# INVERSE TRIGONOMETRIC FUNCTIONS

## IMPORTANT POINTS TO REMEMBER

- $\sin^{-1} x$, $\cos^{-1} x$, ...etc., are angles.
- If sinθ = x and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\theta = \sin^{-1} x$ etc.

| Function | Domain | Range(Principal Value Branch) |
|---|---|---|
| $\sin^{-1} x$ | [-1, 1] | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ |
| $\cos^{-1} x$ | [-1, 1] | $[0, \pi]$ |
| $\tan^{-1} x$ | R | $(-\frac{\pi}{2}, \frac{\pi}{2})$ |
| $\csc^{-1} x$ | R | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ - {0} |
| $\sec^{-1} x$ | R - (-1, 1) | $[0, \pi]$- {π/2} |
| $\cot^{-1} x$ | R - (-1, 1) | (0, π) |

- $\sin^{-1}(\sin x) = x$ , x $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , $\cos^{-1}(\cos x) = x$ , x ε $[0, \pi]$ etc.
- $\sin(\sin^{-1}x) = x$, x [-1, 1], $\cos(\cos^{-1}x) = x$, x [-1, 1] etc.
- $\sin^{-1}x = \text{cosec}^{-1}(1/x)$, x ε [-1, 1], $\tan^{-1} x = \cot^{-1}(1/x)$, x > 0, $\sec^{-1}x = \text{cosec}^{-1}(1/x)$, $|x| \geq 1$.
- $\sin^{-1}(-x) = -\sin^{-1}x$, x ε [-1, 1], $\tan^{-1}(-x) = -\tan^{-1} x$, x ε R, $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \geq 1$.
- $\cos^{-1}(-x) = \pi - \cos^{-1}x$, x ε [-1, 1], $\cot^{-1}(-x) = = \pi - \cot^{-1} x$, x ε R, $\sec^{-1}(-x) = = \pi - \sec^{-1}x$, $|x| \geq 1$
- $\sin^{-1}x + \cos^{-1}x = \pi/2$, x ε [-1, 1], $\tan^{-1}x + \cot^{-1}x = \pi/2$, x ε R, $\sec^{-1}x + \text{cosec}^{-1}x = \pi/2$, $|x| \geq 1$
- $\tan^{-1} x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$; xy < 1.
- $\tan^{-1} x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$; xy > -1.
- $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, -1< x < 1
- $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$, $x \geq 0$
- $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$, $|x| \leq 1$

**Some Important Substitutions**

| Type | Put | Formula |
|---|---|---|
| $2x\sqrt{1-x^2}$ | x = sinθ | sin2θ |
| $3x - 4x^3$ | x = sinθ | sin3θ |
| $4x^3 - 3x$ | x = cosθ | cos3θ |
| $\frac{3x - x^3}{1-3x^2}$ | x = tanθ | tan3θ |
| $\frac{2x}{1+x^2}$ | x = tanθ | sin2θ |
| $\frac{1-x^2}{1+x^2}$ | x = tanθ | cos2θ |
| $\frac{2x}{1-x^2}$ | x = tanθ | tan2θ |
| $\sqrt{1-x^2}$ | x = sinθ | cosθ |
| $\sqrt{a^2-x^2}$ | x = asinθ | a cosθ |
| $\sqrt{1+x^2}$ | x = tanθ | secθ |
| $\sqrt{a^2+x^2}$ | x = atanθ | a secθ |
| $\sqrt{x^2-1}$ | x = secθ | tanθ |
| $\sqrt{x^2-a^2}$ | x = asecθ | a tanθ |
| $\frac{\sqrt{1+x}}{2}$ | x = cos2θ | cosθ, sinθ |
| $\sqrt{\frac{1+x}{1-x}}$ | x = cos2θ | cotθ, tanθ |
| $1 - 2x^2$ | x = sinθ | cos2θ |
| $2x^2 - 1$ | x = cosθ | cos2θ |
| $\frac{1 \pm x^2}{1 \pm x^2}$ | $x^2$ = tanθ | tan(π/4+θ) |
| $\frac{\sqrt{1+x^2} \pm \sqrt{1-x^2}}{\sqrt{1+x^2} \pm \sqrt{1-x^2}}$ | $x^2$ =cos2θ | tan(π/4 +θ) |

**ASSIGNMENT**

1. Evaluate the following: Answers

a. $\sin(\cot^{-1}x)$ — $(+1/(\sqrt{1+x^2})$
b. $\cos(\tan^{-1}x)$ — $(1/(\sqrt{x^2+1})$
c. $\sin(\cos^{-1}4/5)$ — $(3/5)$
d. $\sin(\text{cosec}^{-1}17/8)$ — $(8/17)$

2. Using Principal value, evaluate the following:

a. $\sin^{-1}(-\sqrt{3}/2)$ — $(-\pi/3)$
b. $\cos^{-1}(-1/2)$ — $(2\pi/3)$
c. $\tan^{-1}(-1/\sqrt{3})$ — $(-\pi/6)$
d. $\sec^{-1}(-2)$ — $(2\pi/3)$
e. $\text{cosec}^{-1}(-2/\sqrt{3})$ — $(-\pi/3)$
f. $\sec^{-1}\sec(2\pi/3)$ — $(2\pi/3)$
g. $\cos^{-1}(\cos 2\pi/3) + \sin^{-1}\sin(2\pi/3)$ — $(\pi)$
h. $\sin(\pi/3-\sin^{-1}(-1/2)$ — $(1)$
i. $\sin^{-1}1/2 \pm 2\sin^{-1}(1/\sqrt{2})$ — $(2\pi/3)$ , $(-\frac{\pi}{3})$
j. $\sin^{-1}(-1/2) + 2\cos^{-1}(-\sqrt{3}/2)$ — $(3\pi/2)$
k. $\tan^{-1}(-1) + \cos^{-1}(-1/\sqrt{2})$ — $(\pi/2)$
l. $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\sqrt{3}/2)$ — $(-\pi/6)$

3. Evaluate the following:

a. $\tan(2\tan^{-1}1/5)$ — $(5/12)$
b. $\sin(\tan^{-1}(-\sqrt{3}) + \cos^{-1}(-\sqrt{3}/2)$ — $(1)$
c. $\tan(2\tan^{-1}1/5-\pi/4)$ — $(-7/17)$
d. $\tan(1/2\cos^{-1}\sqrt{5}/3)$ — $(3-\sqrt{5})/2$
e. $\cos(\sec^{-1}x + \text{cosec}^{-1}x)$ — $(0)$

4. Write the following in the simplest form:

a. $\cos^{-1}\dfrac{1-x}{1+x}$ — $(2\tan^{-1}\sqrt{x})$

b. $\tan^{-1}\dfrac{\cos x}{1+\sin x}$ — $(\pi/4-x/2)$

c $\tan\text{-}1\ \dfrac{\sqrt{1+x^2}-1}{x}$ — $(1/2\tan^{-1}x)$

d. $\tan^{-1}\sqrt{(a-x)/(a+x)}$ — $(1/2\cos^{-1}x/a)$

e. $\sin^{-1}\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}$ — $(\sin^{-1}x+\pi/4)$

f. $\tan^{-1}\dfrac{a+bx}{b-ax}$ $(\tan^{-1}a/b + \tan^{-1}x)$

g. $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$ $(\sin^{-1}x - \sin^{-1}\sqrt{x})$

h. $\cos^{-1}\{x/(\sqrt{a^2+x^2})\}$ $(\cot^{-1}x/a)$

i .$\tan^{-1}\{x/(1+6x^2)\}$ $(\tan^{-1}3x - \tan^{-1}2x)$

j. $\tan^{-1}\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$ $(\pi/4 + 1/2\cos^{-1}x^2/a^2)$

k. $\cot^{-1}(1-x+x^2)$ $(\tan^{-1}x + \tan^{-1}(1-x))$

5. Prove the following:

a) $-\sin^{-1}4/5 + \sin^{-1}16/65 = \pi/2$
b) $\sin^{-1}4/5 + \sin^{-1}5/13 + \sin^{-1}16/65 = \pi/2$
c) $2\cos^{-1}3/\sqrt{13} + \cot^{-1}16/63 + 1/2\cos^{-1}7/25 = \pi$
d) $\tan^{-1}1/7 + \tan^{-1}1/13 = \tan^{-1}2/9$
e) $\sin^{-1}4/5 + 2\tan^{-1}1/3 = \pi/2$
f) $\tan^{-1}2/3 = 1/2\tan^{-1}12/5$
g) $\tan^{-1}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \pi/4 + 1/2\cos^{-1}x^2$

6. If $\cos^{-1}x/a + \cos^{-1}y/b = \alpha$, then prove that $\dfrac{x^2}{a^2} - \dfrac{2xy\cos\alpha}{ab} + \dfrac{y^2}{b^2} = \sin^2\alpha$

7. Solve for x:
a. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}8/31$ (x= 1/4)
b. $\tan^{-1}\dfrac{x-1}{x+1} + \tan^{-1}\dfrac{2x-1}{2x+1} = \tan^{-1}\dfrac{23}{36}$ (x = 4/3)
c. $\cos^{-1}x + \sin^{-1}x/2 = \pi/6$ (x = 1)
d. $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$ (x =0, ±1/2)
e. $\cot^{-1}x - \cot^{-1}(x+2) = \pi/12$, x>0 $(x = \sqrt{3})$
f. $\sin^{-1}x + \sin^{-1}2x = \pi/3$ $(x = 1/2\sqrt{3/7})$
g. $\sin(\cot^{-1}\cos(\tan^{-1}x)) = 0$ (not defined)

8. Prove that $\tan^{-1}\dfrac{1-x}{1+x} - \tan^{-1}\dfrac{1-y}{1+y} = \sin^{-1}\dfrac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$

9. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ or $\pi/2$. Prove that $x + y + z = xyz$ or $xy + yz + zx = 1$.