x^m} - \frac{n}{1 - x^n}\right) = \lim_{x \to 1} \frac{m(1 - x^n) - n(1 - x^m)}{(1 - x^n)(1 - x^m)}$

Let $x = 1 + h$ so that $h \to 0$ as $x \to 1$

$$\therefore \lim_{x \to 1} \left(\frac{m}{1 - x^m} - \frac{n}{1 - x^n}\right)$$

$$= \lim_{h \to 0} \frac{m[1 - (1 + h)^n] - n[1 - (1 + h)^m]}{[(1 + h)^m - 1][(1 + h)^n - 1]}$$

$$= \lim_{h \to 0} \frac{m\left(1 - 1 - nh - \binom{n}{2}h^2 - \binom{n}{3}h^3 - \ldots - h^n\right) - n\left(1 - 1 - \binom{m}{1}h - \binom{m}{2}h^2 - \ldots - h^m\right)}{\left(\binom{m}{1}h + \binom{m}{2}h^2 + \ldots + h^m\right)\left(\binom{n}{1}h + \binom{n}{2}h^2 + \ldots + h^n\right)}$$

$$= \lim_{h \to 0} \frac{h\left(-mn - m\binom{n}{2}h - m\binom{n}{3}h^2 - \ldots - mh^{n-1} + nm + n\binom{m}{2}h + n\binom{m}{3}h^2 + \ldots + nh^{m-1}\right)}{h\left(\binom{m}{1} + \binom{m}{2}h + \ldots + h^{m-1}\right) \cdot h\left(\binom{n}{1} + \binom{n}{2}h + \ldots + h^{n-1}\right)}$$

$$= \lim_{h \to 0} \frac{h\left(-m\binom{n}{2} - m\binom{n}{3}h - \ldots - mh^{n-2} + n\binom{m}{2} + n\binom{m}{3}h + \ldots + nh^{m-2}\right)}{h\left(\binom{m}{1} + \binom{m}{2}h + \ldots + h^{m-1}\right)\left(\binom{n}{1} + \binom{n}{2}h + \ldots + h^{n-1}\right)}$$

$$= \frac{-m\binom{n}{2} + n\binom{m}{2}}{\binom{m}{1}\binom{n}{1}}$$

$$= \frac{\frac{-mn(n-1)}{2} + \frac{nm(m-1)}{2}}{mn}$$

$$= \frac{m - 1 - n + 1}{2}$$

$$= \frac{m - n}{2}$$

Exercise 10

Using algebra of limits and the definition of limit prove the following : **(1 to 3)**

1. $\lim\limits_{x \to 2} x^2 = 4$ **2.** $\lim\limits_{x \to 1} |x|^2 = 1$ **3.** $\lim\limits_{x \to 3} x^3 = 27$

Prove following limits do not exist : **(4 to 6)**

4. $\lim\limits_{x \to 0} \frac{|x|}{x}$ **5.** $\lim\limits_{x \to 3} \frac{|x-3|}{x-3}$ **6.** $\lim\limits_{x \to 2} [x]$

7. For $f(x) = \frac{x^2 - 9}{x - 3}$, $x \neq 3$, $f(3) = 6$, prove $\lim\limits_{x \to 3} f(x) = f(3)$.

8. For $f(x) = \frac{x^2 - 1}{x + 1}$, $x \neq -1$, $f(-1) = 5$, prove $\lim\limits_{x \to -1} f(x) \neq f(-1)$.

9. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ where $x \in (a - \delta, a + \delta) - \{a\}$ for some $\delta > 0$, can we say $f(x) = g(x)$ for all $x \in (a - \delta, a + \delta) - \{a\}$.

10. If $x^2 + 1 \leq f(x) \leq 2x^4 + x^2 + 1$, prove $\lim\limits_{x \to 0} f(x) = 1$.

Find following limits : **(11 to 32)**

11. $\lim\limits_{x \to 64} \frac{x^{\frac{1}{6}} - 2}{\sqrt{x} - 8}$

12. $\lim\limits_{x \to 0} \frac{\tan mx}{\tan nx}$

13. $\lim\limits_{x \to \frac{\pi}{3}} \frac{\sqrt{3} \cos x - \sin x}{x - \frac{\pi}{3}}$

14. $\lim\limits_{x \to \alpha} \frac{9 \sin x - 40 \cos x}{x - \alpha}$ where $\tan\alpha = \frac{40}{9}$, $0 < \alpha < \frac{\pi}{2}$

15. $\lim\limits_{h \to 0} \frac{(x+h)^{\frac{1}{4}} - x^{\frac{1}{4}}}{h}$

16. $\lim\limits_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}$

17. $\lim\limits_{x \to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$

18. $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

19. $\lim\limits_{x \to 1+} \frac{\sqrt{x-1}}{\sqrt{x^2 - 1} + \sqrt{x^3 - 1}}$ **(why $x \to 1+$?)**

20. $\lim\limits_{x \to 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2}$, $n \in N$

21. $\lim\limits_{x \to 0} \frac{2 \sin x - \sin 2x}{x^3}$

22. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sin 3x + \cos 3x}{x - \frac{\pi}{4}}$

23. $\lim\limits_{x \to 0} \frac{(1 + mx)^n - (1 + nx)^m}{x^2}$ $m, n \in N$

24. $\lim\limits_{x \to \pi} \dfrac{\sqrt{10 + cosx} - 3}{(\pi - x)^2}$

25. $\lim\limits_{x \to 0} \dfrac{cos5x - cos7x}{x^2}$

26. $\lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + cosx}}{x^2}$

27. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{secx - tanx}{\frac{\pi}{2} - x}$

28. $\lim\limits_{h \to 0} \dfrac{sin(a + 3h) - 3sin(a + 2h) + 3sin(a + h) - sina}{h^3}$

29. $\lim\limits_{x \to 0} \dfrac{sin(3 + x) - sin(3 - x)}{x}$

30. $\lim\limits_{x \to a} \dfrac{\sqrt{2a + 3x} - \sqrt{x + 4a}}{\sqrt{3a + 2x} - \sqrt{4x + a}}$

31. $\lim\limits_{h \to 0} \dfrac{(a + h)^2 \, sin\,(a + h) - a^2 sina}{h}$

32. $\lim\limits_{h \to 0} \dfrac{(x + h)\, sec\,(x + h) - xsecx}{h}$

33. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1 - sinx}{cosx} = ...$ ☐

(a) 1 (b) 0 (c) -1 (d) 2

(2) $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ .... ☐

(a) is 1 (b) is -1 (c) is zero (d) does not exist

(3) $\lim\limits_{x \to \pi} \dfrac{tanx}{\pi - x}$ .... ☐

(a) is 1 (b) is -1 (c) does not exist (d) is 0

(4) If $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 80$, then $n =$ .... ☐

(a) -3 (b) 2 (c) 5 (d) 6

(5) $\lim\limits_{x \to 0} \dfrac{1 - cosmx}{1 - cosnx} = ...$ ☐

(a) $\dfrac{m}{n}$ (b) $\dfrac{m^2}{n^2}$ (c) $\dfrac{m^3}{n^3}$ (d) 0

(6) $\lim\limits_{x \to 0+} \dfrac{|sinx|}{x}$ ...... ☐

(a) is 1 (b) is -1 (c) does not exist (d) is 0

(7) $\lim_{x \to 0-} \frac{\sin[x]}{[x]}$ .... $(-1 < x < 0,\ x \in R)$

(a) is 1 (b) is zero

(c) is -1 (d) is $\sin 1$

(8) $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} =$ ......

(a) 1 (b) 2 (c) 0 (d) -1

(9) $\lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ ....

(a) does not exist (b) is 1 (c) is $\frac{1}{10}$ (d) is $-\frac{1}{10}$

(10) $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x} =$ ......

(a) 5 (b) 6 (c) 0 (d) 10

(11) If $1 \le f(x) \le x^2 + 2x + 2 \ \ \forall x \in R$, $\lim_{x \to -1} f(x) =$ ......

(a) 2 (b) 0 (c) -1 (d) 1

(12) $\lim_{x \to 0-} \left(\frac{1}{x} + \frac{1}{|x|}\right) =$ ......

(a) 2 (b) 1 (c) 0 (d) -1

(13) $\lim_{x \to 2} f(x) =$ ...... where $f(x) = \begin{cases} 2x + 3 & x < 2 \\ 5 & x = 2 \\ 3x + 2 & x > 2 \end{cases}$

(a) 5 (b) 3 (c) 2 (d) does not exist

(14) $\lim_{x \to 0+} f(x) =$ ...... where $f(x) = \begin{cases} 3x^2 - 1 & x < 0 \\ 3x^2 + 1 & x \ge 0 \end{cases}$

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{3}$

(15) $\lim_{x \to 5+} [x] =$ ......

(a) 6 (b) 5 (c) -5 (d) 4

(16) $\lim_{x \to -4-} [x] =$ ......

(a) 5 (b) -5 (c) -4 (d) 4

(17) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a} =$ ......

(a) $\cos a$ (b) $\sin a$ (c) a (d) 0

(18) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ $(a > 0)$ = ...

(a) $\cos a$ (b) $\dfrac{\cos a}{2\sqrt{a}}$ (c) $2\sqrt{a}\cos a$ (d) $2\sqrt{a}$

(19) $\lim\limits_{x \to 0} \dfrac{\tan x - 5x}{7x - \sin x}$ = ...

(a) $\frac{2}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{5}{7}$ (d) $\frac{-5}{7}$

(20) $\lim\limits_{x \to a} \dfrac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{5}} - a^{\frac{1}{5}}}$ $(a > 0)$ = ...

(a) $\frac{1}{3}a^{\frac{3}{5}}$ (b) $\frac{1}{5}a^{\frac{1}{15}}$ (c) $\frac{5}{3}a^{\frac{5}{3}}$ (d) $\frac{5}{3}a^{\frac{2}{15}}$

*

Summary

We studied following points in this chapter :

1. History of limits
2. Graphical and tabulation for inference of limit
3. Formal definition of limit and applications
4. Algebra of limits, if $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x),$$

$$\lim_{x \to a} (f(x)\ g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x),$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \quad \left(\text{where } \lim_{x \to a} g(x) \neq 0\right)$$

5. $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ and rule of substitution
6. Sandwich theorem and trigonometric limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1, \quad \lim_{x \to 0} \sin x = 0, \quad \lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to a} \sin x = \sin a, \quad \lim_{x \to a} \cos x = \cos a$$

Bhaskara I

Bhaskara stated theorems about the solutions of today's so called Pell equations. For instance, he posed the problem : "Tell me, O mathematician, what is that square which multiplied by 8 becomes - together with unity - a square?" In modern notation, he asked for the solutions of the Pell equation $8x^2 + 1 = y^2$. It has the simple solution $x = 1$, $y = 3$, or shortly $(x,y) = (1,3)$, from which further solutions can be constructed, e.g., $(x,y) = (6,17)$.

Chapter 11

DERIVATIVE

Mathematics is as much an aspect of culture as it is a collection of algorithms.

– Carl Boyer (in a Calculus Textbook)

11.1 Introduction

In calculus the derivative is a measure of how a function changes as its input changes. Loosely speaking we can think of a derivative as how much one quantity changes in response to changes in some other quantity. The derivative of the position of a moving object with respect to time is its instantaneous velocity.

The derivative of a function at a chosen input value describes the best linear approximation to the function near the input value. For a real function of a real variable, the derivative at a point is equal to the slope of the tangent line to the graph of the function at that point.

For a 'small' h the line passing through $(a, f(a))$ and $(a + h, f(a + h))$ is called a ***secant*** line. Its slope for a value of h near to zero, gives a good approximation to the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$ and smaller the value of h, we get a better approximation.

Slope m of the secant line at $(a, f(a))$ is given by

$$m = \frac{f(a+h) - f(a)}{h}$$

This is called Newton's difference quotient.

$\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ is the derivative of f at a and is denoted by $f'(a)$, if this limit exists. This represents slope of the tangent to $y = f(x)$ at $(a, f(a))$.

We can also say

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} - f'(a)$$

$$= f'(a) - f'(a)$$

$$= 0$$

This gives best linear approximation $f(a+h) \cong f(a) + hf'(a)$ for f near 'small' h.

If we write $Q(h) = \frac{f(a+h) - f(a)}{h}$, $Q(h)$ is the slope of the secant line joining points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph of $y = f(x)$. If the graph of f is a unbroken curve with no gaps, then $\lim_{h \to 0} Q(h)$, if it exists, is called the derivative of f at a and we say f is differentiable at $x = a$.

Rocket scientists need to compute the accurate velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket. Derivative is a word regularly used in stock market. Financial institutes predict the change in the value of a stock knowing its present value. All these require the knowledge of change in one quantity called dependent variable depending upon the change in another quantity called independent variable.

11.2 Formal Definition and Examples

Definition : Let f be real valued function defined on an interval (a, b). Let $c \in (a, b)$. Let h be sufficiently small so that $c + h \in (a, b)$.

If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, it is called the derivative of f at c and is denoted by $f'(c)$.

Example 1 : Find $f'(1)$ for $f : R \to R$, $f(x) = 3x + 5$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h) + 5 - 8}{h}$ **($f(1) = 8$)**

$$= \lim_{h \to 0} \frac{3h}{h} = 3$$

$\therefore$ $f'(1)$ exists and $f'(1) = 3$

Example 2 : For $f : R \to R$, $f(x) = 2x^2 + 3x - 1$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{2h^2 + 3h - 1 - (-1)}{h}$ **($f(0) = -1$)**

$$= \lim_{h \to 0} \frac{2h^2 + 3h}{h}$$

$$= \lim_{h \to 0} (2h + 3) = 3$$

$\therefore$ $f'(0)$ exists and $f'(0) = 3$

Example 3 : For $f : R \to R$, $f(x) = sinx$, find $f'(0)$, if it exists.

Solution : $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{sinh - 0}{h}$ **($sin0 = 0$)**

$$= \lim_{h \to 0} \frac{sinh}{h} = 1$$

$\therefore$ $f'(0)$ exists and $f'(0) = 1$

Example 4 : For $f : R \to R$, $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, find $f'(0)$, if it exists.

Solution : $\lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h\sin\frac{1}{h}}{h}$ **$(f(0) = 0)$**

$= \lim\limits_{h \to 0} \sin\frac{1}{h}$ does not exist. (Refer chapter 10)

$\therefore \quad f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ has no derivative at $x = 0$.

Example 5 : For $f : R \to R$, $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$, find $f'(0)$, if it exists.

Solution : $\lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h^2\sin\frac{1}{h}}{h}$ **$(f(0) = 0)$**

$= \lim\limits_{h \to 0} h\sin\frac{1}{h}$

Now $0 \leq \left|\sin\frac{1}{h}\right| \leq 1 \Rightarrow 0 \leq \left|h\sin\frac{1}{h}\right| \leq |h|$ and $\lim\limits_{h \to 0} 0 = 0$, $\lim\limits_{h \to 0} |h| = 0$

$\therefore \quad \lim\limits_{h \to 0} \left|h\sin\frac{1}{h}\right| = 0$

$\therefore \quad \lim\limits_{h \to 0} h\sin\frac{1}{h} = 0$

$\therefore \quad f'(0)$ exists and $f'(0) = 0$

Definition : Let f be defined on (a, b). Let $x \in (a, b)$ and h be sufficiently small so that $x + h \in (a, b)$. If $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists, we say f is differentiable at x and call this limit the derivative of f at x. This gives us a function $\dfrac{d}{dx} f(x)$ defined at all points of $x \in (a, b)$ where f is differentiable and we write $f'(x) = \dfrac{d}{dx} f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, at all points of (a, b) where f is differentiable. (Assuming that f is differentiable at at least one point of (a, b).)

If we write $y = f(x)$, $\dfrac{d}{dx} f(x)$ may be written as $\dfrac{dy}{dx}$. Its value at $x = c$ can be written as $\left[\dfrac{d}{dx} f(x)\right]_{x = c}$ or $\left(\dfrac{dy}{dx}\right)_{x = c}$ or sometimes $[D f(x)]_{x = c}$ or $f'(c)$.

Example 6 : For $f : R \to R$, $f(x) = ax^2 + bx + c$, find $f'(x)$ and $f'(0)$.

Solution : $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[a(x^2 + 2hx + h^2) + bx + bh + c] - (ax^2 + bx + c)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2ahx + ah^2 + bh}{h}$

$$= \lim_{h \to 0} (2ax + ah + b)$$

$$= 2ax + b$$

$\therefore$ $f'(x)$ exists for $\forall x \in$ R and $f'(x) = 2ax + b$

$\therefore$ $f'(0) = b$ **(taking $x = 0$ in $f'(x)$)**

Note : If we obtain $f'(0)$ as $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$, then also we will get b as the answer.

Example 7 : For f : R $\to$ R, $f(x) = ax + b$, find $f'(x)$.

Solution : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h}$

$$= \lim_{h \to 0} \frac{ah}{h} = a$$

$\therefore$ $f'(x)$ exists and $f'(x) = a$, $\forall x \in$ R.

Example 8 : For f : R $\to$ R, $f(x) = \frac{ax+b}{cx+d}$, find $f'(x)$ and $f'(0)$. $\left(x \neq -\frac{d}{c}\right)$

Solution : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{a(x+h)+b}{c(x+h)+d} - \frac{ax+b}{cx+d}}{h}$$

$$= \lim_{h \to 0} \frac{(ax + ah + b)(cx + d) - (ax + b)(cx + ch + d)}{(cx + ch + d)(cx + d)h}$$

$$= \lim_{h \to 0} \frac{h(acx + ad - acx - bc)}{(cx + ch + d)(cx + d)h}$$

$$= \lim_{h \to 0} \frac{(ad - bc)}{(cx + ch + d)(cx + d)}$$

$$= \frac{(ad - bc)}{(cx + d)^2}$$

$\therefore$ $f'(x)$ exists and $f'(x) = \frac{ad - bc}{(cx+d)^2}$ **(i)**

$\therefore$ $f'(0) = \frac{ad - bc}{d^2}$

Note : $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$ (taking $a = 0$, $b = 1$, $c = 1$, $d = 0$ in (i))

11.3 Algebra of Derivatives

Let f and g be differentiable in (a, b).

Then **(1)** $f + g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(2) $f - g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

(3) $f \times g$ is also differentiable in (a, b) and

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

(4) $\frac{f}{g}$ is also differentiable in (a, b), if $g(x) \neq 0$, and

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Some Important Results :

(1) $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Let $x + h = t$. Then $t \to x$ as $h \to 0$.

$\therefore \quad f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

(2) The derivative of a constant function is zero.

Let $f(x) = c$, $\forall x \in$ R.

$\therefore \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$

$\therefore \quad \frac{d}{dx} c = 0$

(3) $\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x)$ **($k \in$ R is a constant.)**

$$\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x) + f(x)\frac{d}{dx}k$$

$$= k\frac{d}{dx}f(x) + f(x) \cdot 0 \qquad \textbf{(by (2))}$$

$$= k\frac{d}{dx}f(x)$$

Example 9 : Prove : $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ using

$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ and $\frac{d}{dx} kf(x) = k\frac{d}{dx}f(x)$

Solution : $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x) + (-1)g(x))$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}(-1)g(x)$$

$$= \frac{d}{dx}f(x) + (-1)\frac{d}{dx}g(x)$$

$$= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example 10 : Prove : $\frac{d}{dx} kf(x) = k\frac{d}{dx} f(x)$, $k \in$ R

Solution : $\lim\limits_{h \to 0} \frac{kf(x+h) - kf(x)}{h} = \lim\limits_{h \to 0} k \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ **(Rules of limit)**

$= k\frac{d}{dx} f(x)$

$\therefore \frac{d}{dx} kf(x) = k\frac{d}{dx} f(x)$

Some Standard Forms :

(1) $\frac{d}{dx} x^n = nx^{n-1}$, $n \in$ N, $x \in$ R

Proof : $\frac{d}{dx} x^n = \lim\limits_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$$= \lim_{h \to 0} \frac{\left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n}{h}$$

$$= \lim_{h \to 0} \left(nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \ldots + h^{n-1}\right) = nx^{n-1}$$

Second Proof : Let P(n) : $\frac{d}{dx} x^n = nx^{n-1}$

We have $\frac{d}{dx} x^1 = \lim\limits_{h \to 0} \frac{x+h-x}{h} = \lim\limits_{h \to 0} \frac{h}{h} = 1$. Also $1 \cdot x^{1-1} = 1 \cdot 1 = 1$ **($x \neq 0$)**

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \frac{d}{dx} x^k = kx^{k-1}$

Let $n = k + 1$

$$\frac{d}{dx} x^{k+1} = \frac{d}{dx} x^k \cdot x$$

$$= x^k \frac{d}{dx} x + x \frac{d}{dx} x^k$$

$$= x^k \cdot 1 + x \cdot kx^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1)x^k$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

$\therefore$ By the principle of mathematical induction P(n) is true, $\forall n \in$ N.

Third Proof : $\frac{d}{dx}x^n = \lim_{t \to x} \frac{t^n - x^n}{t - x}$ **(formula (2))**

$$= \lim_{t \to x} \frac{(t-x)(t^{n-1} + t^{n-2}x + t^{n-3}x^2 + ... + x^{n-1})}{t - x}$$

$$= \lim_{t \to x} (t^{n-1} + t^{n-2}x + t^{n-3}x^2 + ... + x^{n-1})$$

$$= x^{n-1} + x^{n-2} \cdot x + x^{n-3} \cdot x^2 + ... + x^{n-1}$$

$$= nx^{n-1}$$

Note : We have given the proof for $n \in N$, $x \in R$, but the result is valid for $n \in R$, $x \in R^+$. We will not prove it.

(2) Derivative of a Polynomial :

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$, $x \in R$, $n \in N$, $a_n \neq 0$, $a_i \in R$ **($i = 0, 1, 2, ..., n$)**

be a polynomial of degree n.

$$\therefore \frac{d}{dx}P(x) = \frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0)$$

$$= \frac{d}{dx}a_n x^n + \frac{d}{dx}a_{n-1}x^{n-1} + \frac{d}{dx}a_{n-2}x^{n-2} + ... + \frac{d}{dx}a_0$$ **(Derivative of sum)**

$$= a_n \frac{d}{dx}x^n + a_{n-1}\frac{d}{dx}x^{n-1} + a_{n-2}\frac{d}{dx}x^{n-2} + ... + \frac{d}{dx}a_0$$

$$= na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + ... + 0)$$

$$\left(\frac{d}{dx}x^n = nx^{n-1}\right)$$

(3) Derivative of a Rational Function :

Let $h(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomial functions. $q(x) \neq 0$.

$\therefore \quad h'(x) = \frac{q(x)\,p'(x) - p(x)\,q'(x)}{[q(x)]^2}$ and $p'(x)$ and $q'(x)$ can be obtained by (2).

(4) $\frac{d}{dx}\sin x = \cos x$, $x \in R$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$$

$$\therefore \frac{d}{dx}\sin x = \cos x$$

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(5) $\frac{d}{dx} \cos x = -\sin x,\ x \in R$

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{-2\sin\left(x + \frac{h}{2}\right) \sin\frac{h}{2}}{h}$$

$$= -\lim_{h \to 0} \frac{\sin\left(x + \frac{h}{2}\right) \sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin x$$

$\therefore\ \frac{d}{dx} \cos x = -\sin x$

(6) $\frac{d}{dx} \tan x = \sec^2 x,\quad x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x}$$

$$= \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{\cos^2 x} \quad \text{(by rule for } \tfrac{d}{dx} \tfrac{f}{g}\text{)}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(7) $\frac{d}{dx} \cot x = -\mathrm{cosec}^2 x,\quad x \in R - \{k\pi \mid k \in Z\}$

$$\frac{d}{dx} \cot x = \frac{d}{dx} \frac{\cos x}{\sin x}$$

$$= \frac{\sin x \frac{d}{dx} \cos x - \cos x \frac{d}{dx} \sin x}{\sin^2 x} \quad \text{(by rule for } \tfrac{d}{dx} \tfrac{f}{g}\text{)}$$

$$= \frac{\sin x(-\sin x) - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\mathrm{cosec}^2 x$$

(8) $\frac{d}{dx} \sec x = \sec x \tan x,\quad x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x}$$

$$= \frac{\cos x \frac{d}{dx} 1 - 1 \frac{d}{dx} \cos x}{\cos^2 x}$$

$$= \frac{cosx \cdot 0 - 1(-sinx)}{cos^2x}$$

$$= \frac{sinx}{cos^2x}$$

$$= secx\ tanx$$

(9) $\frac{d}{dx} cosecx = -cosecx\ cotx, \quad x \in R - \{k\pi \mid k \in Z\}$

$$\frac{d}{dx} cosecx = \frac{d}{dx} \frac{1}{sinx}$$

$$= \frac{sinx \frac{d}{dx} 1 - 1 \frac{d}{dx} sinx}{sin^2x}$$

$$= \frac{sinx \cdot 0 - 1(cosx)}{sin^2x}$$

$$= \frac{-cosx}{sin^2x}$$

$$= -cosecx\ cotx$$

Note : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called the derivative of $f(x)$ obtained using definition or from first principle. Above standard forms can also be obtained from first principle.

Also we can extended the rule $\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$ as $\frac{d}{dx}(f_1(x) + f_2(x) + ... + f_n(x)) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + ... + \frac{d}{dx} f_n(x)$, using principle of mathematical induction and we have used it in obtaining derivative of a polynomial.

Also we note that this result is true only for a finite sum of n terms and for infinite sum $\frac{d}{dx}(f_1(x) + f_2(x) + ...) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + ...$ may not be valid. This would require advance discussion on convergence and **uniform convergence** of a series which we are not able to do here at this stage.

Some Miscellaneous Problems :

Example 11 : Find the derivative of $f(x) = cos^2x$.

Solution : $\frac{d}{dx} cos^2x = \frac{d}{dx} cosx\ cosx$

$$= cosx \frac{d}{dx} cosx + cosx \frac{d}{dx} cosx$$

$$= 2cosx\ (-sinx)$$

$$= -2sinx\ cosx$$

$$= -sin2x$$

Example 12 : Find the derivative of $xsin\,x$ from first principle.

Solution : $\frac{d}{dx} xsin\,x = \lim_{h \to 0} \frac{(x+h)\,sin(x+h) - xsin\,x}{h}$

$$= \lim_{h \to 0} \frac{(x+h)\sin(x+h) - (x+h)\sin x + (x+h)\sin x - x\sin x}{h}$$

$$= \lim_{h \to 0} (x + h)\left(\frac{\sin(x+h) - \sin x}{h}\right) + \lim_{h \to 0} \frac{(x+h-x)\sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)\, 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} + \lim_{h \to 0} \sin x$$

$$= \lim_{h \to 0} \frac{(x+h)\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}} + \sin x \qquad \left(\lim_{h \to 0} c = c\right)$$

$$= x\cos x + \sin x$$

Example 13 : Find $\frac{d}{dx}\tan x$ from first principle. $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

Solution : $\frac{d}{dx}\tan x = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$

$$= \lim_{h \to 0} \frac{\tan(x+h-x)}{h}(1 + \tan x \tan(x+h)) \qquad (\tan(A + B) \text{ formula})$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \lim_{h \to 0} (1 + \tan x \tan(x+h))$$

$$= 1 \cdot (1 + \tan^2 x)$$

$$= \sec^2 x$$

Example 14 : Find $\frac{d}{dx}\sec x$ from first principle. $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$

Solution : $\frac{d}{dx}\sec x = \lim_{h \to 0} \frac{\sec x(x+h) - \sec x}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h\cos x\cos(x+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{-h}{2}\right)\sin\left(x+\frac{h}{2}\right)}{h\cos x\cos(x+h)}$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \frac{\sin\left(x+\frac{h}{2}\right)}{\cos x\cos(x+h)}$$

$$= \frac{1 \cdot \sin x}{\cos x \cos x}$$

$$= \sec x \tan x$$

Example 15 : Find $\frac{d}{dx} sin2x$.

Solution : $\frac{d}{dx} sin2x = \frac{d}{dx} 2sinx \ cosx$

$= 2\left[sinx \frac{d}{dx} cosx + cosx \frac{d}{dx} sinx\right]$

$= 2[sinx(-sinx) + cosx \cdot cosx]$

$= 2(cos^2x - sin^2x)$

$= 2cos\,2x$

Example 16 : Find the derivative of $\frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \frac{x^{n-2}}{n-2} + ... + x + 1$

Solution : $\frac{d}{dx}\left(\frac{x^n}{n} + \frac{x^{n-1}}{n-1} + ... + x + 1\right) = \frac{nx^{n-1}}{n} + \frac{(n-1)x^{n-2}}{n-1} + \frac{(n-2)x^{n-3}}{n-2} + 1 + 0$

$= x^{n-1} + x^{n-2} + x^{n-3} + ... + 1$

$= \frac{x^n - 1}{x - 1}$ as a sum of G.P.

Example 17 : Find $\frac{d}{dx}(ax + b)^n$. Deduce value of $\frac{d}{dx}(ax + b)^m(cx + d)^n$.

Solution :

$\frac{d}{dx}(ax + b)^n = \frac{d}{dx}\left((ax)^n + \binom{n}{1}(ax)^{n-1}b + \binom{n}{2}(ax)^{n-2}b^2 + ... + \binom{n}{n-1} ax \cdot b^{n-1} + b^n\right)$

$= a^n n \cdot x^{n-1} + n(n-1)x^{n-2}a^{n-1}b + (n-2)\binom{n}{2}x^{n-3}a^{n-2}b^2$

$+ ... + \binom{n}{n-1} a \cdot 1 \cdot b^{n-1} + 0$

$= na\left[(ax)^{n-1} + (n-1)(ax)^{n-2}b + \frac{(n-2)(n-1)}{2}(ax)^{n-3}b^2 + ... + b^{n-1}\right]$

$= na\left((ax)^{n-1} + \binom{n-1}{1}(ax)^{n-2}b + \binom{n-1}{2}(ax)^{n-3}b^2 + ... + b^{n-1}\right)$

$= na(ax + b)^{n-1}$

Now, $\frac{d}{dx}(ax + b)^m(cx + d)^n = (cx + d)^n \frac{d}{dx}(ax + b)^m + (ax + b)^m \frac{d}{dx}(cx + d)^n$

$= (cx + d)^n ma(ax + b)^{m-1} + (ax + b)^m nc(cx + d)^{n-1}$

$= (ax + b)^{m-1}(cx + d)^{n-1}[ma(cx + d) + nc(ax + b)]$

Example 18 : Find $\frac{d}{dx}\left(\frac{a + bsinx}{c + dsinx}\right)$. **$(c + dsinx \neq 0)$**

Solution : $\frac{d}{dx}\left(\frac{a + bsinx}{c + dsinx}\right) = \frac{(c + dsinx)\frac{d}{dx}(a + bsinx) - (a + bsinx)\frac{d}{dx}(c + dsinx)}{(c + dsinx)^2}$

$= \frac{(c + dsinx)\,bcosx - (a + bsinx)\,dcosx}{(c + dsinx)^2}$

$$= \frac{bc\cos x + bd\sin x\cos x - ad\cos x - bd\sin x\cos x}{(c + d\sin x)^2}$$

$$= \frac{-(ad - bc)\cos x}{(c + d\sin x)^2}$$

Example 19 : Find $\frac{d}{dx}\frac{x}{\sin^n x}$. ($\sin x \neq 0$), $n \in$ N

Solution : First of all we prove $\frac{d}{dx}\sin^n x = n\sin^{n-1}x\cos x$ by P.M.I.

For $n = 1$, $\frac{d}{dx}\sin x = \cos x = 1 \cdot \sin^0 x\cos x$

$\therefore$ P(1) is true.

Let P(k) be true for some $k \in$ N. So $\frac{d}{dx}\sin^k x = k\sin^{k-1}x\cos x$

For $n = k + 1$, $\frac{d}{dx}\sin^{k+1}x = \frac{d}{dx}\sin^k x \cdot \sin x$

$$= \sin x\frac{d}{dx}\sin^k x + \sin^k x\frac{d}{dx}\sin x$$

$$= \sin x \cdot k\sin^{k-1}x\cos x + \sin^k x\cos x$$

$$= k \cdot \sin^k x\cos x + \sin^k x\cos x$$

$$= (k + 1)\sin^k x\cos x$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Now, $$\frac{d}{dx}\frac{x}{\sin^n x} = \frac{\sin^n x\frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n}x}$$

$$= \frac{\sin^n x - x \cdot n\sin^{n-1}x\cos x}{\sin^{2n}x}$$

$$= \frac{\sin^{n-1}x(\sin x - nx\cos x)}{\sin^{2n}x}$$

$$= \frac{\sin x - nx\cos x}{\sin^{n+1}x}$$

Example 20 : Find the derivative of $\sqrt{\sin x}$ from first principle. **($\sin x > 0$)**

Solution : $$\frac{d}{dx}\sqrt{\sin x} = \lim_{t \to x}\frac{\sqrt{\sin t} - \sqrt{\sin x}}{t - x}$$

$$= \lim_{t \to x}\frac{\sin t - \sin x}{(\sqrt{\sin t} + \sqrt{\sin x})(t - x)}$$

$$= \lim_{t \to x}\frac{2\cos\frac{t+x}{2}\sin\frac{t-x}{2}}{(\sqrt{\sin t} + \sqrt{\sin x})\left(\frac{t-x}{2}\right)2}$$

$$= \frac{\cos x \cdot 1}{2\sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

Example 21 : Find $\frac{d}{dx} x^2 sinx$ by definition and verify using rules.

Solution :

$$\frac{d}{dx} x^2 sinx = \lim_{h \to 0} \frac{(x+h)^2 \sin(x+h) - x^2 sinx}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \sin(x+h) - (x+h)^2 sinx + (x+h)^2 \sin x - x^2 sinx}{h}$$

$$= \lim_{h \to 0} (x+h)^2 \left(\frac{\sin(x+h) - sinx}{h}\right) + \lim_{h \to 0} \frac{[(x+h)^2 - x^2] \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \left(2cos\left(x+\frac{h}{2}\right) sin\frac{h}{2}\right)}{2 \cdot \frac{h}{2}} + \lim_{h \to 0} \frac{(2hx + h^2)\ sinx}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \cos\left(x+\frac{h}{2}\right) \sin\frac{h}{2}}{\frac{h}{2}} + \lim_{h \to 0} (2x + h)\ sinx$$

$$= x^2 cosx + 2x\ sinx$$

Now, $\frac{d}{dx} x^2 sinx = x^2 \frac{d}{dx} sinx + sinx \frac{d}{dx} x^2$

$$= x^2\ cosx + 2x\ sinx$$

Example 22 : Find $\frac{d}{dx} \frac{cosx}{1+sinx}$. **($sinx \neq -1$)**

Solution : $\frac{d}{dx} \frac{cosx}{1+sinx} = \frac{(1+sinx)\frac{d}{dx} cosx - cosx \frac{d}{dx}(1+sinx)}{(1+sinx)^2}$

$$= \frac{(1+sinx)(-sinx) - cosx \cdot cosx}{(1+sinx)^2}$$

$$= \frac{-sinx - sin^2 x - cos^2 x}{(1+sinx)^2}$$

$$= \frac{-(1+sinx)}{(1+sinx)^2} \qquad (sin^2 x + cos^2 x = 1)$$

$$= \frac{-1}{1+sinx}$$

Example 23 : For $f(x) = x^{100} + x^{99} + x^{98} + ... + 1$, find $f'(1)$.

Solution : $f(x) = x^{100} + x^{99} + x^{98} + ... + 1$

$f'(x) = 100x^{99} + 99x^{98} + ... + 0$

$\therefore \quad f'(1) = 100 + 99 + 98 + ... + 1$

$$= \frac{100(101)}{2} = 5050 \qquad \left(\Sigma n = \frac{n(n+1)}{2}\right)$$

Example 24 : Find $\frac{d}{dx}\left(\frac{sinx + cosx}{sinx - cosx}\right)$. $(sinx \neq cosx)$

Solution :

$$\frac{d}{dx}\,\frac{sinx + cosx}{sinx - cosx} = \frac{(sinx - cosx)\frac{d}{dx}(sinx + cosx) - (sinx + cosx)\frac{d}{dx}(sinx - cosx)}{(sinx - cosx)^2}$$

$$= \frac{(sinx - cosx)(cosx - sinx) - (sinx + cosx)(cosx + sinx)}{(sinx - cosx)^2}$$

$$= \frac{-[(sinx - cosx)^2 + (sinx + cosx)^2]}{(sinx - cosx)^2}$$

$$= \frac{-2}{(sinx - cosx)^2} \qquad (sin^2x + cos^2x = 1)$$

Example 25 : For $f(x) = |x|$, find $f'(0)$, if it exists.

Solution : We want $\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{|h|}{h}$

$$\lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = 1,$$

$$\lim_{h \to 0-} \frac{|h|}{h} = \lim_{h \to 0-} \frac{-h}{h} = -1$$

$\therefore$ $\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h}$ does not exist.

$\therefore$ $f(x) = |x|$ is not differntiable at $x = 0$.

Example 26 : $f : R \to Z$, $f(x) = [x]$. Find $f'(1)$, if it exists. Find $f'\left(\frac{1}{2}\right)$, if it exists.

Solution : $f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad 1 \le x < 2 \end{cases}$

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$ (since $h > 0$, $1 + h > 1$ and $[1 + h] = 1$)

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{0-1}{h}$$ does not exist. ($h < 0$, $1 + h < 1$)

$\therefore$ $f'(1)$ does not exist.

For $\frac{1}{2} - h < x < \frac{1}{2} + h$ $\left(h < \frac{1}{2}\right)$, $f(x) = 0$

$\therefore$ $f'(x) = 0$ since f is a constant function in

$\left(\frac{1}{2} - h, \frac{1}{2} + h\right)$.

$\therefore$ $f'(x) = 0$. Look at the graph.

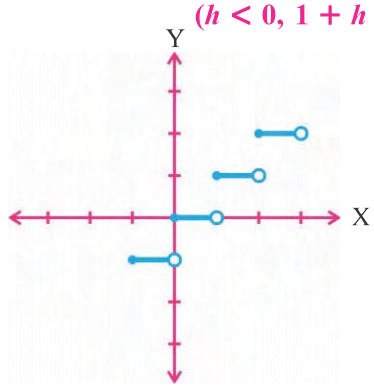

Figure 11.1

Exercise 11

1. Find following derivatives from first principle at given point :

(1) $sinx$ at $x = 0$ (2) $\frac{1}{x}$ at $x = 1$

(3) $2x + 3$ at $x = 2$ (4) $\frac{3x+2}{2x+3}$ at $x = 1$

(5) $3x^2 - 2x + 1$ at $x = -1$ (6) $cosx$ at $x = \frac{\pi}{2}$

(7) $tanx$ at $x = \frac{\pi}{4}$ (8) $secx$ at $x = \frac{\pi}{3}$

(9) $cotx$ at $x = \frac{5\pi}{4}$ (10) $cosecx$ at $x = \frac{\pi}{6}$

2. Find following derivatives from definition : (on proper domain)

(1) $10x$ (2) $secx + tanx$ (3) $cosecx - cotx$

(4) $2sin^2x + 3cosx + 1$ (5) $cos2x$ (6) $sin2x$

(7) $tan2x$ (8) $\frac{1-cosx}{sinx}$ (9) $\frac{cosx}{1-sinx}$

(10) x^3 (11) x^4 (12) x^6

(13) sin^4x (14) cos^4x (15) sec^2x

3. If $f(x) - g(x)$ is a constant function, prove that $f'(x) = g'(x)$.

4. Find $\frac{d}{dx}cos2x$ by definition and also verify by using $cos2x = cos^2x - sin^2x$.

5. Find $\frac{d}{dx}\frac{x^n-1}{x-1}$, $x \neq 1$

6. $\frac{d}{dx}\frac{x^n-1}{x-1} = \frac{d}{dx}(x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1)$

$= (n-1)x^{n-2} + (n-2)x^{n-3} + (n-3)x^{n-4} + ... + 1 + 0$

$\therefore \frac{d}{dx}\frac{x^n-1}{x-1}$ at $x = 1$ is $(n-1) + (n-2) + (n-3) + ... + 1 = \frac{n(n-1)}{2}$.

Comment !

Obtain following derivatives where the function is defined :

7. $\frac{x^2-1}{x^2+1}$ **8.** $\frac{x^n-a^n}{x-a}$ $(x \neq a)$ **9.** $x^{-5}(7 + 3x)$

10. $x^{-6}(4x^2 - 8x^3)$ **11.** $2secx - 3tanx + 5sinx\ cosx$ **12.** $\frac{secx-1}{secx+1}$

13. $\frac{4x+7sinx}{5x+8cosx}$ **14.** $\frac{x}{1+cotx}$ **15.** $(x^2 - 1)sin^2x + (x^2 + 1)cos^2x$

16. $(ax^2 + bx + sinx)(p + qtanx)$ **17.** $sin(x + a)$

18. $\frac{sin(x+a)}{cosx}$ **19.** $tan(x + a)$

20. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For $f(x) = sin^2x$, $f'\left(\frac{\pi}{2}\right)$ = ...

(a) -1 (b) 0 (c) 1 (d) $\frac{1}{2}$

(2) For $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f'(1)$ = ...

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1

(3) If $f(x) = 1 + x + x^2 + x^3 + ... + x^{99} + x^{100}$, then $f'(-1)$ = ...

(a) -50 (b) 50 (c) 5050 (d) -5050

(4) $\frac{d}{dx}\ cos^nx$ = ...

(a) $ncos^{n-1}x$ (b) $nsin^{n-1}x$

(c) $ncos^{n-1}x\ sinx$ (d) $-ncos^{n-1}x\ sinx$

(5) $\frac{d}{dx}(sin^2x + cos^2x)$ = ...

(a) $sin2x + cos2x$ (b) $sin2x - cos2x$ (c) 0 (d) $sinx + cosx$

(6) If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$, then $\frac{dy}{dx}$ = ...

(a) y (b) $y - x$ (c) $y - \frac{x^n}{n!}$ (d) $y - \frac{x^n}{(n-1)!}$

(7) If $y = \sqrt{\frac{1-cos2x}{1+cos2x}}$, $x \in \left(\frac{\pi}{2}, \pi\right)$ then $\frac{dy}{dx}$ = ...

(a) sec^2x (b) $-sec^2x$ (c) cos^2x (d) $|tanx|$

(8) If f is differentiable at a, $\lim\limits_{x \to a} \frac{xf(a) - af(x)}{x - a}$ = ...

(a) $af'(a)$ (b) $f(a) - af'(a)$ (c) $f'(a)$ (d) $\frac{f'(a)}{a}$

(9) If $f(x) = x^{n-1} + x^{n-2} + ... + 1$, $-1 < x < 1$, then $f'(x)$ = ...

(a) $\frac{1}{(x-1)^2}$ (b) $\frac{1}{x-1}$

(c) $\frac{1}{x^n - 1}$ (d) $\frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$

(10) If $f(4) = 16$, $f'(4) = 2$ and f is differentiable at 4, $\lim\limits_{x \to 4} \frac{\sqrt{f(x)} - 4}{x - 4}$ = ...

(a) 2 (b) 1 (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

(11) If $f(x) = \frac{x^2}{|x|}$, $x \in [3, 5]$, then $f'(x) = ...$

(a) 1 (b) -1 (c) does not exist (d) 0

(12) If $\pi < x < \frac{3\pi}{2}$, $\frac{d}{dx}\sqrt{\frac{1+\cos 2x}{2}} = ...$

(a) $-\sin x$ (b) $\sin x$ (c) $\cos x$ (d) $\sin 2x$

(13) If $\pi < x < 2\pi$, $\frac{d}{dx}\sqrt{\frac{1-\cos 2x}{2}} = ...$

(a) $\sin x$ (b) $\cos x$ (c) $-\cos x$ (d) $-\sin 2x$

(14) $\frac{d}{dx}\sqrt{x^2 - 2x + 1}$ $(x \in [-2, -1]) = ...$

(a) does not exist (b) 0 (c) 1 (d) -1

(15) $\frac{d}{dx}(x + |x|)|x|$ $(x < 0) = ...$

(a) 1 (b) 0 (c) 2 (d) 4

(16) $\frac{d}{dx}(x + |x|)|x|$ $(x > 0) = ...$

(a) $-4x$ (b) $4x$ (c) $2x^2$ (d) x^2

(17) $\frac{d}{dx}|x|^2 = ...$ (at $x = 0$)

(a) 0 (b) does not exist (c) 2 (d) 1

(18) $\frac{d}{dx}x|x|$ $(x > 0) = ...$

(a) x^2 (b) $-2x$ (c) $2x$ (d) 0

(19) $\frac{d}{dx}(\cos^2 x - \sin^2 x) = ...$

(a) $\sin 2x$ (b) $\cos 2x$ (c) $-\cos 2x$ (d) $-2\sin 2x$

(20) $\frac{d}{dx}(3\sin x - 4\sin^3 x) = ...$

(a) $3\cos 3x$ (b) $\cos 3x$ (c) $3\sin 3x$ (d) $-3\cos 3x$

(21) $\frac{d}{dx}\sin 18^\circ = ...$

(a) $\cos 18^\circ$ (b) $-\sin 18^\circ$ (c) $-\cos 18^\circ$ (d) 0

(22) $\frac{d}{dx}\sin x^\circ = ...$

(a) $\cos x^\circ$ (b) $-\sin x^\circ$ (c) $\frac{\pi}{180}\cos x^\circ$ (d) 0

(23) $\frac{d}{dx}(2x + 3)^n = ...$

(a) $n(2x + 3)^{n-1}$ (b) $2n(2x + 3)^{n-1}$ (c) $3n(2x + 3)^{n-1}$ (d) $2^n n(2x + 3)^{n-1}$

(24) $\frac{d}{dx}\sqrt{sinx}$, $(0 < x < \frac{\pi}{2})$ = ... ☐

(a) $\sqrt{cosx}$ (b) $\sqrt{sinx}$ (c) $\frac{cosx}{2\sqrt{sinx}}$ (d) $\frac{sinx}{2\sqrt{cosx}}$

(25) $\frac{d}{dx}\ tan^2x$ = ... ☐

(a) $2tanx$ (b) sec^2x (c) cot^2x (d) $2tanx\ sec^2x$

*

Summary

We studied following points in this chapter :

1. Formal definition of derivative and examples based on it.

2. Algebra of derivatives and examples based on rules.

If $f(x)$ and $g(x)$ are differentiable in (a, b),

(1) $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

(2) $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

(3) $\frac{d}{dx}(f(x)\ g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$

(4) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

(5) $\frac{d}{dx}\ kf(x) = k\frac{d}{dx}f(x)$, $k \in R$

3. Some standard forms :

(1) $\frac{d}{dx}c = 0$

(2) $\frac{d}{dx}x^n = nx^{n-1}$, $n \in N$, $x \in R$

(3) $\frac{d}{dx}sinx = cosx$

(4) $\frac{d}{dx}cosx = -sinx$

(5) $\frac{d}{dx}tanx = sec^2x$

(6) $\frac{d}{dx}cotx = -cosec^2x$

(7) $\frac{d}{dx}secx = secx\ tanx$

(8) $\frac{d}{dx}cosecx = -cosecx\ cotx$

4. Derivative of a polynomial and a rational function.

Bhaskara I

Bhaskara wrote three astronomical contributions. In 629 he created the Aryabhatiya, written in verses, about mathematical astronomy. The comments referred exactly to the 33 verses dealing with mathematics. There he considered variable equations and trigonometric formulae.

His work *Mahabhaskariya* is divided into eight chapters about mathematical astronomy. In chapter 7, he gives a remarkable approximation formula for *sin x*, that is

$$sinx \sim \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}\ \left(0 \le x \le \frac{\pi}{2}\right)$$

ANSWERS

(Answers to questions involving some calculations only are given.)

Exercise 1

31. (1) d (2) d (3) b (4) a (5) d (6) b (7) d (8) d (9) d (10) d

Exercise 2.1

1. (1) $-2i$ (2) $-1 + 8i$ (3) $2 + i$ (4) $\frac{5}{13} + \frac{14}{13}i$ (5) $-\frac{2}{5}$ (6) $-\frac{1}{2}$ (7) -4

(8) $2i$ (9) $\frac{77}{25} + \frac{36}{25}i$ (10) $-\frac{7}{\sqrt{2}}i$

2. (1) $x = 4, y = 1$ (2) $x = -\frac{16}{23}, y = \frac{29}{23}$ (3) $x = 4, y = -2$

(4) $\left\{(2, 3)\left(-2, \frac{1}{3}\right)\right\}$ (5) $x = \frac{14}{15}, y = \frac{1}{5}$

3. (1) $\frac{3}{13} + \frac{2}{13}i$ (2) $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$ (3) $\frac{11}{25} - \frac{27}{25}i$ (4) $-\frac{5}{169} + \frac{12}{169}i$ (5) i

Exercise 2.2

1. (1) $\sqrt{2}, \frac{3\pi}{4}$ (2) $\frac{1}{2}, \frac{\pi}{2}$ (3) $2, -\frac{\pi}{6}$ (4) $2, \frac{5\pi}{6}$ (5) $6, \frac{3\pi}{4}$

6. z_1 may not be equal to z_2 **8.** 40 **12.** $-2\sqrt{3} + 2i$ **13.** $z_1 = 2 + i, z_2 = 2 - i$ **15.** $\frac{2}{5}$

Exercise 2.3

1. (1) $\pm\sqrt{2}\,i$ (2) $\frac{-1 \pm \sqrt{3}\,i}{2}$ (3) $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$ (4) $\frac{-1 \pm \sqrt{2\sqrt{2} - 1}\,i}{2}$ (5) $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$ (6) $\frac{2 \pm 4i}{3}$

2. (1) $\pm\sqrt{2}(\sqrt{3} + i)$ (2) $\pm(3 - 2i)$ (3) $\pm(1 + 7i)$ (4) $\pm(2\sqrt{2} - \sqrt{5}\,i)$

(5) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{2} - 1} - i\sqrt{\sqrt{2} + 1}\right)$ (6) $\pm\sqrt{2}(1 + i)$ (7) $\pm(2\sqrt{2} - 2\sqrt{2}\ i)$ (8) $\pm 5\,i$ (9) $\pm\sqrt{10}\,i$

Exercise 2

1. (1) $2 - 2i$ (2) $\frac{307 + 599i}{442}$ **2.** 2 **4.** $-\frac{3}{20}, \frac{1}{20}$ **7.** 1 **8.** $b = \frac{-\beta}{(\alpha - 1)^2 + \beta^2}$

10. (1) $1 \pm \frac{\sqrt{2}}{2}i$ (2) $\frac{5 \pm \sqrt{2}i}{27}$ (3) $\frac{2}{3} \pm \frac{\sqrt{14}}{21}i$

11. Maximum value is 5, Minimum value is 1 **12.** -48 **13.** 4 **15.** $\frac{3}{2} - 2i$

21. (1) c (2) b (3) a (4) d (5) c (6) c (7) a (8) c (9) b (10) b
(11) d (12) b (13) a (14) c (15) b

Exercise 3.1

1. (1) $x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}$ (2) $1 - 8x + 24x^2 - 32x^3 + 16x^4$

(3) $729x^6 - 2916x^5 + 4860x^4 - 4320x^3 + 2160x^2 - 576x + 64$

(4) $x^5 - \frac{5}{2}x^3 + \frac{5}{2}x - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$

2. (1) $x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$
(2) $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$

3. (1) 0.92236816 (2) 96059601 (3) 1061520150601 **4.** $(1.01)^{10000}$ is larger.

Exercise 3.2

1. (1) 672 (2) 1365 **2.** (1) $\frac{5}{81}$ (2) $\frac{7}{18}$ **3.** $n = 55$

4. (1) $\frac{280}{81}x^{12}$, $\frac{-560}{27}x^9$ (2) $\frac{2835}{8}x^4y^4$ (3) $\binom{20}{10}x^{10}$ (4) $720\,x^2y^3$, $1080\,x^3y^2$

5. $n = 6$ **6.** $n = 14$ or 7

Exercise 3

1. 2 : 1 **2.** $r = 3$ or 15 **3.** $n = 6, x = 2, y = 5$ **4.** $a = 2, b = 3, n = 5$ **6.** $n = 11$ **7.** 135 **8.** $n = 10$

12. (1) c (2) b (3) a (4) c (5) a (6) c (7) b (8) d (9) a (10) b

Exercise 4.1

1. (1) $-\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{3}}$ (3) $-\frac{1}{2}$ (4) $\frac{2}{\sqrt{3}}$ (5) $-\sqrt{2}$ (6) $-\frac{1}{\sqrt{3}}$ **17.** (1) 3 (2) 0 (3) $\frac{1}{2}$ (4) 1

18. (1) Negative (2) Positive (3) Negative (4) Negative **19.** $\frac{3}{7}$

Exercise 4.2

1. (1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{-1}{2\sqrt{2}}$ (3) $\frac{\sqrt{6}-\sqrt{2}}{4}$ **4.** (1) Fourth quadrant (2) Fourth quadrant

5. $\frac{2}{11}$, First quadrant **6.** (1) $[-25, 25]$ (2) $[0, 2]$ **8.** $r = 2$, $\alpha = \frac{\pi}{3}$

9. $r = 2$, $\theta = -\frac{\pi}{3}$ **20.** -1, $\frac{1}{7}$

Exercise 4.3

1. (1) $\sin 10\theta + \sin 4\theta$ (2) $\sin 3\theta - \sin 2\theta$ (3) $\sin 8\theta - \sin 2\theta$ (4) $\sin 6\theta + \sin\theta$
(5) $\cos 14\theta + \cos 8\theta$ (6) $\cos 4\theta + \cos\theta$ (7) $\frac{1}{2}(\cos 2\theta - \cos 20\theta)$ (8) $\cos\theta - \cos 8\theta$ (9) $\sin 2\theta$

2. (1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) $\frac{2+\sqrt{3}}{2}$ (4) $\frac{\sqrt{3}-2}{2}$ (5) $\sqrt{2}$ (6) 1 **5.** 1

Exercise 4.4

1. (1) $2\sin 5\theta\,\cos 2\theta$ (2) $2\sin\theta\,\cos\frac{\theta}{2}$ (3) $-2\cos 4\theta\,\sin\theta$ (4) $2\cos\frac{5\theta}{2}\,\sin\theta$
(5) $2\cos 10\theta\,\cos\theta$ (6) $2\cos 4\theta\,\cos\frac{3\theta}{2}$ (7) $2\sin 8\theta\,\sin 3\theta$ (8) $2\sin\theta\,\sin\frac{\theta}{2}$
(9) $-2\sin^2\frac{\theta}{2}$ (10) $2\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ (11) $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$ (12) $\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$

Exercise 4

9. $\sqrt{19}$, $-\sqrt{19}$

14. (1) c (2) a (3) d (4) d (5) c (6) b (7) c (8) a (9) a (10) a
(11) b (12) c (13) b (14) c (15) d (16) c (17) d (18) d (19) d

Exercise 5.1

20. $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}, \frac{336}{625}$

Exercise 5.2

1. $\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, -3$ **2.** $\frac{1}{65}, \frac{64}{65}$

Exercise 5

23. (1) a (2) b (3) c (4) b (5) c (6) d (7) a (8) a (9) d (10) a
(11) b (12) b (13) a (14) c (15) d (16) a (17) b (18) c (19) a (20) d

Exercise 6.1

1. $\left\{k\pi \pm \frac{3\pi}{8} \mid k \in Z\right\}$ **2.** $\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\} \cup \left\{2k\pi \pm \frac{5\pi}{6} \mid k \in Z\right\}$

3. $\{2k\pi \mid k \in Z\} \cup \left\{2k\pi \pm \frac{\pi}{3} \mid k \in Z\right\}$ **4.** $\left\{2k\pi \pm \frac{\pi}{3} \mid k \in Z\right\}$

5. $\left\{\frac{k\pi}{3} + (-1)^k \frac{\pi}{12} \mid k \in Z\right\}$ **6.** $\{k\pi \mid k \in Z\} \cup \left\{k\pi + (-1)^k \frac{\pi}{6} \mid k \in Z\right\}$

7. $\left\{k\pi + (-1)^k \frac{\pi}{6} \mid k \in Z\right\} \cup \left\{k\pi + (-1)^k \frac{\pi}{2} \mid k \in Z\right\}$

8. $\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\} \cup \left\{k\pi - (-1)^k \frac{\pi}{6} \mid k \in Z\right\}$

9. $\left\{\frac{k\pi}{3} \mid k \in Z\right\} \cup \left\{\frac{k\pi}{2} \pm \frac{\pi}{12} \mid k \in Z\right\}$

10. $\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\} \cup \{2k\pi \mid k \in Z\}$

11. $\left\{\frac{k\pi}{2} + \frac{\pi}{6} \mid k \in Z\right\}$ **12.** $\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{6} \mid k \in Z\right\}$

13. $\left\{k\pi + \frac{\pi}{4} \mid k \in Z\right\} \cup \left\{k\pi + \frac{\pi}{3} \mid k \in Z\right\}$

14. $\{2k\pi \mid k \in Z\} \cup \left\{2k\pi + \frac{\pi}{2} \mid k \in Z\right\}$

15. $\left\{2k\pi + \frac{5\pi}{12} \mid k \in Z\right\} \cup \left\{2k\pi - \frac{13\pi}{12} \mid k \in Z\right\}$

16. $\emptyset$ **17.** $\left\{\frac{k\pi}{5} \pm \frac{\pi}{30} \mid k \in Z\right\}$ **18.** $\left\{\frac{k\pi}{2} \pm \frac{\pi}{8} \mid k \in Z\right\}$

19. $\left\{(8k \pm 3)\frac{\pi}{16} \mid k \in Z\right\}$ **20.** $\left\{(2k+1)\frac{\pi}{4} \mid k \in Z\right\}$

Exercise 6.2

16. $\frac{2\pi}{3}$ **17.** $1 : \sqrt{3} : 2$ **18.** $\frac{5\pi}{12}$ **20.** $\frac{\pi}{3}$

Exercise 6

1. $\left\{2k\pi \pm \frac{\pi}{4} \mid k \in Z\right\} \cup \left\{2k\pi \pm \frac{3\pi}{4} \mid k \in Z\right\}$ **2.** $\left\{(4k+1)\frac{\pi}{2} \mid k \in Z\right\}$

3. $\{k\pi \mid k \in Z\} \cup \left\{(3k \pm 1)\frac{\pi}{9} \mid k \in Z\right\}$ **4.** $\left\{2k\pi \pm \frac{\pi}{3} \mid k \in Z\right\}$

5. $\left\{(6k+1)\frac{\pi}{30} \mid k \in Z\right\}$ **6.** $\left\{2k\pi + \frac{\pi}{2} \mid k \in Z\right\}$ **7.** $\left\{(4k \pm 1)\frac{\pi}{8} \mid k \in Z\right\}$

8. $\left\{(4k+1)\frac{\pi}{12} \,\middle|\, k \in Z\right\}$ **9.** $\left\{(4k+1)\frac{\pi}{8} \,\middle|\, k \in Z\right\}$ **10.** $\left\{(12k \pm 5)\frac{\pi}{6} \,\middle|\, k \in Z\right\}$

21. (1) a (2) c (3) d (4) c (5) c (6) b (7) c (8) a (9) d (10) b
(11) a (12) a (13) b (14) d (15) b

Exercise 7.1

1. (1) 4, 7, 10, 13, 16 (2) 1, $\frac{1}{2}$, 2, $\frac{3}{2}$, 3 (3) 2, 3, 5, 7, 11

2. 2, 3, 5, 8 **3.** (1) −5, −9, −17 (2) $\frac{5}{2}$, $\frac{13}{2}$, $\frac{41}{2}$ **4.** (1) 0, 3, 5, 19 (2) 1, 2, 3, 10

5. (1) $a_n = ar^{n-1}$, $n \in$ N (2) $a_1 = 0$, $a_n = 16(-3)^{n-2}$, $n \ge 2$

Exercise 7.2

1. (1) 43 (2) −49 (3) $\frac{33}{2}$ **2.** 510 **3.** 23,700 **4.** $d = -4$, $t_8 = -24$

5. 27 **6.** $-(m+n)$ **7.** 0 **8.** 1 : 2 **9.** 5 : 11 **10.** 6000 **11.** 1 **12.** −1, 3, 7

13. 2, 6, 10, 14 **14.** ₹ 7800 **15.** $n = 10$, ₹ 1287.50 **16.** 660 *cm*

Exercise 7.3

1. (1) 256 (2) $\frac{7}{1024}$ (3) $-16\sqrt{2}$ **2.** (1) 768 (2) 13 (3) 5 (4) $\frac{405}{4}$

3. 93 **4.** $\frac{3}{2}$, 3, 6, 12, 24, ... **5.** (1) $\frac{7}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$ (2) $3\left[n + \frac{10}{9}(10^{n-1} - 1)\right]$

6. $\frac{a^2(a^{2n}-1)}{a^2-1} + \frac{ab(a^n b^n - 1)}{ab-1}$ **7.** $\frac{2}{9}$, $\frac{2}{3}$, 2, 6, 18 **8.** $\sqrt{mn}$ **9.** $2\sqrt{2}$

12. $\frac{1}{4}$, 1, 4, 16 **13.** ₹ 39,366

Exercise 7.4

1. $\frac{19}{6}$, $\frac{10}{3}$, $\frac{7}{2}$, $\frac{11}{3}$, $\frac{23}{6}$ **2.** 5, 13, 21 **3.** $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4 **4.** $\sqrt{2}$, 1, $\frac{1}{\sqrt{2}}$

5. 45, 5 **6.** $x^2 - 20x + 64 = 0$

Exercise 7.5

1. (1) 800 (2) 465 (3) 1070 (4) −2704

2. (1) $\frac{n}{3}(16n^2 + 12n - 1)$ (2) $\frac{n}{4}(27n^3 - 18n^2 - 9n + 4)$ (3) $\frac{n}{2}(4n^2 + n - 1)$

(4) $\frac{10n}{3}(n^2 + 6n + 11)$ (5) $12n(n+1)(9n^2 + 9n + 8)$ (6) $\frac{n}{36}(4n^2 + 15n + 17)$

(7) $2n^2 + n$ (8) $\frac{n(n+1)}{12}(3n^2 + 11n + 10)$ (9) $\frac{n^2(n^2-1)}{4}$

3. (1) −6479 (2) −465

Exercise 7

1. −140, 42 **2.** −2, 4, 10, 16, ... **3.** 9 hr **4.** 16 rows, 5 blocks **6.** 1 : 2 : 3

7. $\frac{20n}{3} - \frac{20}{27} + \frac{20}{27} \times 10^{-n}$ **9.** 740 **10.** $\frac{1}{2} - \frac{1}{\sqrt{2}}$ **11.** $\frac{n}{2}(1 - 5n)$

12. 11, 14, 17, 20, ... **13.** $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$ **14.** $\frac{25025}{2}$

15. 3, 5, 7, 9, 11, 13 **16.** 48, 12, 3, $\frac{3}{4}$, $\frac{3}{16}$

17. (1) c (2) d (3) a (4) a (5) c (6) c (7) b (8) a (9) b (10) d

(11) a (12) d (13) c (14) a (15) c (16) b (17) a (18) c

Exercise 8.1

1. (1) $x^2 + y^2 + 4x - 6y - 12 = 0$ (2) $x^2 + y^2 + 2x - 2y = 0$

(3) $x^2 + y^2 + 8x\cos\alpha - 8y\sin\alpha - 9 = 0$ (4) $x^2 + y^2 + 2\sqrt{2}x + 2\sqrt{5}y + 2 = 0$

(5) $x^2 + y^2 - 2x = 0$

2. $x^2 + y^2 - 6x + 4y - 12 = 0$ **3.** $x^2 + y^2 + 4x + 10y + 25 = 0$

4. $x^2 + y^2 + 6x + 6y + 9 = 0$ **5.** $x^2 + y^2 - 2\sqrt{5}x = 0$

Exercise 8.2

1. (1) Not a circle. (2) Circle, Centre (0, 0), radius 1

(3) Circle, Centre (1, 1), radius 1 (4) Not a circle.

(5) Not a circle. (6) Not a circle.

(7) Circle, Centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$, radius $= \frac{1}{\sqrt{2}}$ (8) Not a circle.

(9) Circle, Centre $= (\tan\alpha, -\sec\alpha)$, radius $= 1$

(10) Case-1 : $\alpha = 0$ Centre $(0, -1)$, radius $= 1$

Case-2 : $\alpha \neq 0$ Not a circle.

2. $x^2 + y^2 - 6x - 8y = 0$ **3.** $x^2 + y^2 - 10y - 15 = 0$

4. $x^2 + y^2 + 6x - 6y + 9 = 0$ and $x^2 + y^2 + 30x - 30y + 225 = 0$

Exercise 8.3

1. (1) Focus $\left(\frac{1}{8}, 0\right)$, directrix $8x + 1 = 0$ (2) Focus $(0, -1)$, directrix $y = 1$

(3) Focus $\left(0, -\frac{1}{16}\right)$, directrix $16y - 1 = 0$ (4) Focus $(3, 0)$, directrix $x + 3 = 0$

2. (1) $x^2 = -8y$ (2) $y^2 = 16x$

3. (1) $x^2 + y^2 + 2xy + 2x - 6y + 9 = 0$ (2) $16x^2 + 9y^2 + 24xy + 180x + 160y + 600 = 0$

4. 4, $y + 3 = 0$ **5.** 18 **6.** $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ **7.** $(3, \pm 6)$

Exercise 8.4

1. (1) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (2) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (3) $\frac{x^2}{100} + \frac{y^2}{36} = 1$ (4) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(5) $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$ (6) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (7) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

2. $\frac{x^2}{18} + \frac{y^2}{9} = 1$

3.

No.	e	Foci	Directrices	Length of a Latus-rectum
(1)	$\frac{\sqrt{5}}{3}$	$(0, \pm\sqrt{5})$	$y = \pm\frac{9}{\sqrt{5}}$	$\frac{8}{3}$
(2)	$\frac{2}{3}$	$(\pm 4, 0)$	$x = \pm 9$	$\frac{20}{3}$
(3)	$\frac{1}{\sqrt{2}}$	$(\pm 5\sqrt{2}, 0)$	$x = \pm 10\sqrt{2}$	10
(4)	$\frac{3}{4}$	$\left(0, \pm\frac{3\sqrt{43}}{\sqrt{7}}\right)$	$y = \pm\frac{16\sqrt{43}}{3\sqrt{7}}$	$\frac{\sqrt{301}}{2}$
(5)	$\frac{2}{3}$	$\left(\pm\frac{6}{\sqrt{5}}, 0\right)$	$x = \pm\frac{27}{2\sqrt{5}}$	$2\sqrt{5}$

4. $e = \frac{1}{\sqrt{3}}$ 5. $5 : 3, x = \pm\frac{50}{3}$ 7. $7x^2 + 15y^2 = 247$ 8. $4x^2 + 3y^2 - 24x - 6y + 27 = 0$

9. Foci : $(2, 1 \pm \sqrt{5})$, Directrices : $y = 1 \pm \frac{9}{\sqrt{5}}$

Exercise 8.5

In answer 1, $\theta \in (-\pi, \pi]$

1. (1) $x = 4\cos\theta, y = 3\sin\theta$ (2) $x = 4\cos\theta, y = 2\sqrt{3}\sin\theta$
(3) $x = 2\cos\theta, y = \sqrt{3}\sin\theta$ (4) $x = 4\cos\theta, y = \sqrt{7}\sin\theta$ (5) $x = 3\sqrt{2}\cos\theta, y = 3\sin\theta$

2. (1) $e = \frac{\sqrt{5}}{3}$, Foci : $(0, \pm\sqrt{5})$ (2) $e = \frac{\sqrt{184}}{25}$, Foci : $\left(\pm\frac{\sqrt{184}}{15}, 0\right)$ (3) $e = \frac{\sqrt{7}}{4}$, Foci : $(\pm\sqrt{7}, 0)$

3. $\frac{x^2}{16} + \frac{y^2}{15} = 1$

Exercise 8.6

1.

No.	Foci	Directrices	Length of a latus-rectum	Length of transverse axis	Length of conjugate axis
(1)	$(\pm 5\sqrt{5}, 0)$	$x = \pm 4\sqrt{5}$	5	20	10
(2)	$(\pm 8\sqrt{2}, 0)$	$x = \pm 4\sqrt{2}$	16	16	16
(3)	$\left(\pm\frac{5}{\sqrt{6}}, 0\right)$	$x = \pm\sqrt{\frac{3}{2}}$	$\frac{2\sqrt{10}}{3}$	$\sqrt{10}$	$2\sqrt{\frac{5}{3}}$
(4)	$(0, \pm 5)$	$y = \pm\frac{16}{5}$	$\frac{9}{2}$	8	6
(5)	$(0, \pm 8)$	$y = \pm\frac{25}{8}$	$\frac{78}{5}$	10	$2\sqrt{39}$

In answer 2 and 4, $\theta \in (-\pi, \pi] - \left\{\frac{-\pi}{2}, \frac{\pi}{2}\right\}$

2. (1) $\frac{y^2}{49} - \frac{9x^2}{343} = 1; x = \frac{\sqrt{343}}{3}\tan\theta, y = 7\sec\theta$ (2) $\frac{x^2}{9} - \frac{y^2}{4} = 1; x = 3\sec\theta, y = 2\tan\theta$
(3) $\frac{x^2}{25} - \frac{y^2}{20} = 1; x = 5\sec\theta, y = \sqrt{20}\tan\theta$ (4) $\frac{y^2}{32} - \frac{x^2}{32} = 1; x = 4\sqrt{2}\tan\theta, y = 4\sqrt{2}\sec\theta$
(5) $\frac{y^2}{16} - \frac{x^2}{9} = 1; x = 3\tan\theta, y = 4\sec\theta$

4. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ 5. $x = 4\tan\theta, y = 3\sec\theta$

Exercise 8

1. $x^2 + y^2 - 3x + y - 4 = 0$ 2. $x^2 + y^2 - 6y - 16 = 0$
3. $x^2 + y^2 - 4x - 6y + 4 = 0$ 4. Focus : $\left(\frac{1}{4}, 0\right)$, Length of lectus-rectum = 1
5. $\frac{x^2}{144} + \frac{y^2}{128} = 1$ 6. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
7. $y^2 = -12(x + 1)$ 8. (a) $y^2 = 10x$ (b) $2\sqrt{110}$ 9. (6, 0) 10. 3.2 m
11. Ellipse, $\frac{x^2}{25} + \frac{y^2}{9} = 1$
12. (1) a (2) d (3) a (4) b (5) d (6) b (7) c (8) c (9) b (10) a
(11) b (12) a (13) b (14) b (15) c (16) a (17) a (18) b (19) c (20) d

Exercise 9.1

1. (1) (x_1, x_2) (2) (x, y, z) (3) (5, −2, 2) (4) (4, −4, −4) (5) (−1, −4, −7) (6) (1, −5, −2)
2. (1) $x = 1, y = -1$ (2) $x = 0, y = 0$ (3) $x = \frac{1}{5}, y = \frac{8}{5}$ (4) $x = 0, y = 0$
3. (1) $\sqrt{3}$ (2) $\sqrt{3}$ (3) 5 (4) $\sqrt{14}$ (5) $\sqrt{38}$
4. (1) $|\bar{x} + \bar{y}| < |\bar{x}| + |\bar{y}|$ (2) $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$ 5. $k = 1$ 6. $\left(\frac{-11}{6}, \frac{47}{15}, 0\right)$

Exercise 9.2

1. (1) OXYZ (2) OXY'Z' (3) OXYZ' (4) OX'Y'Z (5) OX'Y'Z' 2. (0, 0, 0)

Exercise 9.3

1. (1) Same directions (2) Different directions (3) Opposite directions (4) Different directions
2. (1) $\left(\frac{3}{5}, \frac{-4}{5}\right)$ (2) $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ (3) $\left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right)$ (4) $\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$ (5) (1, 0, 0) (6) $\left(\frac{-5}{13}, \frac{12}{13}\right)$
3. $\alpha = \frac{2x_2 - x_1}{3}$, $\beta = \frac{2x_1 - x_2}{3}$

Exercise 9.4

1. (1) 0 (2) $2\sqrt{3}$ (3) 6 (4) 4 (5) 5 (6) 1
2. (1) Non-collinear (2) Collinear (3) Non-collinear (4) Non-collinear
3. Isosceles right triangle 4. (0, 0, 0) or (0, 0, 6) 5. $x^2 + y^2 + z^2 - 2x - 6y - 12z + 52 = k^2$

Exercise 9.5

1. $\left(\frac{4}{3}, \frac{10}{3}, \frac{-5}{3}\right)$ and $\left(\frac{5}{3}, \frac{11}{3}, \frac{-4}{3}\right)$
2. (1) Non-collinear (2) Non-collinear (3) Non-collinear (4) Collinear (5) Non-collinear

Exercise 9

1. Parallelogram, not a rectangle 2. Isosceles right triangle 3. $x = 2z$
4. (1) $\frac{3}{2}, \frac{3}{\sqrt{2}}, \frac{3}{2}$; (1, 1, 1) (2) $\frac{3\sqrt{3}}{2}, \frac{3\sqrt{5}}{2}, \frac{3}{\sqrt{2}}$; (0, 1, 2) (3) $3\sqrt{5}, \sqrt{21}, \sqrt{6}$; $\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$
5. $\left(1, 1, \frac{2}{3}\right)$
6. (1) Non-collinear

(2) Collinear, A divides in the ratio −2 : 1 from B and −1 : 2 from C
B divides in the ratio −2 : 1 from A and −1 : 2 from C
C divides in the ratio 1 : 1 from A and 1 : 1 from B

(3) Non-collinear

(4) Collinear, L divides in the ratio −1 : 3 from M and −3 : 1 from N
M divides in the ratio 1 : 2 from L and 2 : 1 from N
N divides in the ratio −3 : 2 from L and −2 : 3 from M

(5) Collinear, P divides in the ratio 1 : 1 from Q and 1 : 1 from R
Q divides in the ratio −1 : 2 from P and −2 : 1 from R
R divides in the ratio −1 : 2 from P and −2 : 1 from Q

7. (1) b (2) d (3) b (4) c (5) c (6) a (7) c (8) a (9) d (10) c (11) a (12) a (13) a (14) a (15) c (16) c (17) a (18) b (19) c (20) c

Exercise 10

11. $\frac{1}{12}$ **12.** $\frac{m}{n}$ **13.** -2 **14.** 41 **15.** $\frac{1}{4} \cdot x^{\frac{-3}{4}}$ **16.** $\frac{1}{3} \cdot x^{\frac{-2}{3}}$ **17.** $\frac{5}{4}$ **18.** 0

19. $\sqrt{3} - \sqrt{2}$ **20.** $\frac{n(n+1)}{2}$ **21.** 1 **22.** $-3\sqrt{2}$ **23.** $\frac{mn(n-m)}{2}$ **24.** $\frac{1}{12}$

25. 12 **26.** $\frac{1}{4\sqrt{2}}$ **27.** $\frac{1}{2}$ **28.** $-cosa$ **29.** $2cos3$ **30.** -1

31. $2asina + a^2cosa$ **32.** $secx(xtanx + 1)$

33. (1) b (2) d (3) b (4) c (5) b (6) a (7) d (8) a (9) d (10) c (11) d (12) c (13) d (14) a (15) b (16) b (17) a (18) c (19) b (20) d

Exercise 11

1. (1) 1 (2) −1 (3) 2 (4) $\frac{1}{5}$ (5) −8 (6) −1 (7) 2 (8) $2\sqrt{3}$ (9) −2 (10) $-2\sqrt{3}$

2. (1) 10 (2) $secx\ tanx + sec^2x$ (3) $cosec^2x - cosecx\ cotx$

(4) $4sinx\ cosx - 3sinx$ (5) $-2sin2x$ (6) $2cos2x$ (7) $2sec^22x$

(8) $\frac{1}{1+cosx}$ (9) $\frac{1}{1-sinx}$ (10) $3x^2$ (11) $4x^3$ (12) $6x^5$ (13) $4sin^3x\ cosx$

(14) $-4cos^3x\ sinx$ (15) $2sec^2x\ tanx$

4. $-2sin2x$ **5.** $\frac{(n-1)x^n - n \cdot x^{n-1} + 1}{(x-1)^2}$ **7.** $\frac{4x}{(x^2+1)^2}$ **8.** $\frac{(n-1)x^n - a \cdot nx^{n-1} + a^n}{(x-a)^2}$

9. $-35x^{-6} - 12x^{-5}$ **10.** $-16x^{-5} + 24x^{-4}$ **11.** $2secx\ tanx - 3sec^2x + 5cos2x$

12. $\frac{2secx\ tanx}{(secx+1)^2}$ **13.** $\frac{56 + 35(xcosx - sinx) + 32(cosx + sinx)}{(5x - 8cosx)^2}$ **14.** $\frac{1 + cotx + xcosec^2x}{(1+cotx)^2}$

15. $2(x - sin2x)$ **16.** $(p + qtanx)(2ax + b + cosx) + (ax^2 + bx + sinx)qsec^2x$

17. $cos(x + a)$ **18.** $cosa \cdot sec^2x$ **19.** $sec^2(x + a)$

20. (1) b (2) c (3) a (4) d (5) c (6) c (7) b (8) b (9) d (10) c (11) a (12) b (13) c (14) d (15) b (16) b (17) a (18) c (19) d (20) a (21) d (22) c (23) b (24) c (25) d

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TERMINOLOGY

(In Gujarati)

Addition Formulae	સરવાળાનાં સૂત્રો
Allied Numbers	સંબંધિત સંખ્યાઓ
Argand diagram	આર્ગન્ડ આકૃતિ
Argument	કોણાંક
Arithmetic Progression (A.P.)	સમાંતર શ્રેણી
Binomial Theorem	દ્વિપદી પ્રમેય
Bound Vector	નિયત સદિશ
Branch	શાખા
Calculus	કલનશાસ્ત્ર
Central Conic	કેન્દ્રીય શાંકવ
Centroid	મધ્યકેન્દ્ર
Chord	જીવા
Circumcentre	પરિકેન્દ્ર
Common Difference	સામાન્ય તફાવત
Complex Numbers	સંકર સંખ્યાઓ
Conic / Conic Section	શાંકવ
Conjugate Axis	અનુબદ્ધ અક્ષ
Conjugate Hyperbola	અનુબદ્ધ અતિવલય
Conjugate of a Complex Number	અનુબદ્ધ સંકર સંખ્યા
Coordinate	યામ
Coordinate Axis	યામાક્ષ
Derivative	વિકલિત
Differentiation	વિકલન
Direction	દિશા
Directrices	નિયામિકાઓ
Directrix	નિયામિકા
Divisible	વિભાજ્ય
Eccentricity	ઉત્કેન્દ્રતા
Ellipse	ઉપવલય
Factor Formulae	અવયવ સૂત્રો
Focal Chord	નાભિજીવા
Foci	નાભિઓ
Focus	નાભિ
Free Vector	મુક્ત સદિશ
Geometric Progression (G.P.)	સમગુણોત્તર શ્રેણી
Graph	આલેખ
Hyperbola	અતિવલય
Imaginary Part	કાલ્પનિક ભાગ
Incentre	અંતઃકેન્દ્ર
Instantaneous Velocity	તાત્ક્ષણિક વેગ
Latera-recta	નાભિલંબો
Latus-rectum	નાભિલંબ
Law of Trichotomy	ત્રિવિધ વિકલ્પનો નિયમ
Limit	લક્ષ
Magnitude	માન
Major Axis	પ્રધાન અક્ષ
Mathematical Induction	ગણિતીય અનુમાન
Mean	મધ્યક
Minor Axis	ગૌણ અક્ષ
Modulus of a Complex Number	સંકર સંખ્યાનો માનાંક
Multiple	ગુણિત
Parabola	પરવલય
Parameter	પ્રચલ
Polar Form	ધ્રુવીય સ્વરૂપ
Position Vector	સ્થાન સદિશ
Projection Formula	પ્રક્ષેપ સૂત્ર
Purely Imaginary Number	શુદ્ધ કાલ્પનિક સંખ્યા
Real Part	વાસ્તવિક ભાગ
Rectangular Hyperbola	લંબાતિવલય
Recurrence Relation	આવૃત્ત સંબંધ
Rule of Substitution	આદેશનો નિયમ
Scalar	અદિશ
Secant	છેદિકા
Sequence	શ્રેણી
Series	શ્રેઢી
Slope	ઢાળ
Space	અવકાશ
Square Root	વર્ગમૂળ
Submultiple	ઉપગુણિત
Symmetric	સંમિત
Tangent	સ્પર્શક
Transverse Axis	મુખ્ય અક્ષ
Vector	સદિશ
Vertex	શિરોબિંદુ